120 DEGREE AND 60 DEGREE TRIANGLE PRESERVING MATRICES.

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1. 120 DEGREE PRIMITIVE TRIPLES,(A,B,C), WHERE A - B=1.

(a, b, c) is a primitive solution in positive integers to the 120 degree equation $a^2 + b^2 + ab = c^2$ if and only if there exists relatively prime, positive integers m and n, $m > n$, $m \not\equiv n \pmod 3$ such that

$$a = m^2 - n^2, \quad b = 2mn + n^2, \quad \text{and} \quad c = m^2 + n^2 + mn. \tag{1}$$

Finding solutions where a-b=1

(We'll need these first four *recursive* solutions to find the first matrix.)

Let $a - b = \left(m^2 - n^2\right) - \left(2mn + n^2\right) = (m - n)^2 - 3n^2 = 1$, a Pell equation. Hence, solutions are

$$m - n = \frac{(2 + \sqrt{3})^s + (2 - \sqrt{3})^s}{2} = f(s) \quad \text{and} \quad n = \frac{(2 + \sqrt{3})^s - (2 - \sqrt{3})^s}{2\sqrt{3}} = g(s).$$

Then $m = f(s) + g(s)$ and $n = g(s)$. Thus

$$\begin{aligned} a =& \big(f(s) + g(s)\big)^2 - g(s)^2\,, \\ b =& 2\big(f(s) + g(s)\big)g(s) + g(s)^2\,, \\ \text{and} \quad c =& \big(f(s) + g(s)\big)^2 + g(s)^2 + \big(f(s) + g(s)\big)\, g(s)\,. \end{aligned}$$

Examples

TABLE 1. $a - b = 1$.

s	f(s)	g(s)	m	n	a	b	c
1	2	1	3	1	8	7	13
2	7	4	11	4	105	104	181
3	26	15	41	15	1456	1455	2521
4	97	56	153	56	20273	20272	35113

1.1. **A 120 degree triangle preserving matrix.** Let

$$\mathbf{M} = \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{pmatrix}.$$

From table (1), set

$$\mathbf{A} = \begin{pmatrix} 8 & 7 & 13 \\ 105 & 104 & 181 \\ 1456 & 1455 & 2521 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 105 & 104 & 181 \\ 1456 & 1455 & 2521 \\ 20273 & 20272 & 35113 \end{pmatrix}.$$

So solving for $\mathbf{M}$ in the matrix equation

$$\mathbf{A}\,\mathbf{M} = \mathbf{B}$$

we find that

$$\boxed{\mathbf{M} = \begin{pmatrix} 4 & 3 & 6 \\ 3 & 4 & 6 \\ 4 & 4 & 7 \end{pmatrix}.} \tag{2}$$

If $a^2 + b^2 + ab = c^2$ then

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 4 & 3 & 6 \\ 3 & 4 & 6 \\ 4 & 4 & 7 \end{pmatrix} = \begin{pmatrix} u & v & w \end{pmatrix}$$

where $u^2 + v^2 + uv = w^2$. And

$$\begin{pmatrix} m^2 - n^2 & 2mn + n^2 & m^2 + n^2 + mn \end{pmatrix} \mathbf{M} = \begin{pmatrix} s^2 - t^2 & 2st + t^2 & s^2 + t^2 + st \end{pmatrix}$$

where $s = 3m + 2n$ and $t = m + n$. So, if $(m^2 - n^2, 2mn + n^2, m^2 + n^2 + mn)$ is a primitive triple, s and t are, relatively prime, positive integers, $s > t$, where, $3 \nmid s - t = 2m + n$. That is, $\mathbf{M}$ takes a primitive triple and outputs a primitive triple.

Note that $u - v = a - b$. For example.

$$\begin{pmatrix} 3 & 5 & 7 \end{pmatrix} \mathbf{M} = \begin{pmatrix} 55 & 57 & 97 \end{pmatrix}.$$

If n is a positive integer then

$$\begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \mathbf{M}^n = \begin{pmatrix} a_n & b_n & c_n \end{pmatrix},$$

where (a_n, b_n, c_n) is the n^{th} solution to the primitive 120 degree triangle $a^2 + b^2 + ab = c^2$ such that $a_n - b_n = 1$.

1.2. **60 degree triangle preserving matrices.** The 120 degree triangle preserving matrix

$$\mathbf{M} = \begin{pmatrix} 4 & 3 & 6 \\ 3 & 4 & 6 \\ 4 & 4 & 7 \end{pmatrix},$$

we found in the previous section, can be transformed into four 60 degree triangle preserving matrices. To do so, we will need the following identity,

$$x^2 + y^2 + xy = (x + y)^2 + y^2 - (x + y)y = x^2 + (x + y)^2 - x(x + y). \tag{3}$$

We have $a^2 + b^2 - ab = (-a)^2 + b^2 + (-a)b = c^2$. Then using $\mathbf{M}$,

$$\begin{pmatrix} -a & b & c \end{pmatrix} \begin{pmatrix} 4 & 3 & 6 \\ 3 & 4 & 6 \\ 4 & 4 & 7 \end{pmatrix} = \begin{pmatrix} u & v & w \end{pmatrix}$$

where $u^2+v^2+uv = w^2$. However, from equation (3), $(u+v)^2+v^2-(u+v)v = w^2$. Hence

$$\begin{pmatrix} -a & b & c \end{pmatrix} \begin{pmatrix} 4+3 & 3 & 6 \\ 3+4 & 4 & 6 \\ 4+4 & 4 & 7 \end{pmatrix} = \begin{pmatrix} u+v & v & w \end{pmatrix} = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -7 & -3 & -6 \\ 7 & 4 & 6 \\ 8 & 4 & 7 \end{pmatrix}$$

where $a^2+b^2-ab = c^2$ is a 60 degree triangle. Similarly, if $a_1^2+(-b_1)^2+a_1(-b_1) = c_1^2$, then

$$\begin{pmatrix} a_1 & b_1 & c_1 \end{pmatrix} \begin{pmatrix} 7 & 3 & 6 \\ -7 & -4 & -6 \\ 8 & 4 & 7 \end{pmatrix} = \begin{pmatrix} u_1+v_1 & v_1 & w_1 \end{pmatrix}$$

where $a_1^2 + b_1^2 - a_1\, b_1 = c_1^2$ and $(u_1+v_1)^2 + v_1^2 - (u_1+v_1)v_1 = w_1^2$.

Let

$$\boxed{\mathbf{S}_1 = \begin{pmatrix} 7 & 3 & 6 \\ -7 & -4 & -6 \\ 8 & 4 & 7 \end{pmatrix} \quad \text{and} \quad \mathbf{T}_1 = \begin{pmatrix} -7 & -3 & -6 \\ 7 & 4 & 6 \\ 8 & 4 & 7 \end{pmatrix}.}$$

Then, clearly,

$$\boxed{\mathbf{S}_2 = \begin{pmatrix} 4 & 7 & 6 \\ -3 & -7 & -6 \\ 4 & 8 & 7 \end{pmatrix} \quad \text{and} \quad \mathbf{T}_2 = \begin{pmatrix} -4 & -7 & -6 \\ 3 & 7 & 6 \\ 4 & 8 & 7 \end{pmatrix}}$$

are also 60 degree triangle preserving matrices.

For example, since $8^2 + 3^2 - (8)(3) = 7^2$,

$$\begin{pmatrix} 8 & 3 & 7 \end{pmatrix} \mathbf{S}_1 = \begin{pmatrix} 91 & 40 & 79 \end{pmatrix}, \qquad \begin{pmatrix} 8 & 3 & 7 \end{pmatrix} \mathbf{S}_2 = \begin{pmatrix} 51 & 91 & 79 \end{pmatrix},$$
$$\begin{pmatrix} 8 & 3 & 7 \end{pmatrix} \mathbf{T}_1 = \begin{pmatrix} 21 & 16 & 19 \end{pmatrix}, \qquad \begin{pmatrix} 8 & 3 & 7 \end{pmatrix} \mathbf{T}_2 = \begin{pmatrix} 5 & 21 & 19 \end{pmatrix}$$

where

$$91^2 + 40^2 - (91)(40) = 79^2, \qquad 51^2 + 91^2 - (51)(91) = 79^2,$$
$$21^2 + 16^2 - (21)(16) = 19^2, \qquad 5^2 + 21^2 - (5)(21) = 19^2.$$

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