120 DEGREE AND 60 DEGREE TRIPLES FROM FIBONACCI NUMBERS

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A 120 degree triple is a solution, (a, b, c), in positive integers to the 120 degree triangle equation

$$a^2 + b^2 - 2ab\cos 120^\circ = a^2 + b^2 + ab = c^2.$$

If additionally a, b,and c are pairwise relatively prime then (a, b, c) is a primitive 120 degree triple.

(a, b, c) ia a primitive 120 degree triple if and only if there exists relative prime integers u and v, $u > v$ and $3 \nmid u - v$ such that

$$(1) \qquad a = u^2 - v^2, \quad b = 2uv + v^2, \quad \text{and} \quad c = u^2 + v^2 + uv. See (\textbf{??}) for a proof.$$

For $n > 2$, the n^{th} Fibonacci number is given by $F_n = F_{n-2} + F_{n-1}$ where $F_1 = F_2 = 1$. The first few are $1, 1, 2, 3, 5, 8, 13, 21, 34$.

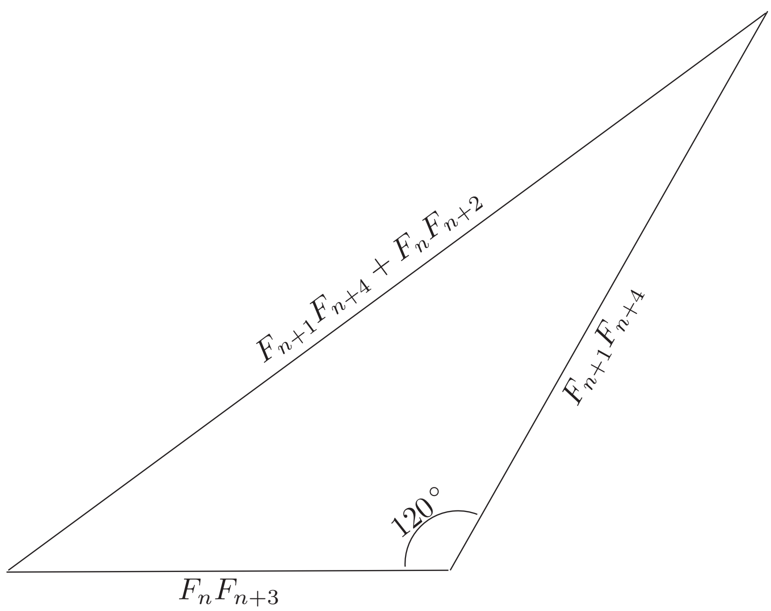

FIGURE 1. 120 degree triangle

Some notation

- $(s,t)=d$ means that the positive integer d is the greatest common divisor of the two integers s and t. If $d=1$ then s and t are relatively prime.
- $s \mid t$ means s divides t.
- $s \nmid t$ means s does not divide t.
- $\Rightarrow$ means implies.

Claim 1. *If F_n, F_{n+1}, F_{n+2}, F_{n+3}, and F_{n+4} are 5 consecutive Fibonacci numbers then*

$$\Big(F_nF_{n+3},\;\; F_{n+1}F_{n+4},\;\; F_{n+1}F_{n+4}+F_nF_{n+2}\Big)$$

is a 120 degree triple. And if $\boxed{3 \nmid F_n}$ *then it's a primitive triple. That is*

$$(F_nF_{n+3})^2+(F_{n+1}F_{n+4})^2+(F_nF_{n+3})(F_{n+1}F_{n+4})=(F_{n+1}F_{n+4}+F_nF_{n+2})^2$$

where each side of the triangle is relatively prime to each of the other two sides.

Proof. First note that any two consecutive Fibonacci numbers are relatively prime. This can easily be proved by induction on n.

(1) $(1,1)=(2,1)=(3,2)=1$. So it's true for F_1, F_2, F_3, and F_4.
(2) Assume that $(F_n,F_{n-1})=1$. We want to show that this implies that $(F_{n+1},F_n)=1$. To do so, let $d=(F_{n+1},F_n)$. This implies that $d \mid F_{n+1}=F_{n-1}+F_n$ and $\boxed{d \mid F_n}$. This implies that $d \mid F_{n-1}+F_n-F_n=\boxed{F_{n-1}} \Rightarrow d=1$ since $(F_n,F_{n-1})=1$.

Hence F_{n+1} and F_n are relatively prime, and therefore, any two consecutive Fibonacci numbers are relatively prime.

Let $u=F_{n+2}$ and $v=F_{n+1}$. Then $3 \nmid F_n \Rightarrow 3 \nmid F_n+F_{n+1}-F_{n+1}=F_{n+2}-F_{n+1}=u-v$. So, we have

(1) $F_n=u-v$.
(2) $F_{n+1}=v$.
(3) $F_{n+2}=u$.
(4) $F_{n+3}=v+u$.
(5) $F_{n+4}=v+2u$.

From equation (1),

$$a=u^2-v^2=(u-v)(u+v)=F_nF_{n+3},$$
$$b=2uv+v^2=v(v+2u)=F_{n+1}F_{n+4},$$
$$\text{and}\;\; c=u^2+v^2+uv=v(v+2u)+(u-v)u=F_{n+1}F_{n+4}+F_nF_{n+2}.$$

□

Example

Let $F_n = 5$, then $F_{n+1} = 8$, $F_{n+2} = 13$, $F_{n+3} = 21$, and $F_{n+4} = 34$. So

$$\begin{aligned} & F_nF_{n+3} = 5 \cdot 21 = 105, \\ & F_{n+1}F_{n+4} = 8 \cdot 34 = 272, \\ \text{and} \quad & F_{n+1}F_{n+4} + F_nF_{n+2} = 8 \cdot 34 + 5 \cdot 13 = 337. \end{aligned}$$

Then

$$105^2 + 272^2 + 105 \cdot 272 = 337^2$$

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This works for generalized Fibonacci numbers also. That is, choose any two positive integers N_0 and N_1, then obtain integers N_2, N_3, and N_4 thusly,

(1) $N_0 + N_1 = N_2$.
(2) $N_1 + N_2 = N_3$.
(3) $N_2 + N_3 = N_4$.

Set $N_1 = v$ and $N_2 = u$. We have

(1) $N_0 = u - v$.
(2) $N_1 = v$.
(3) $N_2 = u$.
(4) $N_3 = v + u$.
(5) $N_4 = v + 2u$.

Then $(N_0N_3,\ N_1N_4,\ N_1N_4 + N_0N_2)$ is a 120 degree triple.

Example: If $N_0 = 13$ and $N_1 = 1$ then $N_2 = 14$, $N_3 = 15$, and $N_4 = 29$. Therefore

$$(13 \cdot 15)^2 + (1 \cdot 29)^2 + (13 \cdot 15)(1 \cdot 29) = (1 \cdot 29 + 13 \cdot 14)^2.$$

Sixty degree triangles

Construct equilateral triangles on each of the adjacent legs of the 120° triangle in Figure (1) creating the two 60° triangles ABD and ACD as shown in figure (2). Thus,

$$\begin{aligned} & \big(F_nF_{n+3},\ \ F_nF_{n+3} + F_{n+1}F_{n+4},\ \ F_{n+1}F_{n+4} + F_nF_{n+2}\big) \\ \text{and} \quad & \big(F_nF_{n+3} + F_{n+1}F_{n+4},\ \ F_{n+1}F_{n+4},\ \ F_{n+1}F_{n+4} + F_nF_{n+2}\big) \end{aligned}$$

are 60° triples. That is,

$$\begin{aligned} & (F_nF_{n+3})^2 + (F_nF_{n+3} + F_{n+1}F_{n+4})^2 - (F_nF_{n+3})(F_nF_{n+3} + F_{n+1}F_{n+4}) \\ =& (F_nF_{n+3} + F_{n+1}F_{n+4})^2 + (F_{n+1}F_{n+4})^2 - (F_nF_{n+3} + F_{n+1}F_{n+4})(F_{n+1}F_{n+4}) \\ =& (F_{n+1}F_{n+4} + F_nF_{n+2})^2. \end{aligned}$$

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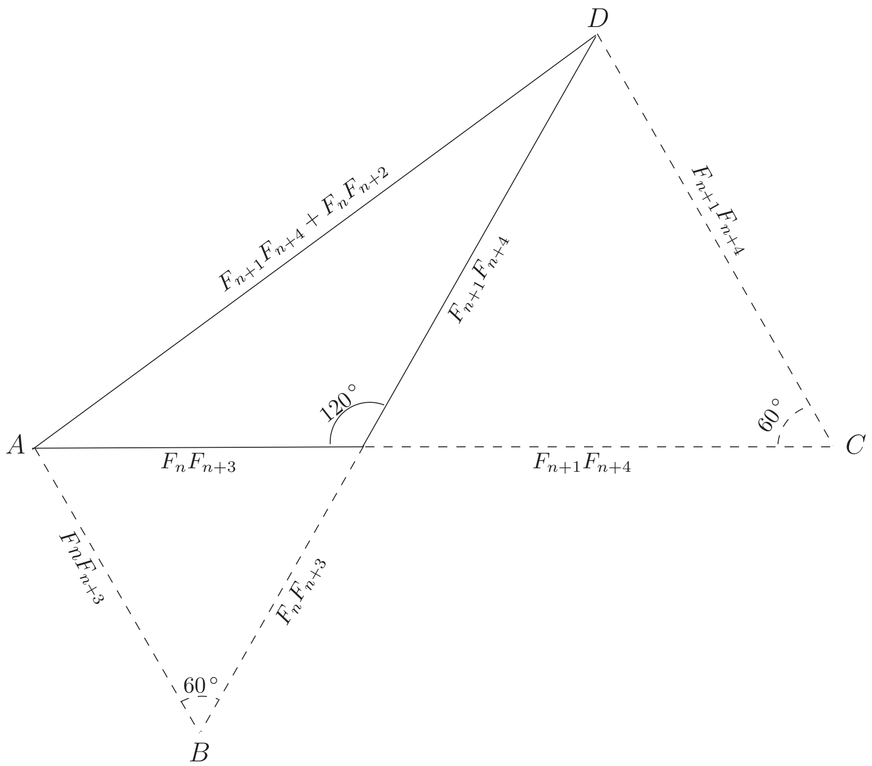

Figure 2. 60 degree triangles