PYTHAGOREAN TRIANGLES, THE HYPOTENUSE TO A POWER.

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It is well-known that if (a, b, c) is a solution to a Pythagorean triangle, where c is the hypotenuse, then the Mersenne prime 3 divides ab, and the Fermat prime 5 divides abc. In this section, I show that if $\left(a, b, c^{2^j}\right)$ is a solution to a primitive Pythagorean triangle, where j is a non-negative integer, then every Mersenne prime less than or equal to $\left(2^{j+2} - 1\right)$ divides ab, and every Fermat prime less than or equal to $\left(2^{j+2} + 1\right)$ divides abc.

1. Introduction

If (a, b, c) is a solution to the Pythagorean triangle $a^2 + b^2 = c^2$ then (a, b, c) is a Pythagorean triple. If, additionally, a, b, and c are pairwise relatively prime then (a, b, c) is a primitive Pythagorean triple (PPT), and $a^2 + b^2 = c^2$ is a Primitive Pythagorean triangle.

All PPT's are given by the parametric equations

$$a = m^2 - n^2, \quad b = 2mn, \quad \text{and} \quad c = m^2 + n^2 \tag{1}$$

where m and n are relatively prime, positive integers of opposite parity, and $m > n$.

When computing the PPT $\left(a, b, c^{2^j}\right)$, it is convenient to express a, b, and c in terms of Gaussian integers $\mathbb{Z}[i]$. To do so, let $m > n$ where m and n are relatively prime, positive integers having opposite parity. Let $z = m + ni$ and $\bar{z} = m - ni$. And let j be a non-negative integer. Then there exists positive integers u and v such that

$$u = \left| \frac{z^{2^j} + \bar{z}^{2^j}}{2} \right| = \left| \frac{(u + vi) + (u - vi)}{2} \right|, \quad \text{and} \quad v = \left| \frac{z^{2^j} - \bar{z}^{2^j}}{2i} \right| = \left| \frac{(u + vi) - (u - vi)}{2i} \right|.$$

- If $u < v$ then interchange the labels.
- Clearly, $\gcd(m, n) = 1$ implies $\gcd(u, v) = 1$.
- The parameters m and n have opposite parity. Therefore $2 \nmid \left(m^2 + n^2\right)^{2^j} = u^2 + v^2$. Hence u and v have opposite parity.

Thus, all primitive Pythagorean triples of the form $\left(a, b, c^{2^j}\right)$ are given by the parametric equations

$$a = u^2 - v^2 = \left| \frac{(m + ni)^{2^{j+1}} + (m - ni)^{2^{j+1}}}{2} \right|, \tag{2}$$

$$b = 2uv = \left| \frac{(m + ni)^{2^{j+1}} - (m - ni)^{2^{j+1}}}{2i} \right|, \tag{3}$$

$$\text{and} \quad c^{2^j} = u^2 + v^2 = \left(m^2 + n^2\right)^{2^j} \quad . \tag{4}$$

Then

$$2ab = \left| \frac{(m+ni)^{2^{j+2}} - (m-ni)^{2^{j+2}}}{2i} \right| . \tag{5}$$

2. Some prime divisors of primitive Pythagorean triangles.

If a, b, c, k and d are positive integers such that $a^2 + b^2 = \left(c^k\right)^2$ is a primitive Pythagorean triangle, and if d divides k, then we are going to show the following:

- If $4d-1$ is a prime p, then p divides exactly one of a and b.
- If $4d+1$ is a prime q, then q divides exactly one of a, b, and c.
- If $d = 2^\alpha$, α a non-negative integer, then $2^{\alpha+2}$ divides exactly one of a and b.

I will, first, state and prove a theorem on the divisors of ab, and of abc, where $\left(a, b, c^k\right)$ is a PPT. Then the case where $k = 2^j$ will be proven in the corollary. If $k = 2^j$ in equation (5) then

$$2ab = \left| \frac{(m+ni)^{4k} - (m-ni)^{4k}}{2i} \right| .$$

Theorem 1. *Let $a^2+b^2 = \left(c^k\right)^2$ be a primitive Pythagorean triangle where $k \in \mathbb{Z}^+$. Let d be a positive integer divisor of k. Let $p = 4d-1$ and $q = 4d+1$.*

(a) *If p is a prime then $p|ab$.*
(b) *If q is a prime then $q|abc$.*

Proof of part (a). Since $m+ni \in \mathbb{Z}[i]$, then $\left((m+ni)^{\frac{k}{d}}\right)^{4d} = (M+Ni)^{4d}$ for some M, $N \in \mathbb{Z}$. If $p = 4d-1$ is a prime, then since $-i = i^{4d-1} = i^p$,

$$p|(M+Ni)^p - (M-Ni) = \left((M+Ni)^p - M^p - (Ni)^p\right) + (M^p - M) - i(N^p - N).$$

Which implies

$$p|(M+Ni)^{p+1} - \left(M^2+N^2\right).$$

Similarly,

$$p|(M-Ni)^{p+1} - \left(M^2+N^2\right).$$

Which implies

$$p \text{ divides } (M+Ni)^{p+1} - (M-Ni)^{p+1} = (m+ni)^{4k} - (m-ni)^{4k} = 4abi.$$

Therefore $p|ab$. □

Proof of part (b). If $q = 4d+1$ is a prime, then since $i = i^{4d+1} = i^q$,

$q|(M+Ni)^q - (M+Ni)$ where $M+Ni = (m+ni)^{\frac{k}{d}}$ and $M-Ni = (m-ni)^{\frac{k}{d}}$. Which implies

$$q|(M+Ni)\left((M+Ni)^{q-1} - 1\right).$$

Similarly

$$q|(M-Ni)\big((M-Ni)^{q-1}-1\big).$$

Then, if

$$q \nmid (M+Ni)(M-Ni) = M^2+N^2 = c^{\frac{k}{d}},$$

$$q \text{ divides } (M+Ni)^{q-1}-(M-Ni)^{q-1} = (m+ni)^{4k}-(m-ni)^{4k} = 4abi.$$

Therefore $q|abc$. □

Corollary 1. *Let $a^2+b^2=\left(c^{2^j}\right)^2$ be a primitive Pythagorean triangle where j is a nonnegative integer. Let M be any Mersenne prime less than or equal to $2^{j+2}-1$. And, let F be any Fermat prime less than or equal to $2^{j+2}+1$. Then*

$$\boxed{M \text{ divides } ab, \text{ and } F \text{ divides } abc.}$$

Proof. Let $k=2^j$ in theorem (1). Then

$$d = \left\{\, 2^0,\, 2^1,\, 2^2, \ldots, 2^{j-1},\, 2^j \,\right\},$$
$$4d-1 = \left\{\, 2^2-1,\, 2^3-1,\, 2^4-1, \ldots, 2^{j+1}-1,\, 2^{j+2}-1 \,\right\},$$
$$\text{and } 4d+1 = \left\{\, 2^2+1,\, 2^3+1,\, 2^4+1, \ldots, 2^{j+1}+1,\, 2^{j+2}+1 \,\right\}.$$

From Theorem (1), part (a), every prime in the set $4d-1$ divides ab. From Theorem (1), part (b), every prime in the set $4d+1$ divides abc. □

3. The divisor $\mathbf{4 = 2^{0+2}}$

Theorem 2. *If*

$$a^2+b^2=\left(c^{2^j}\right)^2$$

is a primitive Pythagorean triangle where j is a nonnegative integer then 2^{j+2} divides ab.

Proof. (by induction on j) If $j=0$ then

$$a^2+b^2=\left(c^{2^0}\right)^2=c^2.$$

So there exists integers m and n, one odd the other even, such that $a=m^2-n^2$ and $b=2mn$. Hence $4=2^{0+2}$ divides ab. Assume true for j, then

$$\left(c^{2^{j+1}}\right)^2 = \left[\left(c^{2^j}\right)^2\right]^2 = \left(a^2+b^2\right)^2 = \left(a^2-b^2\right)^2+(2ab)^2$$

. Let $a_1=a^2-b^2$ and $b_1=2ab$. Then if 2^{j+2} divides ab, $2^{(j+1)+2}$ divides a_1b_1. □

4. Examples

Triangle	k	d	Divisors of a or b	Divisors of a, b, or c
$a^2+b^2=c^2$	1	1	3, 2^2	3, 2^2, 5
$a^2+b^2=\left(c^2\right)^2$	2	1,2	3, 7, 2^3	3, 5, 7, 2^3
$a^2+b^2=\left(c^3\right)^2$	3	1,3	3, 2^2, 11	3, 2^2, 5, 11, 13
$a^2+b^2=\left(c^4\right)^2$	4	1,2,4	3, 7, 2^4	3, 5, 7, 2^4, 17
$a^2+b^2=\left(c^5\right)^2$	5	1,5	3, 2^2, 19	3, 2^2, 5, 19
$a^2+b^2=\left(c^6\right)^2$	6	1,2,3,6	3, 7, 2^3, 11, 23	3, 5, 7, 2^3, 11, 13, 23
$a^2+b^2=\left(c^7\right)^2$	7	1,7	3, 2^2	3, 2^2, 5, 29
$a^2+b^2=\left(c^8\right)^2$	8	1,2,4,8	3, 7, 31, 2^5	3, 5, 7, 17, 31, 2^5
$a^2+b^2=\left(c^9\right)^2$	9	1,3,9	3, 2^2, 11	3, 2^2, 5, 11, 13, 37
$a^2+b^2=\left(c^{10}\right)^2$	10	1,2,5,10	3, 7, 2^3, 19	3, 5, 7, 2^3, 19, 41

Table 1. Divisors of primitive Pythagorean triples

5. Area of a primitive Pythagorean triangle and the perfect numbers

A perfect number is a positive integer that is equal to the sum of its divisors excluding itself. For example 6 is a perfect number since its divisors are 1, 2, 3 and 6 where $1+2+3=6$. All even perfect numbers are of the form $2^{s-1}\left(2^s-1\right)$ where 2^s-1 is a prime. Such primes are called Mersenne primes. The first three perfect numbers are 6, 28, and 496. It's not known if there are any odd perfect numbers.

From corollary (1) and theorem (2) we see that it is no coincidence that the area of the $3-4-5$ triangle, the smallest Pythagorean triangle, is $\frac{1}{2}\left(2^2-1\right)\left(2^2\right)=2^1\left(2^2-1\right)=6$, the smallest perfect number.

Let t be a positive integer such that $2^{t+1}-1$ is a prime (Mersenne prime). And let $a^2+b^2=c^{2^t}$ be a primitive Pythagorean triangle. Then, as table (1) shows, the area $\frac{1}{2}\,ab$ is a multiple of the *perfect number* $2^t\left(2^{t+1}-1\right)$.

5.1. Examples.

Consider the two primitive Pythagorean triangles $3^2+4^2=5^{2^1}$ and $5^2+12^2=13^{2^1}$. In this case $t=1$.

Since $2^{1+1}-1=2^2-1=3$ is a prime, we know that each area must be a multiple of the perfect number $2^1\left(2^2-1\right)=6$. And indeed, $\frac{1}{2}\,(3)(4)=\mathbf{6}$ and $\frac{1}{2}\,(5)(12)=\mathbf{6}\,(5)$.

Consider the two primitive Pythagorean triangles $7^2+24^2=5^{2^2}$ and $119^2+120^2=13^{2^2}$. In this case $t=2$.

Since $2^{2+1} - 1 = 2^3 - 1 = 7$ is a prime, we know that each area must be a multiple of the perfect number $2^2\left(2^3 - 1\right) = 28$. And indeed, $\frac{1}{2}(7)(24) = \mathbf{28}(3)$ and $\frac{1}{2}(119)(120) = \mathbf{28}(255)$.

Consider the two primitive Pythagorean triangles $164833^2 + 354144^2 = 5^{2^4}$ and $815616479^2 + 13651680^2 = 13^{2^4}$. In this case $t = 4$.
Since $2^5 - 1 = 31$ is a prime, we know that each area must be a multiple of the perfect number $2^4\left(2^5 - 1\right) = 496$. And indeed, $\frac{1}{2}(164833)(354144) = \mathbf{496}(58845381)$ and $\frac{1}{2}(815616479)(13651680) = \mathbf{496}(11224329812535)$.

Note: Since $5^{2^4} = \left(5^8\right)^{2^1} = \left(5^4\right)^{2^2}$, $\frac{1}{2}(164833)(354144)$ is also a multiple of each of 6 and 28.

That is,

- If $a^2 + b^2 = c^2$ is a primitive Pythagorean triangle then its area is a multiple of the perfect number **6**.

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- If $a^2 + b^2 = c^4$ is a primitive Pythagorean triangle then its area is a multiple of each of the perfect numbers **6** and **28**.

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- If $a^2 + b^2 = c^{16}$ is a primitive Pythagorean triangle then its area is a multiple of each of the perfect numbers **6**, **28**, and **496**.

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- If $a^2 + b^2 = c^{64}$ is a primitive Pythagorean triangle then its area is a multiple of each of the perfect numbers **6**, **28**, **496**, and **8128**.

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- If $a^2 + b^2 = c^{4096}$ is a primitive Pythagorean triangle then its area is a multiple of each of the perfect numbers **6**, **28**, **496**, **8128**, and **33550336**.

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- And, in general, if $a^2 + b^2 = \left(c^{2^t}\right)$ is a primitive Pythagorean triangle where $2^{t+1} - 1$ is a Mersenne prime then the triangle's area is a multiple of each perfect number less than or equal to the perfect number $2^t\left(2^{t+1} - 1\right)$.

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