MULTIPLYING PYTHAGOREAN TRIPLES

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Usually when people speak of multiplying Pythagorean triples they are referring to multiplying the hypotenuses of of two triples to generate another Pythagorean triple. For example, if (a_1, b_1, c_1) and (a_2, b_2, c_2) are Pythagorean triples then $(a_1a_2 - b_1b_2,\ a_1b_2 + a_2b_1,\ c_1c_2)$ is also a Pythagorean triple. However, Pythagorean triples can be generated by multiplying the other sides of two triples as well. That is, not only does c_1c_2 generate a Pythagorean triple, so does a_1a_2, b_1b_2, b_1a_2, and a_1b_2.

Preliminaries

We will first derive identities for multiplying the sums of two squares and for multiplying the differences of two squares to give the sum of two squares and the difference of two squares respectively.

Let $a \pm b\beta$ and $c \pm d\beta$ be elements of an *integral domain*. Then,

$$\begin{aligned}
\left(a^2 - b^2\beta^2\right)\left(c^2 - d^2\beta^2\right) &= (a+b\beta)(a-b\beta)(c+d\beta)(c-d\beta) \\
&= \Big((a+b\beta)(c+d\beta)\Big)\Big((a-b\beta)(c-d\beta)\Big) \\
&= \Big(\left(ac+bd\beta^2\right)+(bc+ad)\beta\Big)\Big(\left(ac+bd\beta^2\right)-(bc+ad)\beta\Big) \\
&= \boxed{\left(ac+bd\beta^2\right)^2 - \left(bc+ad\right)^2\beta^2} \\
&= \Big((a+b\beta)(c-d\beta)\Big)\Big((a-b\beta)(c+d\beta)\Big) \\
&= \Big(\left(ac-bd\beta^2\right)+(bc-ad)\beta\Big)\Big(\left(ac-bd\beta^2\right)-\left(bc-ad\right)\beta\Big) \\
&= \boxed{\left(ac-bd\beta^2\right)^2 - \left(bc-ad\right)^2\beta^2}\,.
\end{aligned}$$

Hence, we have

(1)
$$\left(a^2 - b^2\beta^2\right)\left(c^2 - d^2\beta^2\right) = \left(ac+bd\beta^2\right)^2 - \left(bc+ad\right)^2\beta^2 = \left(ac-bd\beta^2\right)^2 - \left(bc-ad\right)^2\beta^2.$$

Set $\beta = i = \sqrt{-1}$, and substitute into equation (1) to get Fibonacci's Identity for multiplying the sums of two squares,

(2) $$\left(a^2+b^2\right)\left(c^2+d^2\right) = (ac-bd)^2 + (bc+ad)^2 = (ac+bd)^2 + (bc-ad)^2.$$

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Set $\beta = 1$, to get an analogous identity for multiplying the differences of two squares,

$$(3)\qquad \left(a^2 - b^2\right)\left(c^2 - d^2\right) = (ac + bd)^2 - (bc + ad)^2 = (ac - bd)^2 - (bc - ad)^2.$$

Each of these identities is a special case of Brahmagupta's identity,

$$(4)\qquad \left(-na^2 + b^2\right)\left(-nc^2 + d^2\right) = (nac + bd)^2 - n\,(bc + ad)^2 = (nac - bd)^2 - n\,(bc - ad)^2$$

Set $n = -1$ and $n = 1$ in (4) to get equations (2) and (3) respectively.

Multiplying Pythagorean triples

Note that $a^2 = c^2 - b^2$, $b^2 = c^2 - a^2$, and $a^2 + b^2 = c^2$ are equivalent methods of writing a Pythagorean triangle.

If (a_1, b_1, c_1) and (a_2, b_2, c_2) are two Pythagorean triples then, from equations(2) and (3), we have

(a) $c_1^2 c_2^2 = \left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right) = (a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2 = (a_1a_2 + b_1b_2)^2 + (a_1b_2 - b_1a_2)^2$

(b) $a_1^2 a_2^2 = \left(c_1^2 - b_1^2\right)\left(c_2^2 - b_2^2\right) = (c_1c_2 + b_1b_2)^2 - (c_1b_2 + b_1c_2)^2 = (c_1c_2 - b_1b_2)^2 - (c_1b_2 - b_1c_2)^2$.

(c) $b_1^2 b_2^2 = \left(c_1^2 - a_1^2\right)\left(c_2^2 - a_2^2\right) = (c_1c_2 + a_1a_2)^2 - (c_1a_2 + a_1c_2)^2 = (c_1c_2 - a_1a_2)^2 - (c_1a_2 - a_1c_2)^2$.

(d) $b_1^2 a_2^2 = \left(c_1^2 - a_1^2\right)\left(c_2^2 - b_2^2\right) = (c_1c_2 + a_1b_2)^2 - (c_1b_2 + a_1c_2)^2 = (c_1c_2 - a_1b_2)^2 - (c_1b_2 - a_1c_2)^2$.

(e) $a_1^2 b_2^2 = \left(c_1^2 - b_1^2\right)\left(c_2^2 - a_2^2\right) = (c_1c_2 + b_1a_2)^2 - (c_1a_2 + b_1c_2)^2 = (c_1c_2 - b_1a_2)^2 - (c_1a_2 - b_1c_2)^2$.

From which we will define the following multiplications where (a_1, b_1, c_1) and (a_2, b_2, c_2) are Pythagorean triples. And the over bars designate multiplier and multiplicand.

(1) $(a_1, b_1, \bar{c_1}) \otimes (a_2, b_2, \bar{c_2}) = (a_1a_2 - b_1b_2,\ a_1b_2 + a_2b_1,\ c_1c_2)$.

(2) $(\bar{a_1}, b_1, c_1) \otimes (\bar{a_2}, b_2, c_2) = (a_1a_2,\ c_1b_2 + c_2b_1,\ c_1c_2 + b_1b_2)$.

(3) $\left(a_1, \bar{b_1}, c_1\right) \otimes \left(a_2, \bar{b_2}, c_2\right) = (c_1a_2 + c_2a_1,\ b_1b_2,\ c_1c_2 + a_1a_2)$.

(4) $\left(a_1, \bar{b_1}, c_1\right) \otimes (\bar{a_2}, b_2, c_2) = (c_1b_2 + a_1c_2,\ b_1a_2,\ c_1c_2 + a_1b_2)$.

(5) $(\bar{a_1}, b_1, c_1) \otimes \left(a_2, \bar{b_2}, c_2\right) = (c_1a_2 + b_1c_2,\ a_1b_2,\ c_1c_2 + b_1a_2)$.

(6) $(a_1, b_1, \bar{c_1}) \odot (a_2, b_2, \bar{c_2}) = (a_1a_2 + b_1b_2,\ a_1b_2 - a_2b_1,\ c_1c_2)$.

(7) $(\bar{a_1}, b_1, c_1) \odot (\bar{a_2}, b_2, c_2) = (a_1a_2,\ c_1b_2 - c_2b_1,\ c_1c_2 - b_1b_2)$.

(8) $\left(a_1, \bar{b_1}, c_1\right) \odot \left(a_2, \bar{b_2}, c_2\right) = (c_1a_2 - c_2a_1,\ b_1b_2,\ c_1c_2 - a_1a_2)$.

(9) $\left(a_1, \bar{b_1}, c_1\right) \odot (\bar{a_2}, b_2, c_2) = (c_1b_2 - a_1c_2,\ b_1a_2,\ c_1c_2 - a_1b_2)$.

(10) $(\bar{a_1}, b_1, c_1) \odot \left(a_2, \bar{b_2}, c_2\right) = (c_1a_2 - b_1c_2,\ a_1b_2,\ c_1c_2 - b_1a_2)$.

If $a_1 = a_2 = a,\ b_1 = b_2 = b$, and $c_1 = c_2 = c$ then, from items $1 - 10$,

i. $(a, b, \bar{c}) \otimes (a, b, \bar{c}) = \left(a^2 - b^2,\ 2ab,\ c^2\right)$.
ii. $(\bar{a}, b, c) \otimes (\bar{a}, b, c) = \left(a^2,\ 2cb,\ c^2 + b^2\right)$.
iii. $\left(a, \bar{b}, c\right) \otimes \left(a, \bar{b}, c\right) = \left(b^2,\ 2ca,\ c^2 + a^2\right)$.
iv. $\left(a, \bar{b}, c\right) \otimes (\bar{a}, b, c) = (\bar{a}, a, c) \otimes \left(a, \bar{b}, c\right) = \left(ca + cb,\ ab,\ c^2 + ab\right)$.
v. $\left(a, \bar{b}, c\right) \odot (\bar{a}, b, c) = (\bar{a}, a, c) \odot \left(a, \bar{b}, c\right) = \left(ca - cb,\ ab,\ c^2 - ab\right)$.

Examples

The ordered triples $(3, 4, 5)$ and $(5, 12, 13)$ are Pythagorean triples, hence from $1 - 10$, we get the resultant Pythagorean triples,

- $(3, 4, \bar{5}) \otimes (5, 12, \bar{13}) = (33, 56, 65)$.
- $(\bar{3}, 4, 5) \otimes (\bar{5}, 12, 13) = (15, 112, 113)$.
- $(3, \bar{4}, 5) \otimes (5, \bar{12}, 13) = (64, 48, 80)$.
- $(3, \bar{4}, 5) \otimes (\bar{5}, 12, 13) = (99, 20, 101)$.
- $(\bar{3}, 4, 5) \otimes (5, \bar{12}, 13) = (77, 36, 85)$.
- $(3, 4, \bar{5}) \odot (5, 12, \bar{13}) = (63, 16, 65)$.
- $(\bar{3}, 4, 5) \odot (\bar{5}, 12, 13) = (15, 8, 17)$.
- $(3, \bar{4}, 5) \odot (5, \bar{12}, 13) = (14, 48, 50)$.
- $(3, \bar{4}, 5) \odot (\bar{5}, 12, 13) = (21, 20, 29)$.
- $(\bar{3}, 4, 5) \odot (5, \bar{12}, 13) = (27, 36, 45)$.

Let a be odd. Then clearly, from items i through v, if (a, b, c) is a primitive Pythagorean triple (PPT) then so are

$$\left(\left|a^2 - b^2\right|,\ 2ab,\ c^2\right),\ \left(a^2,\ 2cb,\ c^2 + b^2\right),\ \left(ca,\ \frac{b^2}{2},\ \frac{c^2 + a^2}{2}\right),$$
$$\left(ca + cb,\ ab,\ c^2 + ab\right),\ \text{and}\ \left(\left|ca - cb\right|,\ ab,\ c^2 - ab\right).$$

Example: Since $(3, 4, 5)$ is a PPT, so are

$$(7, 24, 25),\ (9, 40, 41),\ (15, 8, 17),\ (35, 12, 37),\ \text{and}\ \ (5, 12, 13).$$

Supplemental

Claim 1. *Let (a_1, b_1, c_1) and (a_2, b_2, c_2) be vectors in 3-space. Then, if (a_1, b_1, c_1) and (a_2, b_2, c_2) are also primitive Pythagorean triples, where a_1 and a_2 are of opposite parity, the scalar product*

$$\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = a_1a_2 + b_1b_2 + c_1c_2$$

is a perfect square.

Proof. Without loss of generality let a_1 be odd and a_2 even. So, since both vectors are also PPTs, there exists positive integers m_1, n_1, m_2, and n_2 such that

$$a_1 = m_1^2 - n_1^2, \quad b_1 = 2m_1n_1, \quad \text{and} \quad c_1 = m_1^2 + n_1^2$$
$$a_2 = 2m_2n_2, \quad b_2 = m_2^2 - n_2^2, \quad \text{and} \quad c_2 = m_2^2 + n_2^2$$

Where m_i and n_i are relative prime, have opposite parity, and $m_i > n_i$ for $i = 1$ and 2.

Hence we have,

$$\begin{aligned} a_1a_2 + b_1b_2 + c_1c_2 &= \left(m_1^2 - n_1^2\right) a_2 + 2m_1n_1b_2 + \left(m_1^2 + n_1^2\right) c_2 \\ &= m_1^2(c_2 + a_2) + 2m_1n_1\sqrt{c_2 + a_2}\sqrt{c_2 - a_2} + n_1^2(c_2 - a_2) \\ &= \left(m_1\sqrt{c_2 + a_2} + n_1\sqrt{c_2 - a_2}\right)^2 \\ &= \big(m_1(m_2 + n_2) + n_1(m_2 - n_2)\big)^2 \\ &= \big((m_1 + n_1)m_2 + (m_1 - n_1)n_2\big)^2. \end{aligned}$$

Similarly it can be shown that the sums $(-a_1a_2+b_1b_2+c_1c_2)$, $(a_1a_2-b_1b_2+c_1c_2)$, and $(-a_1a_2 - b_1b_2 + c_1c_2)$ are also perfect squares. □

Example: Let (a, b, c) be a PPT where a is even, then each of the sums $(3a + 4b + 5c)$, $(-3a + 4b + 5c)$, $(3a - 4b + 5c)$, and $(-3a - 4b + 5c)$ is a perfect square.

Theorem 1. *Let $(a, b, c) = (b, a, c)$ be greater than $(3, 4, 5)$. Then (a, b, c) is a primitive Pythagorean triple (PPT) if and only if there exists another PPT, (u, v, w), such that (a, b, c) equals either $(\bar{3}, 4, 5) \otimes (\bar{u}, v, w)$ or $(\bar{3}, 4, 5) \odot (\bar{u}, v, w)$.*

Proof. We know that since (a, b, c) is a PPT then a and b have opposite parity. That is, one of a and b is odd and the other is even. We also know that 3 divides exactly one of a and b. Label (a, b, c) such that 3 divides a. Then there are two cases: a is even, or a is odd.

Case 1, a is even: So, there exists relatively prime positive integers of opposite parity, m and n, $m > n$, such that

$$a = 2mn, \quad b = m^2 - n^2, \quad \text{and} \quad c = m^2 + n^2$$

where 3 divides exactly one of m and n.

If $3 \mid m$, set s equal to the greater of $m/3$ and n and t equal to the lesser. Then choose u, v, and w such that

$$u = 2st, \quad v = s^2 - t^2, \quad \text{and} \quad w = s^2 + t^2.$$

Then we have

Case 1a, $s = m/3 > n = t$**:**

$$\begin{aligned} a =& 2mn = 2 \cdot 3(m/3)n = 3 \cdot 2st = 3u, \\ b =& m^2 - n^2 = 9(m/3)^2 - n^2 = 5\left((m/3)^2 - n^2\right) + 4\left((m/3)^2 + n^2\right) \\ =& 4\left(s^2 + t^2\right) + 5\left(s^2 - t^2\right) = 4w + 5v, \\ c =& m^2 + n^2 = 9(m/3)^2 + n^2 = 5\left((m/3)^2 + n^2\right) + 4\left((m/3)^2 - n^2\right) \\ =& 5\left(s^2 + t^2\right) + 4\left(s^2 - t^2\right) = 5w + 4v. \end{aligned}$$

Therefore

$$(a, b, c) = (3u, 5v + 4w, 5w + 4v) = (\bar{3}, 4, 5) \otimes (\bar{u}, v, w).$$

And since m and n are relatively prime positive integers of opposite parity so are s and t. Hence (u, v, w) is a PPT.

Case 1b, $s = n > m/3 = t$**:**

$$\begin{aligned} a =& 2mn = 2 \cdot 3n(m/3) = 3 \cdot 2st = 3u, \\ b =& m^2 - n^2 = 9(m/3)^2 - n^2 = 4\left(n^2 + (m/3)^2\right) + 5\left((m/3)^2 - n^2\right) \\ =& 4\left(s^2 + t^2\right) - 5\left(s^2 - t^2\right) = 4w - 5v, \\ c =& m^2 + n^2 = n^2 + 9(m/3)^2 = 5\left(n^2 + (m/3)^2\right) - 4\left(n^2 - (m/3)^2\right) \\ =& 5\left(s^2 + t^2\right) - 4\left(s^2 - t^2\right) = 5w - 4v. \end{aligned}$$

Therefore

$$(a, b, c) = (3u, 5v + 4w, 5w + 4v) = (\bar{3}, 4, 5) \otimes (\bar{u}, v, w).$$

Where (u, v, w) is a PPT.

Case 2, a **is odd:** So, there exists relatively prime positive integers of opposite parity, m and n, $m > n$, such that

$$a = m^2 - n^2 = (m + n)(m - n), \quad b = 2mn, \quad \text{and} \quad c = m^2 + n^2.$$

Note that since m and n are relatively prime then so are $m + n$ and $m - n$.

Case 2a, 3 divides $m + n$
set $s = \frac{2m-n}{3}$ and $t = \frac{2n-m}{3}$. Then choose u, v, and w such that

$$u = s^2 - t^2, \quad v = 2st, \quad \text{and} \quad w = s^2 + t^2.$$

We have,

$$\begin{aligned}
a =& m^2 - n^2 = 3\left(\left(\frac{2m-n}{3}\right)^2 - \left(\frac{2n-m}{3}\right)^2\right) = 3\left(s^2 - t^2\right) = 3u. \\
b =& 2mn = 4\left(\left(\frac{2m-n}{3}\right)^2 + \left(\frac{2n-m}{3}\right)^2\right) + 5 \cdot 2\left(\frac{2m-n}{3}\right)\left(\frac{2n-m}{3}\right) \\
=& 4\left(s^2+t^2\right) + 5(2st) = 4w + 5v. \\
c =& m^2 + n^2 = 5\left(\left(\frac{2m-n}{3}\right)^2 + \left(\frac{2n-m}{3}\right)^2\right) + 4 \cdot 2\left(\frac{2m-n}{3}\right)\left(\frac{2n-m}{3}\right) \\
=& 5\left(s^2+t^2\right) + 4(2st) = 5w + 4v.
\end{aligned}$$

Therefore, if $2n - m > 0$, $(a, b, c) = (3u, 4w + 5v, 5w + 4v) = (\bar{3}, 4, 5) \otimes (\bar{u}, v, w)$. And (u, v, w) is a PPT. And if $2n - m < 0$ then $v = -2st$ and $(a, b, c) = (3u, 4w - 5v, 5w - 4v) = (\bar{3}, 4, 5) \odot (\bar{u}, v, w)$.

Case 2b, 3 divides $m - n$ set $s = \frac{2m+n}{3}$ and $t = \frac{2n+m}{3}$. Then choose u, v, and w such that

$$u = s^2 - t^2, \quad v = 2st, \quad \text{and} \quad w = s^2 + t^2.$$

We have,

$$\begin{aligned}
a =& m^2 - n^2 = 3\left(\left(\frac{2m+n}{3}\right)^2 - \left(\frac{2n+m}{3}\right)^2\right) = 3\left(s^2 - t^2\right) = 3u. \\
b =& 2mn = 4\left(\left(\frac{2m+n}{3}\right)^2 + \left(\frac{2n+m}{3}\right)^2\right) - 5 \cdot 2\left(\frac{2m+n}{3}\right)\left(\frac{2n+m}{3}\right) \\
=& 4\left(s^2+t^2\right) - 5(2st) = 4w - 5v. \\
c =& m^2 + n^2 = 5\left(\left(\frac{2m+n}{3}\right)^2 + \left(\frac{2n+m}{3}\right)^2\right) - 4 \cdot 2\left(\frac{2m+n}{3}\right)\left(\frac{2n+m}{3}\right) \\
=& 5\left(s^2+t^2\right) - 4(2st) = 5w - 4v.
\end{aligned}$$

Therefore $(a, b, c) = (3u, 4w - 5v, 5w - 4v) = (\bar{3}, 4, 5) \odot (\bar{u}, v, w)$. And (u, v, w) is a PPT.

And going in the other direction, if (u, v, w) is a PPT then either $(\bar{3}, 4, 5) \otimes (\bar{u}, v, w)$ or $(\bar{3}, 4, 5) \odot (\bar{u}, v, w)$ is a PPT.

Proof. If $(3u, 4w + 5v,\ 5w + 4v)$ and $(3u, 4w - 5v,\ 5w - 4v)$ or both not primitive then $(4w + 5v,\ 5w + 4v) = d_1 > 1$ implies $d_1 \mid 4w + 5v + 5w + 4v = 3^2(w + v)$. And $d_1 \mid -4w - 5v + 5w + 4v = w - v$. Since $1 < d_1 \mid w - v$, $d_1 > 1$ can not divide $w + v$ since $(w, v) = 1$. Hence $d_1 = 3$ or 3^2.

Similarly, if $(4w - 5v,\ 5w - 4v) = d_2 > 1$ then $d_2 = 3$ or 3^2. That is, $3 \mid 4w + 5v + 4w - 5v = 4w$ and $3 \mid 4w + 5v - (4w - 5v) = 10v$. This implies that $3 \mid w$ and $3 \mid v$, a contradiction. Hence either $(\bar{3}, 4, 5) \otimes (\bar{u}, v, w)$ or $(\bar{3}, 4, 5) \odot (\bar{u}, v, w)$ is primitive. $\square$

$\square$

That is, if 3 divides $\mathbf{a} > \mathbf{3}$ then $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a primitive Pythagorean triple if and only if there exists a primitive Pythagorean triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ such that

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{3u},\ \mathbf{4w} + \mathbf{5v},\ \mathbf{5w} + \mathbf{4v})\ \textbf{ or }\ (\mathbf{3u},\ \mathbf{4w} - \mathbf{5v},\ \mathbf{5w} - \mathbf{4v}).$$

Examples

$(780, 1421, 1621)$ and $(780, 731, 1069)$ are primitive Pythagorean triples where 3 divides the even side.

Problem 1: Find a PPT (u, v, w) such that

$$(\bar{3}, 4, 5) \otimes (\bar{u}, v, w) = (780, 1421, 1621) \quad \text{and} \quad (\bar{3}, 4, 5) \odot (\bar{u}, v, w) = (780, 731, 1069).$$

Solution: Since $(780, 1421, 1621)$ is primitive, $m/n = 780/(1621 - 1421)$ (reduced to lowest terms) equals $39/10$. So $s = m/3 = 39/3 = 13$ and $t = n = 10$. Then

$$(u, v, w) = \left(2st, s^2 - t^2, s^2 + t^2\right) = (260, 69, 269).$$

And

$$(\bar{3}, 4, 5) \otimes (\bar{u}, v, w) = (\bar{3}, 4, 5) \otimes (2\bar{6}0, 69, 269) = (3{\cdot}260, 4{\cdot}269{+}5\dot{6}9, 5{\cdot}269{+}4{\cdot}69) = (780, 1421, 1621).$$

Similarly, since $(780, 731, 1069)$ is primitive, $m/n = 780/(1069 - 731) = 30/13$. So $s = 30/3 = 10$ and $t = n = 13$. Since $s < t$, we have

$$(\bar{3}, 4, 5) \odot (\bar{u}, v, w) = (\bar{3}, 4, 5) \odot (2\bar{6}0, 69, 269) = (3{\cdot}260, 4{\cdot}269{-}5{\cdot}69, 5{\cdot}269{-}4{\cdot}69) = (780, 731, 1069).$$

$(1365, 18988, 19037)$ is a PPT where 3 divides the odd side.

Problem 2: Find a PPT (u, v, w) such that $(\bar{3}, 4, 5) \otimes (\bar{u}, v, w) = (1365, 18988, 19037)$.

Solution: Since $(1365, 18988, 19037)$ is a PPT, $m/n = 18988/(19037 - 1365) = 101/94$. And 3 divides $m + n = 101 + 94 = 195$, therefore $s = (2 \cdot 101 - 94)/3 = 36$, and $t = 2 \cdot 94 - 101) = 29$. So,

$$\begin{aligned}
(\bar{3}, 4, 5) \otimes (\bar{u}, v, w) &= (\bar{3}, 4, 5) \otimes \left(36^2 \bar{-} 29^2,\ 2 \cdot 29 \cdot 36,\ 36^2 + 29^2\right) \\
&= (\bar{3}, 4, 5) \otimes (4\bar{5}5, 2088, 2137) \\
&= (3 \cdot 455,\ 5 \cdot 2088 + 4 \cdot 2137,\ 5 \cdot 2137 + 4 \cdot 2088) \\
&= (1365, 18988, 19037).
\end{aligned}$$

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