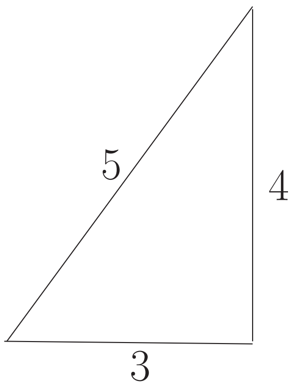
PYTHAGOREAN TRIPLES, ETC.

FRED BARNES

1. Generating all Pythagorean Triples



This section in pdf form. triples2.pdf

If (a, b, c) is a positive integer solution to the equation

$$a^2 + b^2 = c^2 \tag{1}$$

then (a, b, c) is a Pythagorean triple. If a, b, and c have no common divisors greater than 1, then (a, b, c) is a primitive Pythagorean triple (PPT). Similarly, if $a^2 + b^2 = c^2$ where (a, b, c) is a PPT then $a^2 + b^2 = c^2$ is primitive. In the following list only the second triangle is not primitive.

(1) $3^2 + 4^2 = 5^2$
(2) $6^2 + 8^2 = 10^2$
(3) $5^2 + 12^2 = 13^2$
(4) $7^2 + 24^2 = \left(5^2\right)^2$
(5) $\left(15^2\right)^2 + 272^2 = 353^2$

Clearly, if k divides any two of a, b, and c it divides all three. And if $a^2 + b^2 = c^2$ then $k^2a^2 + k^2b^2 = k^2c^2$. That is, for a positive integer k, if (a, b, c) is a Pythagorean triple then so is (ka, kb, kc). Hence, to find all Pythagorean triples, it's sufficient to find all primitive Pythagorean triples.

Let a, b, and c be relatively prime positive integers such that $a^2 + b^2 = c^2$. Set

$$\frac{m}{n} = \frac{c + a}{b}$$

reduced to lowest terms, That is, $\gcd(m,n) = 1$. From the triangle inequality $m > n$. Then

$$\frac{m}{n}\,b - a = c. \tag{2}$$

Squaring both sides of (2) and multiplying through by n^2 we get

$$m^2b^2 - 2mnab + n^2a^2 = n^2a^2 + n^2b^2;$$

which, after canceling and rearranging terms, becomes

$$b\left(m^2 - n^2\right) = a(2mn). \tag{3}$$

There are two cases, either m and n are of opposite parity, or they or both odd. Since $\gcd(m,n) = 1$, they can not both be even.

Case 1. m and n of opposite parity, i.e., $m \not\equiv \pm n(\,mod\; 2\,)$. So 2 divides b since $m^2 - n^2$ is odd. From equation (2), n divides b. Since $\gcd(m,n) = 1$ then $\gcd(m, m^2 - n^2) = 1$, therefore m also divides b. And since $\gcd(a,b) = 1$, b divides $2mn$. Therefore $b = 2mn$. Then

$$a = m^2 - n^2, \quad b = 2mn, \quad \text{and from (2), } c = \frac{m}{n}\,2mn - (m^2 - n^2) = m^2 + n^2. \tag{4}$$

Case 2. m and n both odd, i.e., $m \equiv \pm n(\,mod\; 2\,)$. So 2 divides $m^2 - n^2$. Then by the same process as in the first case we have

$$a = \frac{m^2 - n^2}{2}, \quad b = mn, \quad \textit{and} \quad c = \frac{m^2 + n^2}{2}. \tag{5}$$

The parametric equations in (4) and (5) appear to be different but they generate the same solutions. To show this, let

$$u = \frac{m + n}{2} \quad \text{and} \quad v = \frac{m - n}{2}.$$

Then $m = u + v$, and $n = u - v$. Substituting those values for m and n into (5) we get

$$a = 2uv, \;\; b = u^2 - v^2, \quad \text{and} \quad c = u^2 + v^2 \tag{6}$$

where $u > v$, $gcd(u,v) = 1$, and u and v are of opposite parity. Therefore (6), with the labels for a and b interchanged, is identical to (4). Thus since $\left\{m^2 - n^2, 2mn, m^2 + n^2\right\}$, as in (4), is a primitive Pythagorean triple, we can say that (a,b,c) is a primitive pythagorean triple if and only if there exists relatively prime, positive integers m and n, $m > n$, such that $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$. And (a,b,c) is a Pythagorean triple if and only if

$$a = k\left(m^2 - n^2\right), \quad b = k(2mn), \quad \text{and} \quad c = k\left(m^2 + n^2\right)$$

where k is a positive integer.

Alternatively, (a,b,c) is a Pythagorean triple if and only if there exists relatively prime, positive integers u and v, $u > v$, $u \equiv v \pmod 2$ such that

$$a = k\left(\frac{u^2 - v^2}{2}\right), \quad b = k(uv), \quad \text{and} \quad c = k\left(\frac{u^2 + v^2}{2}\right).$$

Another Method (using Gaussian Integers).

Let

$$a^2 + b^2 = c^2 \tag{7}$$

where a, b, and c are pairwise, relatively prime, positive integers. Let $z = a + bi$ and $\bar{z} = a - bi$, where $i = \sqrt{-1}$. Then z and $\bar{z}$ are Gaussian Integers, and $\bar{z}$ is the conjugate of z. Note that

$$a = \frac{z + \bar{z}}{2}, \quad b = \frac{z - \bar{z}}{2i}, \quad \text{and} \quad c = \sqrt{z\bar{z}}. \tag{8}$$

Since $\gcd(a, b) = 1$ then $\gcd(z, \bar{z}) = 1$. Hence each of z and $\bar{z}$ is a square. That is, there exists integers m and n such that $(m + ni)^2 = z$ and $(m - ni)^2 = \bar{z}$. So, from equation (8),

$$\begin{aligned} a &= \frac{(m + ni)^2 + (m - ni)^2}{2} = m^2 - n^2, \\ b &= \frac{(m + ni)^2 - (m - ni)^2}{2i} = 2mn, \\ \text{and} \quad c &= \sqrt{(m + ni)^2(m - ni)^2} = m^2 + n^2. \end{aligned}$$

Since a is a positive integer, m and n must be positive integers, $m > n$. And since $\gcd(a, b) = 1$, m and n must be relatively prime and of opposite parity.

Equation (8) illustrates a method for finding primitive Pythagorean triangles where the hypotenuse is to a power. We have the identity,

$$\left|\frac{z^{2k} + \bar{z}^{2k}}{2}\right|^2 + \left|\frac{z^{2k} - \bar{z}^{2k}}{2i}\right|^2 = \left((z\bar{z})^k\right)^2. \tag{9}$$

The absolute values are necessary since the terms on the left, depending on k, may, or may not, be positive.

Example

Let $a = 3$, $b = 4$, and $k = 3$. Then, from equation (9), we have,

$$\left|\frac{(3 + 4i)^6 + (3 - 4i)^6}{2}\right|^2 + \left|\frac{(3 + 4i)^6 - (3 - 4i)^6}{2i}\right|^2 = \left(\left((3 + 4i)(3 - 4i)\right)^3\right)^2$$

$$= 11753^2 + \left|-10296\right|^2 = \left(25^3\right)^2.$$

Still another method — the difference method.

Let (a, b, c) be a primitive Pythagorean triple. Without loss of generality, let a be odd. Equation (7) can be written as,

$$a^2 = c^2 - b^2.$$

Set $w = c + b$ and $\bar{w} = c - b$. Thus,

$$a = \sqrt{w\bar{w}}, \quad c = \frac{w + \bar{w}}{2}, \quad \text{and} \quad b = \frac{w - \bar{w}}{2}. \tag{10}$$

Since (a, b, c) is primitive, $\gcd(c, b) = \gcd(w, \bar{w}) = 1$. This implies that each of w and $\bar{w}$ is a square. So, since any odd positive integer greater than 1 can be written as the difference of two positive integer squares, there exists positive integers m

and n, $m > n$, such that $(m+n)^2 = w$ and $(m-n)^2 = \bar{w}$. Hence, from equation (10),

$$\begin{aligned} a &= \sqrt{(m+n)^2(m-n)^2} = m^2 - n^2, \\ c &= \frac{(m+n)^2 + (m-n)^2}{2} = m^2 + n^2, \\ \text{and} \quad b &= \frac{(m+n)^2 - (m-n)^2}{2} = 2mn. \end{aligned}$$

And, since $\gcd(a, b) = 1$, m and n must be relatively prime and of opposite parity.

Equation (10) illustrates an efficient method for finding primitive Pythagorean triples where the odd leg is to a power. We have the identity,

$$\left((w\bar{w})^k\right)^2 = \left(\frac{w^{2k} + \bar{w}^{2k}}{2}\right)^2 - \left(\frac{w^{2k} - \bar{w}^{2k}}{2}\right)^2. \tag{11}$$

Example

Let $c = 5$, $b = 4$, and $k = 3$. Then, from equation (11), we have,

$$\left(((5+4)(5-4))^3\right)^2 = \left(\frac{(5+4)^6 + (5-4)^6}{2}\right)^2 - \left(\frac{(5+4)^6 - (5-4)^6}{2}\right)^2.$$

$$\text{That is,} \quad \left(9^3\right)^2 = 265721^2 - 265720^2.$$

E-mail address: fredlb@centurytel.net