120 DEGREE AND 60 DEGREE TRIPLES AND THE DIVISORS 3, 5, AND 7

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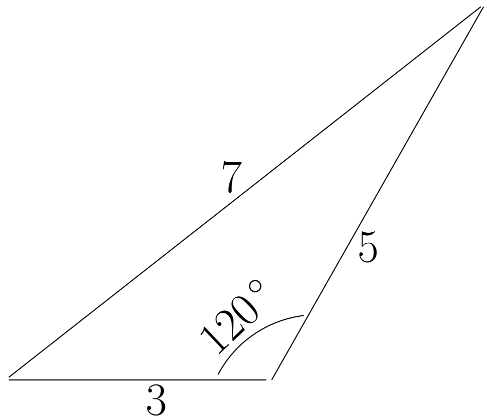


1. INTRODUCTION

It's well known that if (a, b, c) is a a Pythagorean triple, that is, if (a, b, c) is a solution in positive integers to the 90 degree triangle equation $a^2 + b^2 = c^2$, then 3 and 4 each divides a or b, and 5 divides a, b or c where, of course, $(3, 4, 5)$ is the smallest such solution.

A 120 degree triple, (a, b, c), is a solution in positive integers to the 120 degree triangle equation

$$a^2 + b^2 - 2ab \cos 120\,^\circ = a^2 + b^2 + ab = c^2.$$

So, naturally, one wonders if a similar relationship exists between the positive integer solutions of 120 degree triangles and the smallest such solution, $(3, 5, 7)$. To find such a relationship it's necessary to look more closely at the 3,4,5-ness of Pythagorean triangles. We will look at all integer solutions, both positive and negative.

$$a^2 + b^2 = (-a)^2 + b^2 = (-a)^2 + (-b)^2 = a^2 + (-b)^2 = c^2.$$

These 4 solutions are plotted in figure (1).

Hence, if (a,b,c) is a Pythagorean triple, saying 3 divides one of ab, $(-a)b$, $(-a)(-b)$, or $a(-b)$ is saying 3 divides a or b. And since 3,4,5-ness holds for Primitive triples, the case is the same for the divisor 4.

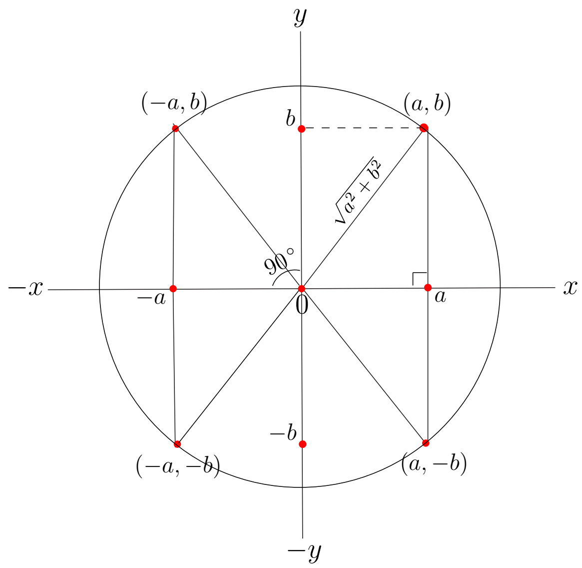


FIGURE 1. solutions to a 90 degree triangle equation

Note that:

$$a^2 + b^2 + ab = (-a-b)^2 + b^2 + (-a-b)b = (a+b)^2 + (-b)^2 + (a+b)(-b) = (-a)^2 + (-b)^2 + (-a)(-b)$$

.

$$= a^2 + (-a-b)^2 + a(-a-b) = (-a)^2 + (a+b)^2 + (-a)(a+b)$$

As shown in figure (2). So, similarly, if (a, b, c) is a 120 degree triple then saying a prime p divides one of ab, $(-a-b)b$, $(a+b)(-b)$, $(-a)(-b)$, $a(-a-b)$, or $(-a)(a+b)$ is saying p divides one of a, b, or $a+b$.

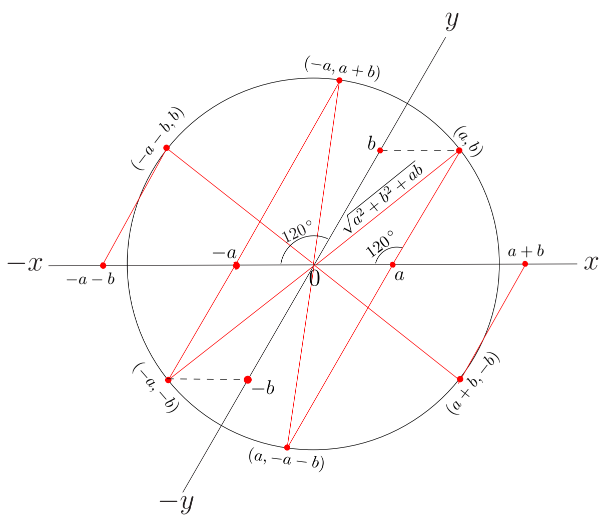


FIGURE 2. solutions to a 120 degree triangle equation

2. 120 degree triples and the divisors 3, 5, and 7.

All primitive solutions to a 120 degree triple (a, b, c), are given by the parametric equations:

$$a = m^2 - n^2, \quad b = 2mn + n^2, \quad \text{and} \quad c = m^2 + n^2 + mn. \tag{1}$$

where m and n are relatively prime, positive integers, $m > n$, and $3 \nmid m - n$. See *http://www.geocities.com/fredlb37/triples10.pdf* for a proof.

If (a, b, c) and (b, a, c) are considered the same solution, then the first 6 primitive solutions in order of smallest value for c are,

(1) $5^2 + 3^2 + 5 \cdot 3 = 7^2$
(2) $8^2 + 7^2 + 8 \cdot 7 = 13^2$
(3) $16^2 + 5^2 + 16 \cdot 5 = 19^2$
(4) $24^2 + 11^2 + 24 \cdot 11 = 31^2$
(5) $33^2 + 7^2 + 33 \cdot 7 = 37^2$
(6) $35^2 + 13^2 + 35 \cdot 13 = 43^2$

Notice that, in each case, 3 and 5, each, divides one of a, b, or $a+b$, and 7 divides one of a, b, $a + b$, or c.

Claim 1. *If (a, b, c) is any 120 degree triple then 3 and 5 divides $ab(a + b)$, and 7 divides $ab(a + b)c$.*

Proof. It's sufficient to show it's true for primitive triples. This claim can be proven directly by looking at residues modulo 3, 5, and 7; however it gets quite messy for the divisor 7. So, instead, I will use the parametric equations from (1) and the following result from Fermat's little theorem. That is, if s and t are integers, and p is a prime, then

$$p \text{ divides } st\left(s^{p-1} - t^{p-1}\right).$$

From (1),

$$\begin{aligned} ab(a + b) &= \left(m^2 - n^2\right)\left(2mn + n^2\right)\left(m^2 + 2mn\right) \\ &= mn\left(m^2 - n^2\right)\left(2\left(m^2 + n^2\right) - 5mn\right) \\ &= 2mn\left(m^4 - n^4\right) - 5\left(m^4n^2 - m^2n^4\right), \end{aligned}$$

and

$$\begin{aligned} ab(a + b)c &= \left(m^2 - n^2\right)\left(2mn + n^2\right)\left(m^2 + 2mn\right)\left(m^2 + n^2 + mn\right) \\ &= 2mn\left(m^6 - n^6\right) - 7\left(m^6n^2 - m^5n^3 + m^3n^5 - m^2n^6\right). \end{aligned}$$

Therefore, from Fermat's little theorem, 3 and 5 divide $ab(a + b)$, and 7 divides $ab(a + b)c$. $\square$

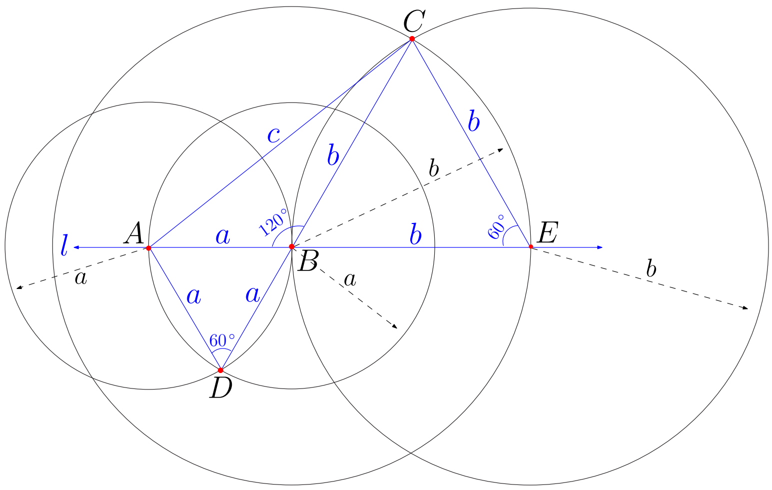
2.1. **120 degree triples and their associated 60 degree triples.** A 60 degree triple, (p, q, r), is a solution in positive integers to the 60 degree triangle equation

$$p^2 + q^2 - 2pq\cos 60^\circ = p^2 + q^2 - pq = r^2.$$

Note that

$$a^2 + b^2 + ab = (a+b)^2 + b^2 - (a+b)b = a^2 + (a+b)^2 - a(a+b).$$

Hence, if (a, b, c) is a 120 degree triple then $(a+b, b, c)$ and $(a, a+b, c)$ are 60 degree triples. Here is a "neat" way to construct these three triangles.



On line l layout line segments AB and BE having lengths a and b respectively, where a and b are the adjacent side lengths of a 120 degree triangle. On and below AB construct equilateral triangle ADB with sides of length a. On and above BE construct equilateral triangle BEC with sides of length b. Hence $\angle ABD$ and $\angle CBE$ are each 60 degrees. So point B lies on line segment DC and $\angle ABC$ is 120 degrees. Draw line segment AC. Thus, the construction shows the 120 degree triangle ABC and its two associated 60 degree triangles AEC and ADC.

3. 60 degree triples and the divisors 3, 5, and 7.

Let $u^2 + v^2 - uv = w^2$. If u, v, and w are positive integers, then (u, v, w) is a 60 degree triple. If, additionally, u, v, and w are pairwise relatively prime, then (u, v, w) is a primitive 60 degree triple. The first seven such triples in order of the smallest value for w are,

(1) $1^2 + 1^2 - 1 \cdot 1 = 1^2$
(2) $8^2 + 5^2 - 8 \cdot 5 = 7^2$
(3) $8^2 + 3^2 - 8 \cdot 3 = 7^2$
(4) $15^2 + 7^2 - 15 \cdot 7 = 13^2$
(5) $15^2 + 8^2 - 15 \cdot 8 = 13^2$

(6) $21^2 + 5^2 - 21 \cdot 5 = 19^2$

(7) $21^2 + 16^2 - 21 \cdot 16 = 19^2$

Notice that, in each case, 3 and 5, each, divides one of u, v, or $u - v$, and 7 divides one of u, v, $u - v$, or w.

Claim 2. *If (u, v, w) is any 60 degree triple then 3 and 5 divides $uv(u - v)$, and 7 divides $uv(u - v)w$.*

Proof. It's sufficient to show the claim is true for primitive triples. Clearly it's true for the triple $(1, 1, 1)$. So let $u^2 + v^2 - uv = w^2$ where (u, v, w) is a primitive triple, $uvw \neq 1$. Without loss of generality, let u be greater than v, then

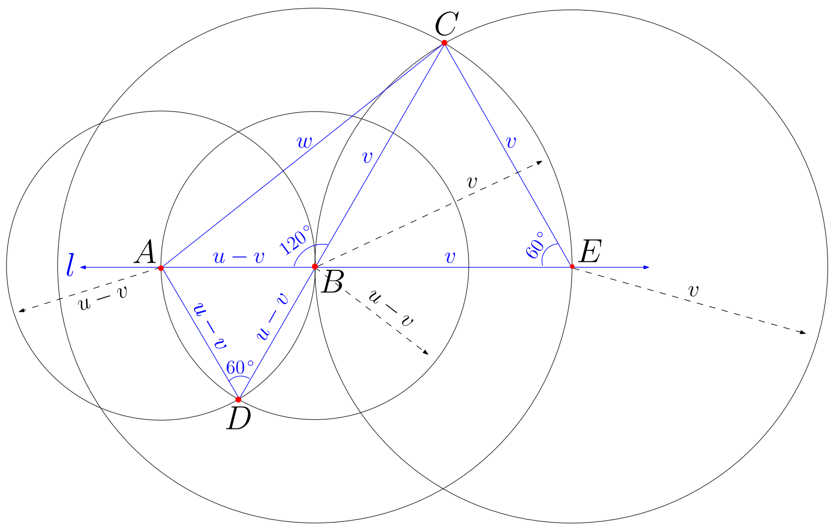
$$(u - v)^2 + v^2 + (u - v)v = u^2 + v^2 - uv = w^2.$$

Hence $(u - v, v, w)$ is a 120 degree triple. So, from claim (1),

$$3{\cdot}5 \mid (u{-}v)v\big((u{-}v){+}v\big) = uv(u{-}v), \quad \text{and} \quad 7 \mid (u{-}v)v\big((u{-}v){+}v\big)w = uv(u{-}v)w.$$

$\square$

The drawing below shows two 60 degree triangles AEC and ADC along with their associated 120 degree triangle ABC.



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