4.6 Determine u_b from Potential-Flow Theory

This development paralleled that of Davies *et al.,* (1950) for determining the rise velocity U_b of gas slug or bubble ascending in an otherwise stagnant inviscid liquid in a vertical tube of radius a. By Superimposing a downwards velocity U_b on the system, the analysis was performed for liquid streaming downwards past a stationary bubble, as shown in Figure 4.20.

The following relationships may be assumed for the velocities potential function, and stream function in the coordinate system shown, with symmetry about the z-axis:

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial z} \tag{13}$$

$$v_z = \frac{\partial \phi}{\partial z} = -\frac{1}{r}\frac{\partial \psi}{\partial r} \tag{14}$$

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{15}$$

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{16}$$

The following forms are assumed for the potential and stream function:

$$\phi = U_b z - Ae^{kz/a} f(r) \tag{17}$$

$$\psi = -\frac{1}{2}U_b r^2 + Are^{kz/a} g(r) \tag{18}$$

Substitute (17) in (15),

Also note that:

$$\frac{\partial \phi}{\partial r} = -Ae^{kz/a} f'(r) \tag{19}$$

$$\frac{\partial^2 \phi}{\partial r^2} = -Ae^{kz/a} f''(r) \tag{20}$$

$$\frac{\partial \phi}{\partial z} = U_b - A\left(\frac{k}{a}\right)e^{kz/a}f(r) \tag{21}$$

$$\frac{\partial \phi}{\partial z} = -A\left(\frac{k}{a}\right)^2 e^{kz/a}f(r) \tag{22}$$

Thus Eqn. (15) becomes

$$-Ae^{kz/a}f''(r) - \frac{1}{r}Ae^{kz/a}f'(r) - A\left(\frac{k}{a}\right)^2 e^{kz/a}f(r) = 0 \tag{23}$$

Deviding both sides by $-Ae^{kz/a}$

$$f''(r) + \frac{1}{r}f'(r) + \left(\frac{k}{a}\right)^2 f(r) = 0 \tag{24}$$

and multiplying both sides r^2

$$r^2 f''(r) + rf'(r) + \left(\frac{kr}{a}\right)^2 f(r) = 0 \tag{25}$$

Also note that for Bessel's equation:

$$x^2 f''(x) + xf'(x) + \left(\lambda x^2 - n^2\right)f(x) = 0 \tag{26}$$

$$\lambda = \alpha^2 \tag{27}$$

The solution is:

$$f(x) = J_n(\alpha x) \tag{28}$$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(1+n+k)}\left(\frac{x}{2}\right)^{2k+n} \tag{29}$$

And the following relations will be needed:

$$\frac{d}{dx}J_0(x) = -J_1(x) \tag{30}$$

$$\frac{d}{dx}[xJ_1(x)] = xJ_0(x) \tag{31}$$

$$J_0(0) = 1 \tag{32}$$

$$J_1(0) = 0 \tag{33}$$

Therefore Eqn. (25) becomes

$$r^2 f''(r) + rf'(r) + \left(\left(\frac{k}{a}\right)^2 r^2 - 0\right) f(r) = 0 \tag{34}$$

So

$$\lambda = \frac{k^2}{a^2} = \alpha^2 \tag{35}$$

$$\alpha = \frac{k}{a} \tag{36}$$

$$n = 0 \tag{37}$$

Therefore the solution of Eqn. (25) is:

$$f(r) = J_0\left(\frac{k}{a} r\right) \tag{38}$$

In a similar way, substitute (18) in (16):

$$r^2 g''(r) + rg'(r) + \left(\left(\frac{k}{a}\right)^2 r^2 - 1\right) g(r) = 0 \tag{39}$$

The solution of Eqn. (39) is:

$$g(r) = J_1\left(\frac{k}{a} r\right) \tag{40}$$

Therefore, the potential and stream functions are:

$$\phi = U_b z - Ae^{kz/a} J_0\left(\frac{k}{a} r\right) \tag{41}$$

$$\psi = -\frac{1}{2} U_b r^2 + Are^{kz/a} J_1\left(\frac{k}{a} r\right) \tag{42}$$

Therefore, Eqn. (13) becomes:

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial z} = A\left(\frac{k}{a}\right) e^{kz/a} J_1\left(\frac{k}{a} r\right) \tag{43}$$

and Eqn. (14) becomes:

$$v_z = \frac{\partial \phi}{\partial z} = -\frac{1}{r}\frac{\partial \psi}{\partial r} = U_b - A\left(\frac{k}{a}\right) e^{kz/a} J_0\left(\frac{k}{a} r\right) \tag{44}$$

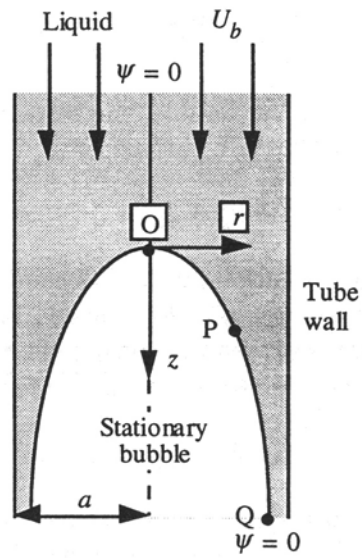

Figure 4.20 Gas slug in a tube.

Far above the nose of the bubble (z is large negative.), check at r =0:

$$v_r = A\left(\frac{k}{a}\right)e^{kz/a}J_1(0) = 0 \tag{45}$$

$$v_z = U_b - A\left(\frac{k}{a}\right)e^{kz/a}J_0(0) = U_b - A\left(\frac{k}{a}\right)e^{kz/a} \tag{46}$$

Because z is large negative, so that $e^{kz/a} \approx 0$.

$$v_z = U_b \tag{47}$$

And

$$v = \sqrt{v_z^2 + v_r^2} = v_z \tag{48}$$

$$v = U_b \tag{49}$$

It shows that the velocity is uniformly U_b downwards at far above the nose of the bubble. Since there is no radial velocity component at the tube wall (r=a).

$$v_r = A\left(\frac{k}{a}\right)e^{kz/a}J_1\left(\frac{k}{a}r\right) = 0 \tag{50}$$

$$A\left(\frac{k}{a}\right)e^{kz/a}J_1(k) = 0 \tag{51}$$

$$A \neq 0 \tag{52}$$

$$k \neq 0 \tag{53}$$

$$\therefore J_1(k) = 0 \tag{54}$$

The lowest root of that equation is k=3.832.

Along r=0, the centerline above the nose of the bubble at the origin O:

$$\psi = -\frac{1}{2}U_b r^2 + Are^{kz/a} J_1\left(\frac{k}{a}r\right) \tag{55}$$

$$\psi = 0 \tag{56}$$

At the wall (r=a) is also streamline, whose value of ψ corresponds to a flow rate through the tube of $\pi a^2 U_b$. The stream function at a point is ψ if the flow rate in the negative z direction, through a circle formed by rotating the point about the axis of symmetry, is $2\pi\psi$.

$$-\pi a^2 U_b = 2\pi\psi \tag{57}$$

$$\psi = -\frac{1}{2}U_b a^2 \tag{58}$$

At the stagnant point O ,z=0, r=0

$$v_r = A\left(\frac{k}{a}\right)e^{kz/a} J_1\left(\frac{k}{a}r\right) = A\left(\frac{k}{a}\right)J_1(0) = 0 \tag{59}$$

$$v_z = U_b - A\left(\frac{k}{a}\right)e^{kz/a} J_0\left(\frac{k}{a}r\right) = U_b - A\left(\frac{k}{a}\right)J_0(0) = U_b - A\left(\frac{k}{a}\right) \tag{60}$$

The requirement that v_z is zero gives

$$v_z = U_b - A\left(\frac{k}{a}\right) = 0 \tag{61}$$

$$A = \frac{aU_b}{k} \tag{62}$$

Note that because of the gas in the bubble, the pressure is uniform along OPQ, so that the Bernoulli condition along the bubble surface is:

$$v_r^2 + v_z^2 = 2gz \tag{63}$$

With only a single term in the approximation for ϕ, this condition can only be observed at a single point, which we shall take at r= a/2. By considering the streamline ψ=0, and assuming that $J_1(3.832/2)=0.580$ and also assuming that $J_0(3.832/2)=0.273$.

Along the streamline ψ=0:

$$\psi = -\frac{1}{2}U_b r^2 + Are^{kz/a} J_1\left(\frac{k}{a}r\right) \tag{64}$$

$$0 = -\frac{1}{2}U_b \frac{a^2}{4} + \frac{aU_b}{k}\left(\frac{a}{2}\right)e^{kz/a} J_1\left(\frac{k}{2}\right) \tag{65}$$

$$0 = -\frac{1}{8}U_b a^2 + \frac{a^2 U_b}{2k} e^{kz/a} J_1\left(\frac{k}{2}\right) \tag{66}$$

$$\therefore \frac{1}{8}U_b a^2 = \frac{a^2 U_b}{2k} e^{kz/a} J_1\left(\frac{k}{2}\right) \tag{67}$$

$$\frac{1}{8}U_b a^2 = \frac{a^2 U_b}{2(3.832)} e^{3.832z/a} J_1\left(\frac{3.832}{2}\right) \tag{68}$$

$$\frac{1}{8}U_b a^2 = \frac{a^2 U_b}{7.664} e^{3.832z/a}(0.580) \tag{69}$$

$$e^{3.832z/a} = 1.651 \tag{70}$$

$$3.832z/a = 0.501 \tag{71}$$

$$z/a = 0.131 \tag{72}$$

From Eqn. (63)

$$v_r^2 + v_z^2 = 2gz \tag{63}$$

$$\left[A\left(\frac{k}{a}\right)e^{kz/a} J_1\left(\frac{k}{a}r\right)\right]^2 + \left[U_b - A\left(\frac{k}{a}\right)e^{kz/a} J_0\left(\frac{k}{a}r\right)\right]^2 = 2gz \tag{73}$$

$$\left[\frac{aU_b}{k}\left(\frac{k}{a}\right)e^{kz/a} J_1\left(\frac{k}{2}\right)\right]^2 + \left[U_b - \frac{aU_b}{k}\left(\frac{k}{a}\right)e^{kz/a} J_0\left(\frac{k}{2}\right)\right]^2 = 2gz \tag{74}$$

$$\left[U_b e^{3.832(0.131)} J_1\left(\frac{3.832}{2}\right)\right]^2 + \left[U_b - U_b e^{3.832(0.131)} J_0\left(\frac{3.832}{2}\right)\right]^2 = 2gz \tag{75}$$

$$[U_b(1.652)(0.580)]^2 + [U_b - U_b(1.652)(0.273)]^2 = 2gz \tag{76}$$

$$0.918U_b^2 + 0.301U_b^2 = 2gz \tag{77}$$

$$1.219U_b^2 = 2gz \tag{78}$$

$$U_b^2 = 1.641gz \tag{79}$$

$$z = 0.131a \tag{80}$$

$$U_b^2 = 0.215ga \tag{81}$$

$$U_b = 0.464\sqrt{ga} \tag{82}$$

and a =D/2

$$U_b = 0.464\sqrt{gD/2} \tag{83}$$

$$U_b = 0.328\sqrt{gD} \tag{84}$$

$$\therefore U_b = c\sqrt{gD} \tag{85}$$

$$c \approx 0.33 \tag{86}$$

$$U_b = 0.33\sqrt{gD} \tag{87}$$