Presented on

Thursday, August 5' 1999, 15:00

In Utrecht University

# Axiomatization of End-Extension Kripke Models

## and Applications to Heyting Arithmetic

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Theorem Each node of an end-extension Kripke model of HA, is a classical model of $B\Sigma_1$.

(that is the principal $\mathcal{B}$:

$$\forall x \leq a \, \exists y \, \varphi(x,y) \to \exists z \, \forall x \leq a \, \exists y \leq z \, \varphi(x,y)$$

restricted to $\Sigma_1$ formulas: $\varphi \in \Sigma_1$)

each node of a kripke model can be considered as a classical structure (with same language), and one can define a satisfaction relation that for the atomic formulas is equivallent to forcing.

Conjecture Every Kripke model of HA is PA-normal i.e. each node is a classical model of PA.

This is proved for kripke models on finite frames, frames are $\omega$, ... (van Dalen et al & wehmeier)

Conjecture By removing some nodes of any kripke model of HA, we can obtain a PA-normal Kripke model that is equivallent to the former one

Theorem Each node of a Kripke model of HA is a classical model of $I\Delta_1$.

It is unclear that

How rich are the end-extension Kripke models of HA, ---?

Is HA complete w.r.t its class of end-ext. models?

**Definition** A Kripke model $\mathcal{K}$ is end-extension if for all nodes $k, k'$

$$k \leq k' \Rightarrow Dk \subseteq_e Dk'$$

$$(a \in Dk,\ b \in Dk',\ b \leq a \rightarrow b \in Dk)$$

for models of HA this def. is OK, since $\leq$ is decidable.

**Definition** A Kripke model, of the language $\Sigma$ which contains a binary predicate $\leq$, is end-extension if for all nodes $k, k'$

$$a \in Dk,\ b \in Dk',\ k' \Vdash b \leq a \rightarrow b \in Dk,\ k \Vdash b \leq a$$

**LO**
$$\begin{cases} x \leq x \\ x \leq y \wedge y \leq x \rightarrow x = y \\ x \leq y \wedge y \leq z \Rightarrow x \leq z \\ x \leq y \vee \neg x \leq y \end{cases}$$

If we want to say that, new elements are strictly greater than old ones, we will need the decidability.

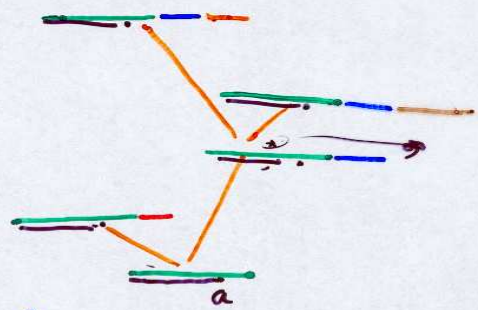
We are going to axiomatize the class of end-extension Kripke models of LO.

Constant domain Kripke models are axiomatized by

CD $\quad \forall x (A \vee B(x)) \longrightarrow A \vee \forall x B(x)$

$x$ is not free in $A$

let $a \in D_k$ in an end-extension Kripke model, and dominate the domains of the upper nodes of $k$, to the elements $\leq a$, What do we get?




A constant domain Kripke model

so it seems that it is enough to restrict all the quantifiers of CD to $\leq a$, to axiomatize end-extension Kripke models

but just one restriction is enough

$$\forall x \leq a (A \vee B(x)) \longrightarrow A \vee \forall x \leq a\, B(x)$$

**Definition** Let EE be the schema

$$\forall y \left[ \forall x \le y \, (A(y) \vee B(x,y)) \rightarrow A(y) \vee \forall x \le y \, B(x,y) \right]$$

in which $x$ is not free in $A$.

**Theorem (soundness)**, The schema EE is valid in any end-extension Kripke model of LO.

**Proof**

$a \in D_k$ $\quad k \Vdash \forall x \le a (A \vee B(x))$

$k \nVdash A \vee \forall x \le a \, B(x)$

$k \nVdash A \quad k \nVdash \forall x \le a \, B(x)$

$\exists k' \ge k \quad b \in D_{k'} \quad k' \Vdash b \le a \quad k' \nVdash B(b)$

$b \in D_k$

(by LO) $\quad k \Vdash b \le a$

(by hypothesis) $\quad k \Vdash A \vee B(b)$ $\quad$ ✱

**Theorem (Completeness)** The class of end-extension Kripke models of LO is sound and strongly complete w.r.t LO + EE.

**Theorem** HA is strongly complete w.r.t its class of end-extension Kripke models.

**Proof** Let

$$\psi(y) \equiv \forall z\left[\forall x\le y\left(A(y+z)\vee B(x,y+z)\right)\to A(y+z)\vee \forall x\le y B(x,y+z)\right]$$

$HA\vdash\psi(0)$ , $HA\vdash\psi(y)\to\psi(y+1)$ ; so $HA\vdash EE$.

for $i\Pi_2$ (and any of its fragments) this doesnot hold:

Consider the first two nodes of Buss's Kripke model, which he constructed to show that $\mathcal{H}(PA)\neq HA$.

that is


$M_0\subseteq M_1$ , $a\in M_0$ , $b\in M_1\setminus M_0$

$M_0\models PA+\neg Con\, I\Sigma_a+\forall x<a\, Con\, I\Sigma_x$

$M_1\models PA+\Delta_0(M_0)+\neg Con\, I\Sigma_b+b<a$

now for the formulas

$$\alpha(x)= Con\, I\Sigma_x\vee\left(0<x\wedge\neg Con\, I\Sigma_{x-1}\right)$$
$$\beta(x)= 0<x\wedge Pr_{I\Sigma_{x-1}}(\bar{0}=\bar{1})$$

we have
$$0\Vdash\forall x\le a\left(\beta(a)\vee\alpha(x)\right)$$
$$0\nVdash\beta(a)\vee\forall x\le a\,\alpha(x)$$

on the other hand $0\Vdash i\Pi_2$ ; so $i\Pi_2\nvdash EE$

So, this is a two node PA-normal Kripke model that is not a model of HA.

We axiomatized the class of end-extension Kripke models of LO by LO+EE. We can remove the decidability of $\leq$ from LO by a small modification in the definition of end-extension Kripke model.

Also we can remove the linearity of $\leq$;

**Theorem** For any formula $F(x,\bar{y})$, the class of Kripke models which satisfy the condition

$$k \leq k',\ \bar{a} \in Dk,\ b \in Dk',\ k' \Vdash F(b,\bar{a}) \Rightarrow b \in Dk,\ k \Vdash F(b,\bar{a})$$

is axiomatized by

$$\forall x,\bar{y}\,(F(x,\bar{y}) \vee \neg F(x,\bar{y})) \quad +$$

$$\forall x\,(F(x,\bar{y}) \to A \vee B(x)) \longrightarrow A \vee \forall x\,(F(x,\bar{y}) \to B(x))$$

in which $x$ is not free in $A$.

One can also remove the condition $k \Vdash F(b,\bar{a})$; that is the class of Kripke models satisfying

$k \leq k'$; $\bar{a} \in Dk$, $b \in Dk'$, $k' \Vdash F(b,\bar{a}) \Rightarrow b \in Dk$ is axiomatized by

$$\forall x\,[(A_0 \to \neg F(x,\bar{y}) \vee \varphi(x,\bar{y})) \vee B_0] \to [A_0 \to \forall x\,(\neg F(x,\bar{y}) \vee \varphi(x,\bar{y}))] \vee B_0$$

$$\forall x\,[(A_1 \to [(A_0 \to \neg F(x,\bar{y}) \vee \varphi(x,\bar{y})) \vee B_0]) \vee B_1] \to (A_1 \to [(A_0 \to \forall x(\neg F(x,\bar{y}) \vee \varphi(x,\bar{y})) \vee B_0)]) \vee B_1$$

$$\vdots$$

$$\forall x\,[(A_n \to \cdots) \vee B_n] \to (A_n \to \cdots) \vee B_n$$

$$\vdots$$

in which $x$ is not free in $A_0, B_0, A_1, B_1, \ldots, A_n, B_n, \ldots$

This class of models is the same of $\neg F$-expansions.