# FINDING THE GENERATING PARAMETERS OF A GIVEN PRIMITIVE PYTHAGOREAN TRIPLE

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## 1. Finding m and n if given a primitive Pythagorean triple

[1]

**Claim 1.** *If $a^2 + b^2 = c^2$ is a primitive Pythagorean triangle, where $a$ is odd, then each of the fractions*

$$\frac{c+a}{b}, \quad \frac{c+b+a}{c+b-a}, \quad \frac{b}{c-a}, \ \textit{and} \ \frac{a+c-b}{a+b-c}$$

*(each fraction reduced to lowest terms) is equal to $\frac{m}{n}$ where*

$$a = m^2 - n^2, \quad b = 2mn, \quad \textit{and} \quad c = m^2 + n^2 .$$

*Proof.*

$$\frac{c+a}{b} = \frac{\left(m^2+n^2\right) + \left(m^2-n^2\right)}{2mn} = \frac{2m^2}{2mn} = \frac{m}{n},$$

$$\frac{c+b+a}{c+b-a} = \frac{\left(m^2+n^2\right) + (2mn) + \left(m^2-n^2\right)}{(m^2+n^2) + (2mn) - (m^2-n^2)} = \frac{2m(m+n)}{2n(n+m)} = \frac{m}{n},$$

$$\frac{b}{c-a} = \frac{2mn}{(m^2+n^2) - (m^2-n^2)} = \frac{2mn}{2n^2} = \frac{m}{n},$$

$$\text{and} \quad \frac{a+c-b}{a+b-c} = \frac{\left(m^2-n^2\right) + \left(m^2+n^2\right) - (2mn)}{(m^2-n^2) + (2mn) - (m^2+n^2)} = \frac{2m(m-n)}{2n(m-n)} = \frac{m}{n}.$$

□

**Claim 2.** *If $a^2 + b^2 = c^2$ is a primitive Pythagorean triangle, where $a$ is even, then each of the fractions*

$$\frac{c+a}{b}, \quad \frac{c+b+a}{c+b-a}, \quad \frac{b}{c-a}, \ \textit{and} \ \frac{a+c-b}{a+b-c}$$

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[1]Apparently, although I was unaware of this fact until recently, Jekuthiel Ginsburg was first to publish the method of finding the generating parameters for a given PPT by setting $\frac{c+a}{b}$(reduced to lowest terms) equal to $\frac{m}{n}$, see The Generators of a Pythagorean triangle, **Scripta Math**. 11 (1945), p. 188.

*(each fraction reduced to lowest terms) is equal to* $\frac{m}{n}$ *where*

$$a = \frac{m^2 - n^2}{2}, \quad b = mn, \quad \textit{and} \quad c = \frac{m^2 + n^2}{2}.$$

*Proof.* Since $a, b, c$ is a PPT, there exists relatively prime, positive integers, $u$ and $v$, $u > v$, $u$ and $v$ of opposite parity such that

$$a = 2uv\,, \quad b = u^2 - v^2\,, \quad \text{and} \quad c = u^2 + v^2\,.$$

So,

$$\frac{c+a}{b} = \frac{u^2 + v^2 + 2uv}{u^2 - v^2} = \frac{u+v}{u-v}\,.$$

$$\frac{c+b+a}{c+b-a} = \frac{u^2 + v^2 + u^2 - v^2 + 2uv}{u^2 + v^2 + u^2 - v^2 - 2uv} = \frac{2u(u+v)}{2u(u-v)} = \frac{u+v}{u-v}\,.$$

$$\frac{b}{c-a} = \frac{u^2 - v^2}{u^2 + v^2 - 2uv} = \frac{(u+v)(u-v)}{(u-v)^2} = \frac{u+v}{u-v}\,.$$

$$\frac{a+c-b}{a+b-c} = \frac{2uv + u^2 + v^2 - u^2 + v^2}{2uv + u^2 - v^2 - u^2 - v^2} = \frac{2v(u+v)}{2v(u-v)} = \frac{u+v}{u-v}\,.$$

Let $m = u + v$ and $n = u - v$. Then

$$u = \frac{m+n}{2} \quad \text{and} \quad v = \frac{m-n}{2}\,.$$

Hence, we have,

$$a = 2uv = \frac{m^2 - n^2}{2}, \quad b = u^2 - v^2 = mn, \quad \text{and} \quad c = u^2 + v^2 = \frac{m^2 + n^2}{2}\,.$$

□

## Examples

$3^2 + 4^2 = 5^2$ is a primitive Pythagorean triple and 3 is **odd**. So, let $a = 3$. Then

$$\frac{5+3}{4} = \frac{5+4+3}{5+4-3} = \frac{4}{5-3} = \frac{3+5-4}{3+4-5} = \frac{2}{1}\,.$$

Hence

$$3 = 2^2 - 1^2, \quad 4 = 2(2)(1), \quad \text{and} 5 = 2^2 + 1^2\,.$$

$20^2 + 21^2 = 29^2$ is a PPT and 21 is **odd**. So

$$\frac{29+21}{20} = \frac{29+20+21}{29+20-21} = \frac{20}{29-21} = \frac{21+29-20}{21+20-29} = \frac{5}{2}\,.$$

Hence,

$$20 = 2(5)(2), \quad 21 = 5^2 - 2^2\,, \quad \text{and} \quad 29 = 5^2 + 2^2\,.$$

$4^2+3^2=5^2$ is a primitive Pythagorean triple and 4 is **even**. So, let $a=4$. Then

$$\frac{5+4}{3}=\frac{5+3+4}{5+3-4}=\frac{3}{5-4}=\frac{4+5-3}{4+3-5}=\frac{3}{1}\,.$$

Thus

$$4=\frac{3^2-1^2}{2},\quad 3=(3)(1),\quad \text{and}\quad 5=\frac{3^2+1^2}{2}\,.$$

$20^2+21^2=29^2$ is a PPT and 20 is **even**. So let a=20. Then

$$\frac{29+20}{21}=\frac{29+21+20}{29+21-20}=\frac{21}{29-20}=\frac{20+29-21}{20+21-29}=\frac{7}{3}\,.$$

Therefore

$$20=\frac{7^2-3^2}{2},\quad 21=(7)(3),\quad \text{and}\quad 29=\frac{7^2+3^2}{2}\,.$$

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