definition:

power spectral density (PSD)

$$\boxed{S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2}$$

of a sampled signal with duration $T = N\Delta t$

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of a sampled signal with duration $T = N\Delta t$

For interpretation, the observed system should be **stationary in time (in wider sense)**.

$$E[s(t)]_{t_1}^{t_2} = E[s(t)]_{t_3}^{t_4} = \mathrm{const}$$

temporal average independent of time window

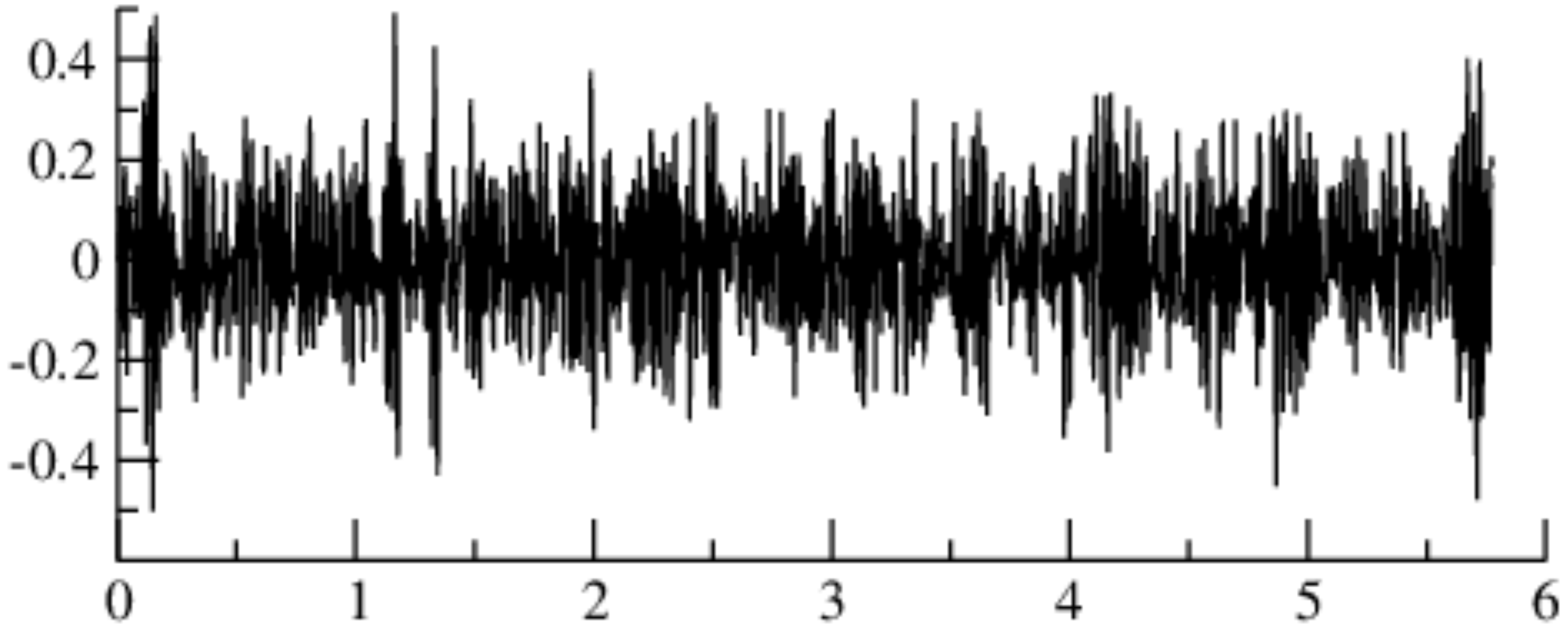
$$Var[s(t)]_{t_1}^{t_2}, Var[s(t)]_{t_3}^{t_4} < \infty$$

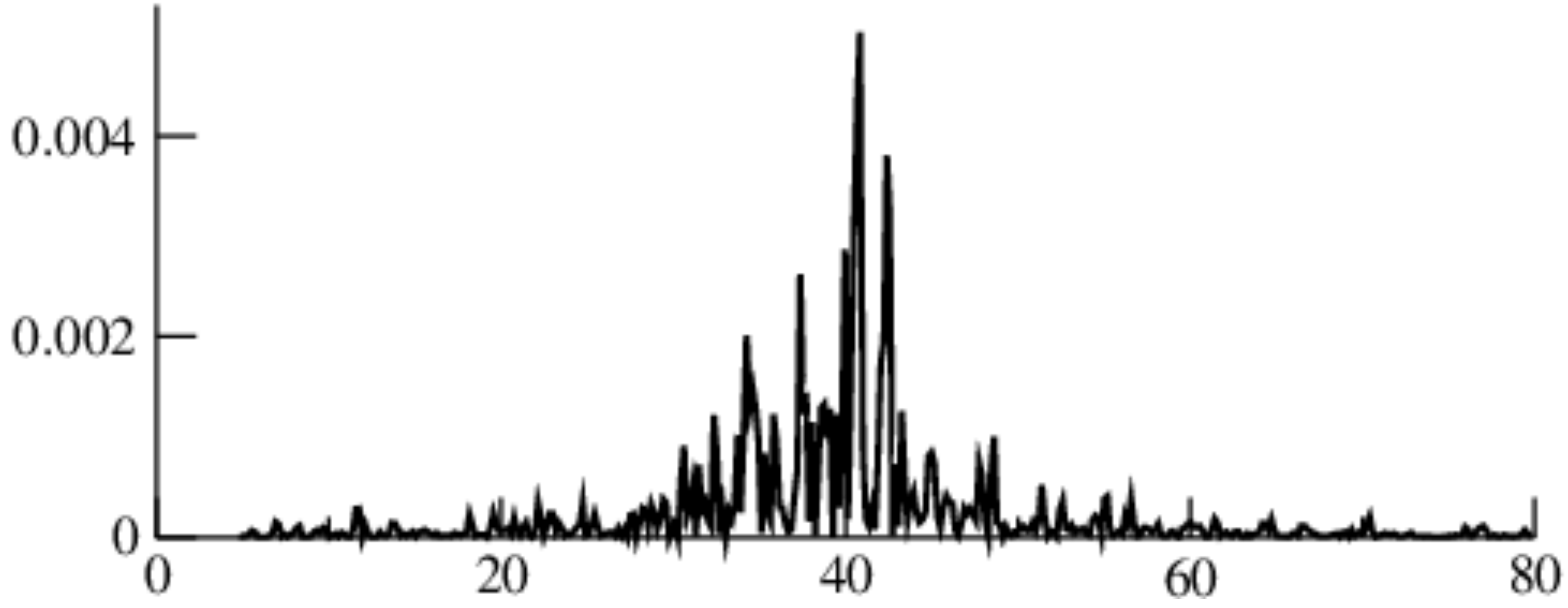
temporal variance is finite

1. Periodogram

$$S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2$$

DFT: computed from full signal





SamplingError_5.py

1. Periodogram

$$S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2$$

DFT: computed from full signal

2. Bartlett method

$$S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2$$

DFT: average over segments

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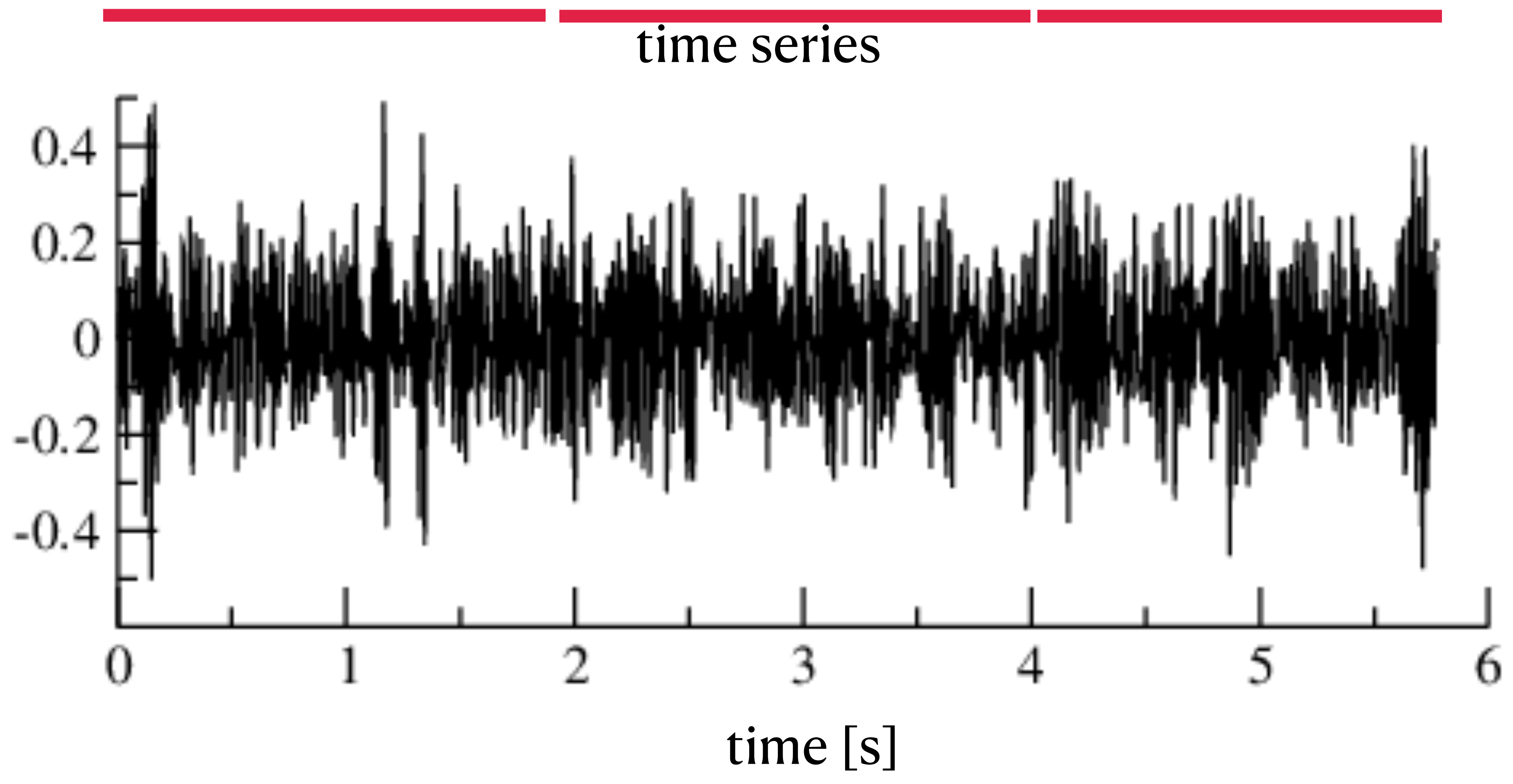

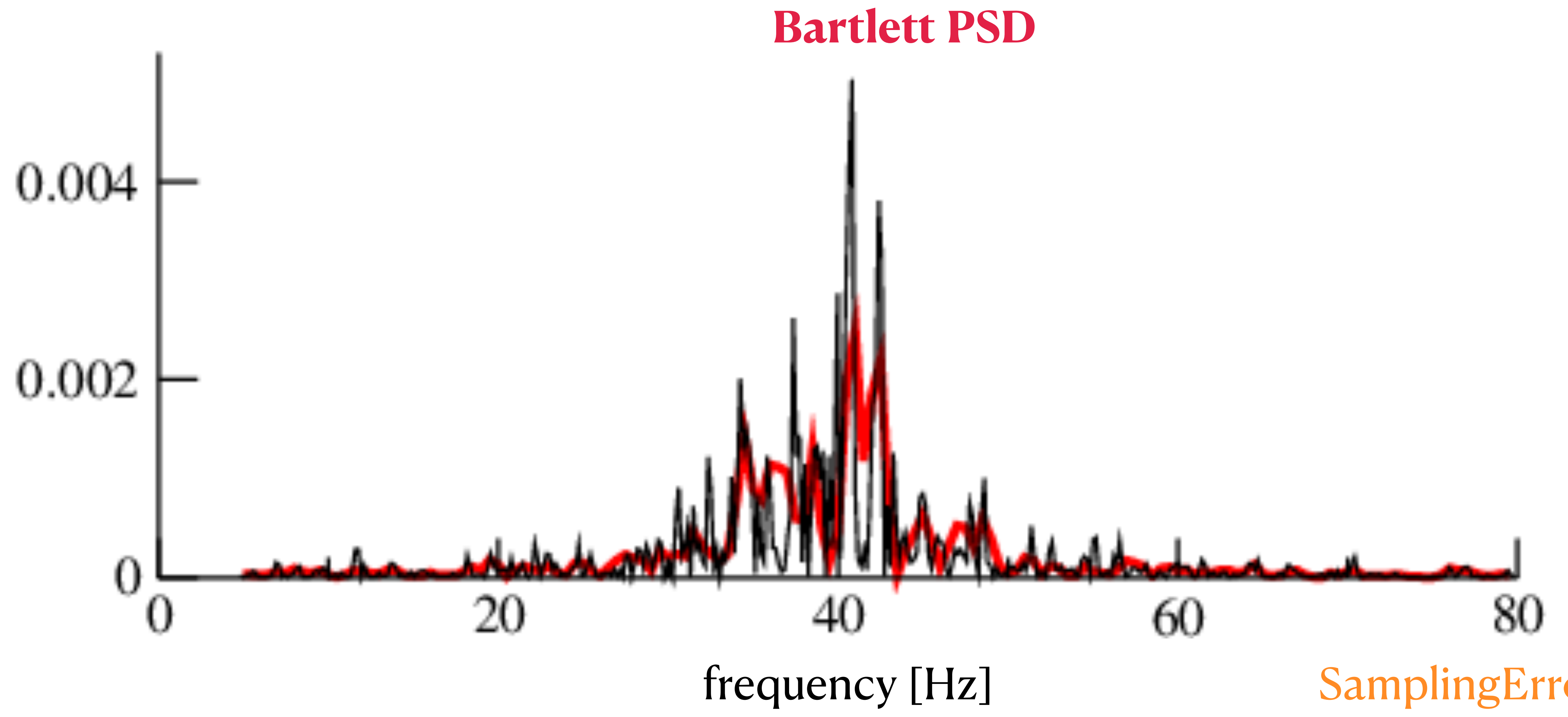

SamplingError_5.py

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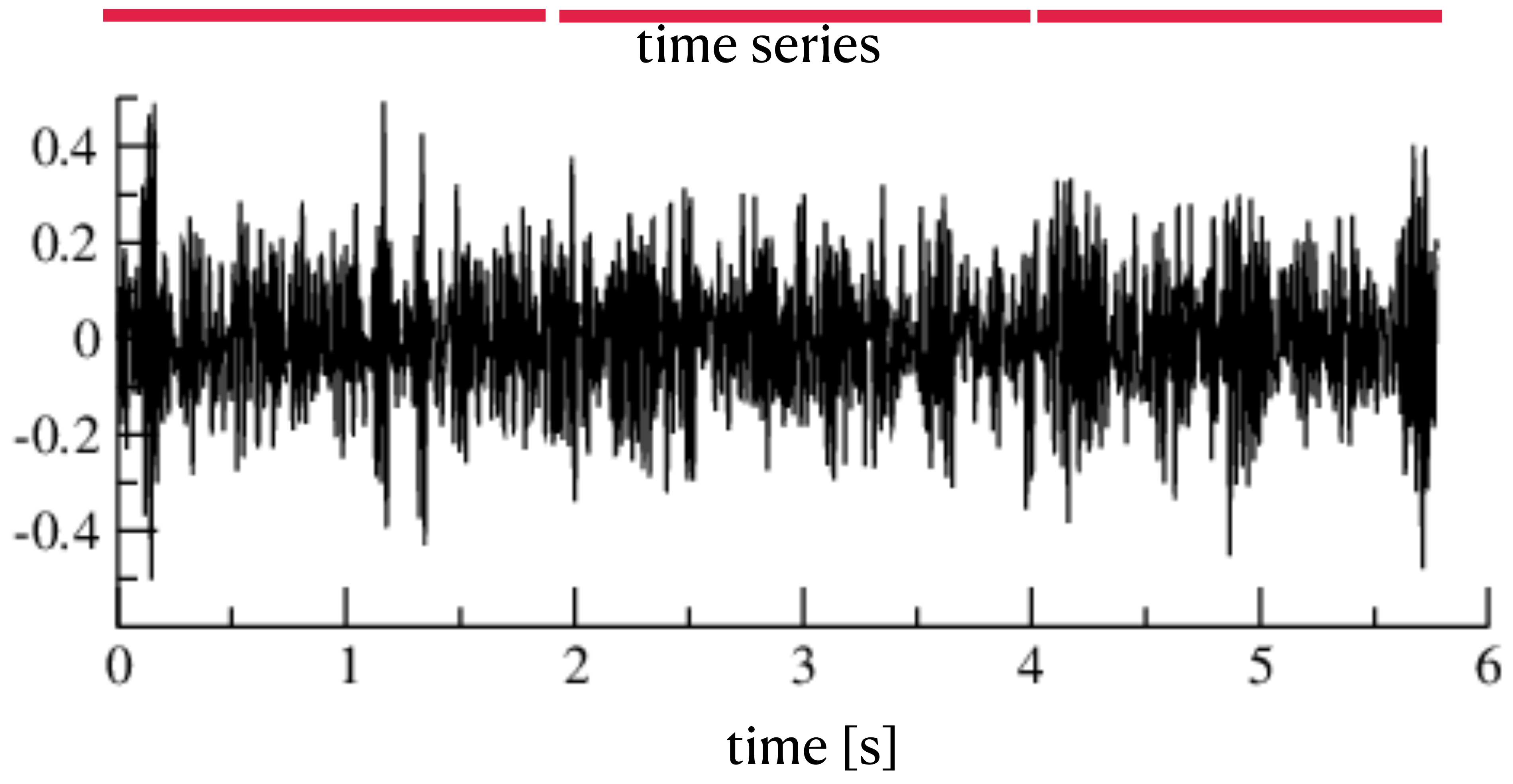
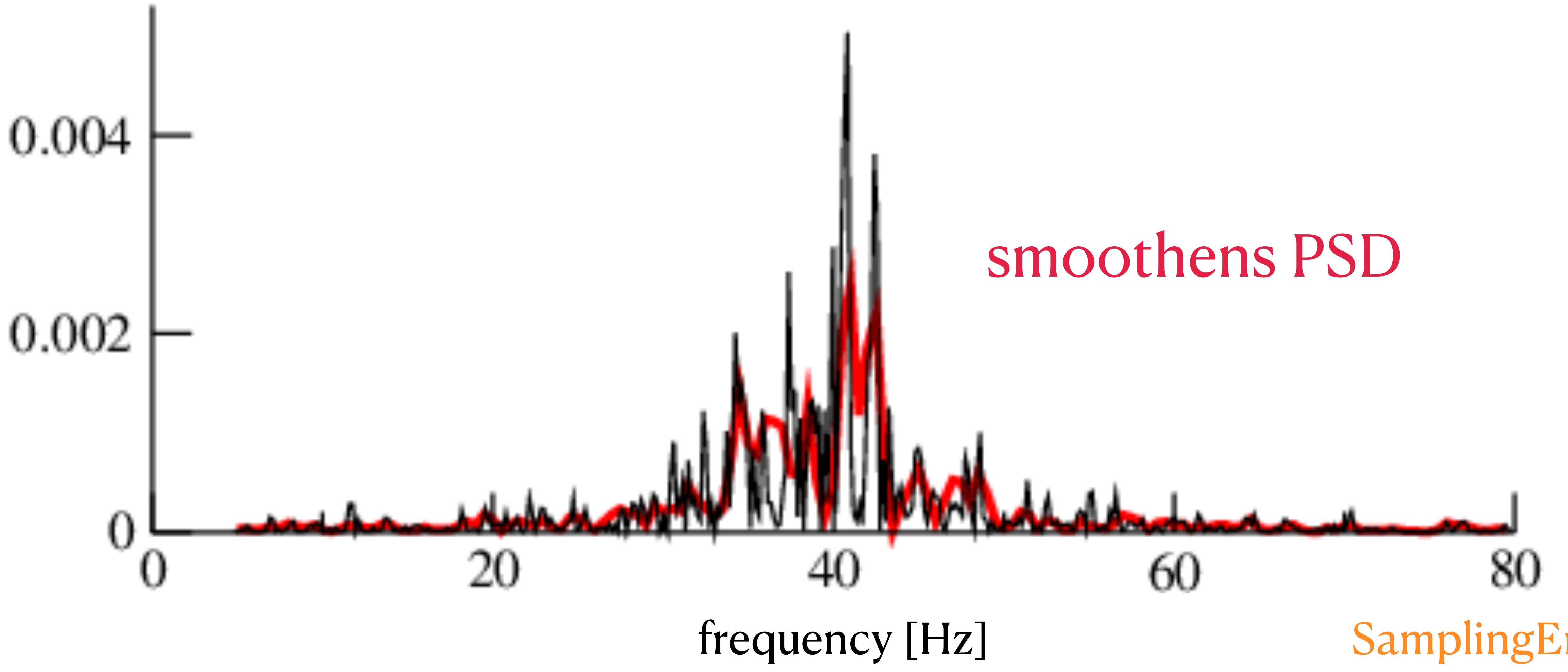


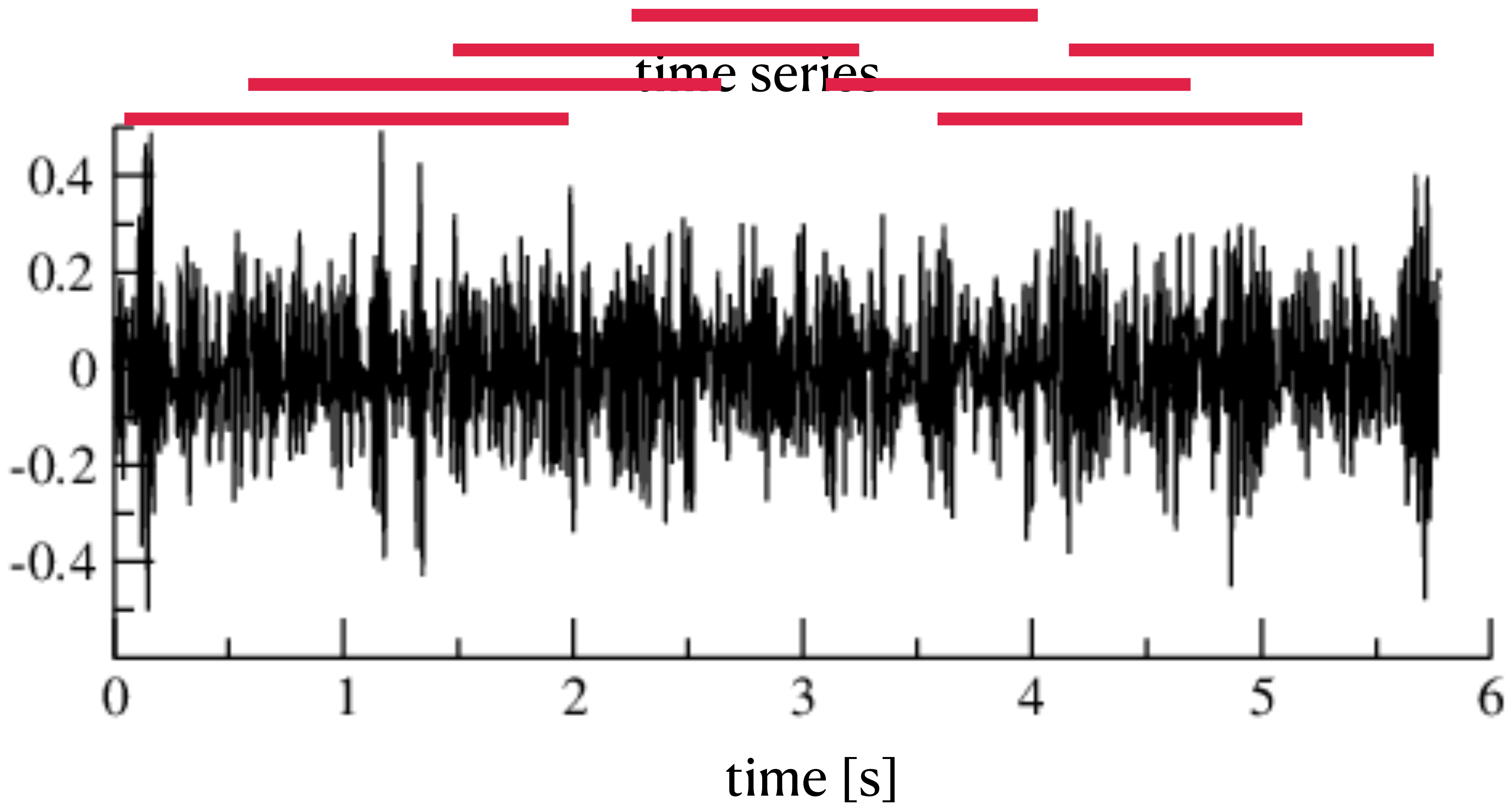


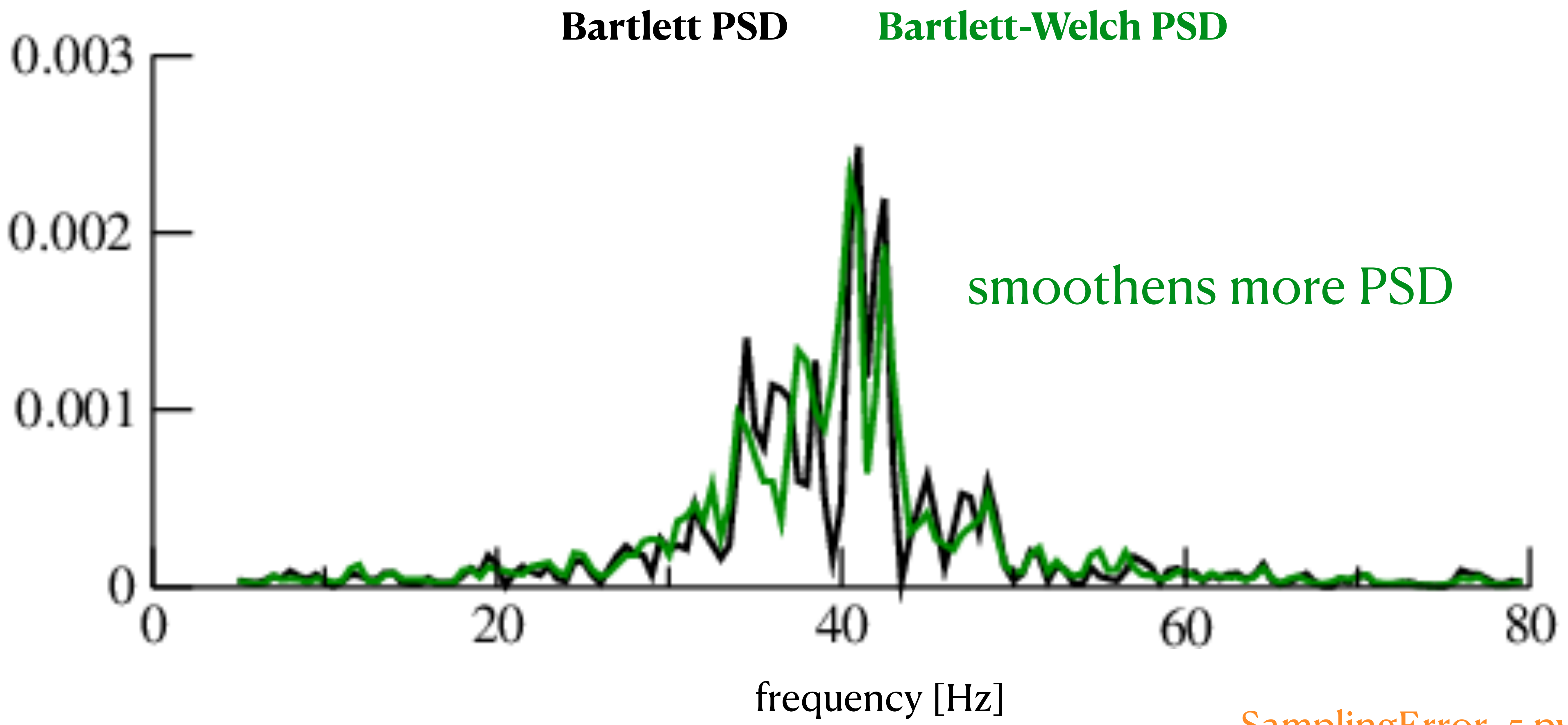
SamplingError_5.py

3. Bartlett-Welch method

$$S(f_n) = \frac{\Delta t}{N} |\mathrm{DFT}(f_n)|^2$$

DFT: average over **overlapping** segments





SamplingError_5.py

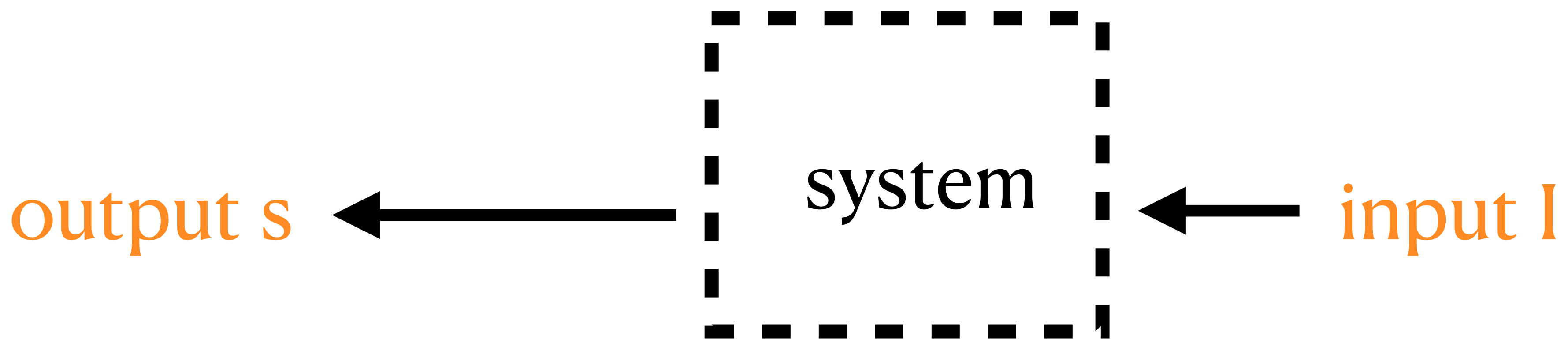
data sampling

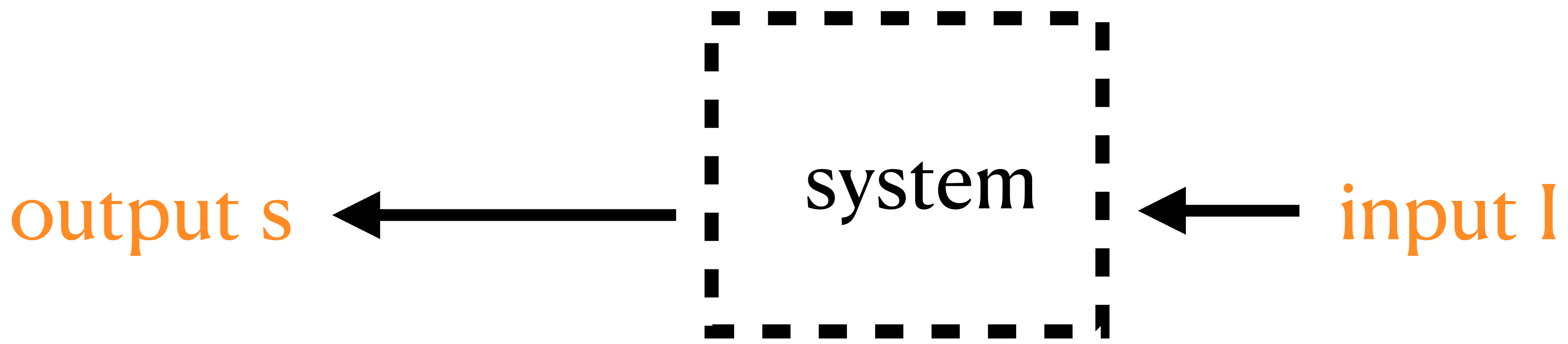
Fourier analysis

errors in analysis

linear filters

time-frequency analysis


output s
system
input I



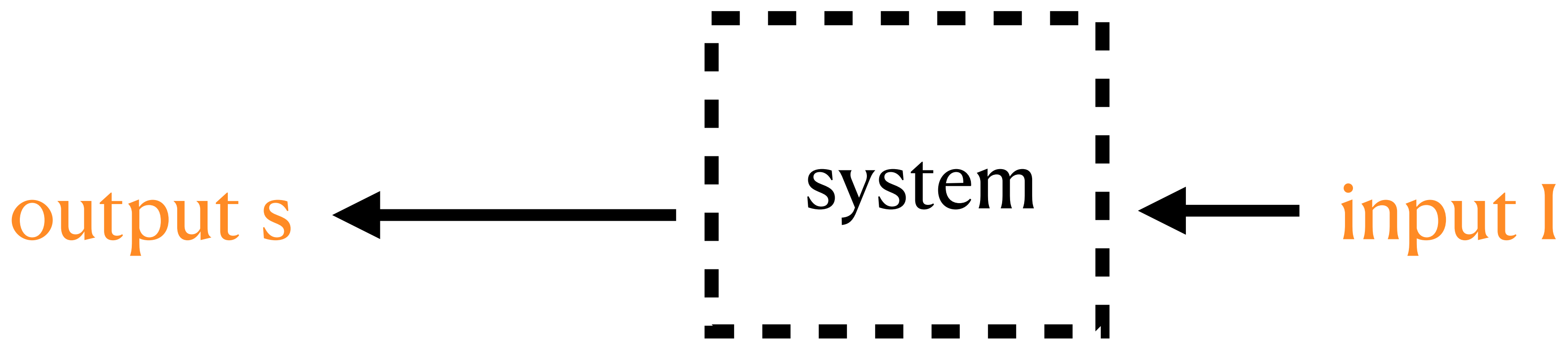
continuous time

$$s(t) = \int_{-\infty}^{t} H(t-\tau) I(\tau) d\tau$$

output

impulse response function

input



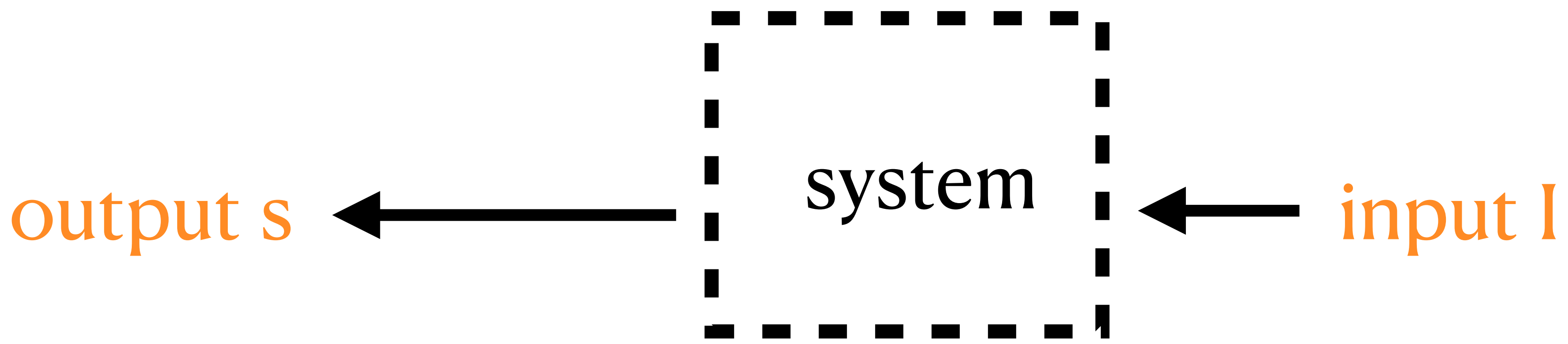
discrete time

$$s(t_k) = \sum_{m=-\infty}^{k} H(t_{k-m}) I(t_m)$$

output

impulse response function

input



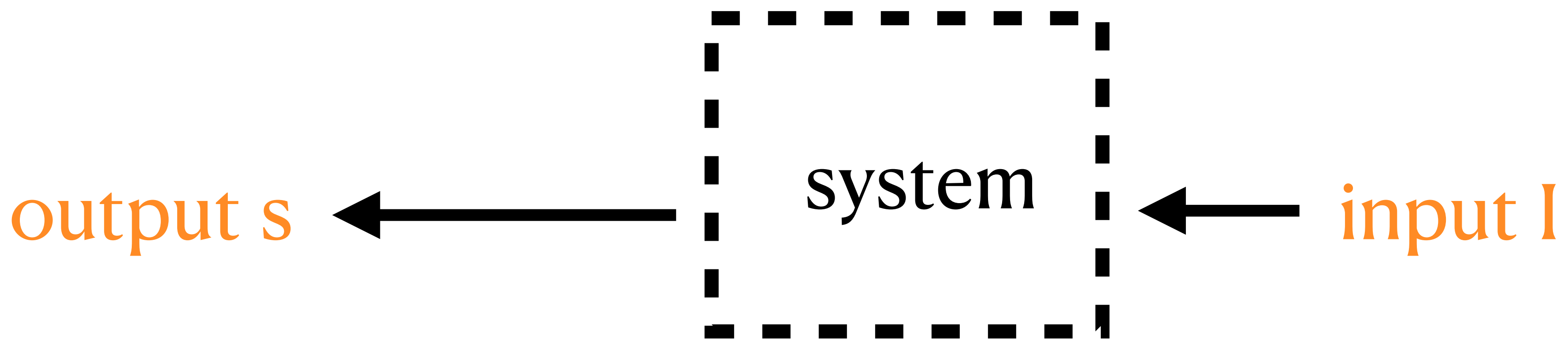
$$s(t_k) = \sum_{m=-\infty}^{k} H(t_{k-m}) I(t_m)$$

output

impulse response function

input

$$\tilde{s}(f) = \tilde{H}(f)\tilde{I}(f)$$



$$s(t_k) = \sum_{m=-\infty}^{k} H(t_{k-m}) I(t_m)$$

output

impulse response function

input

$$\tilde{s}(f) = \tilde{H}(f)\tilde{I}(f)$$

$$\boxed{\mathrm{PSD}(f) = |\tilde{H}(f)|^2 |\tilde{I}(f)|^2}$$

data sampling

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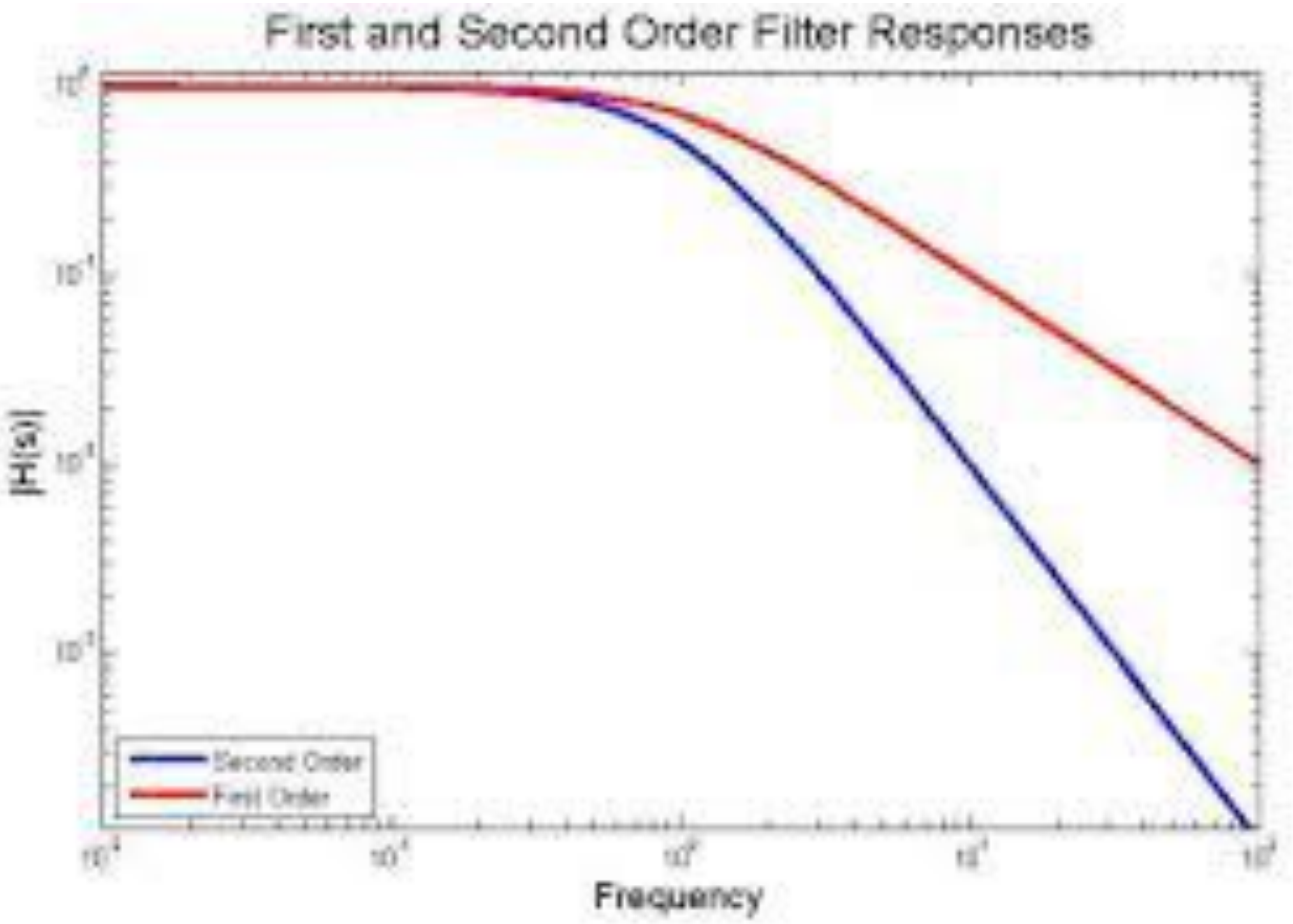
linear filters **frequency pass filter**

time-dependent filters

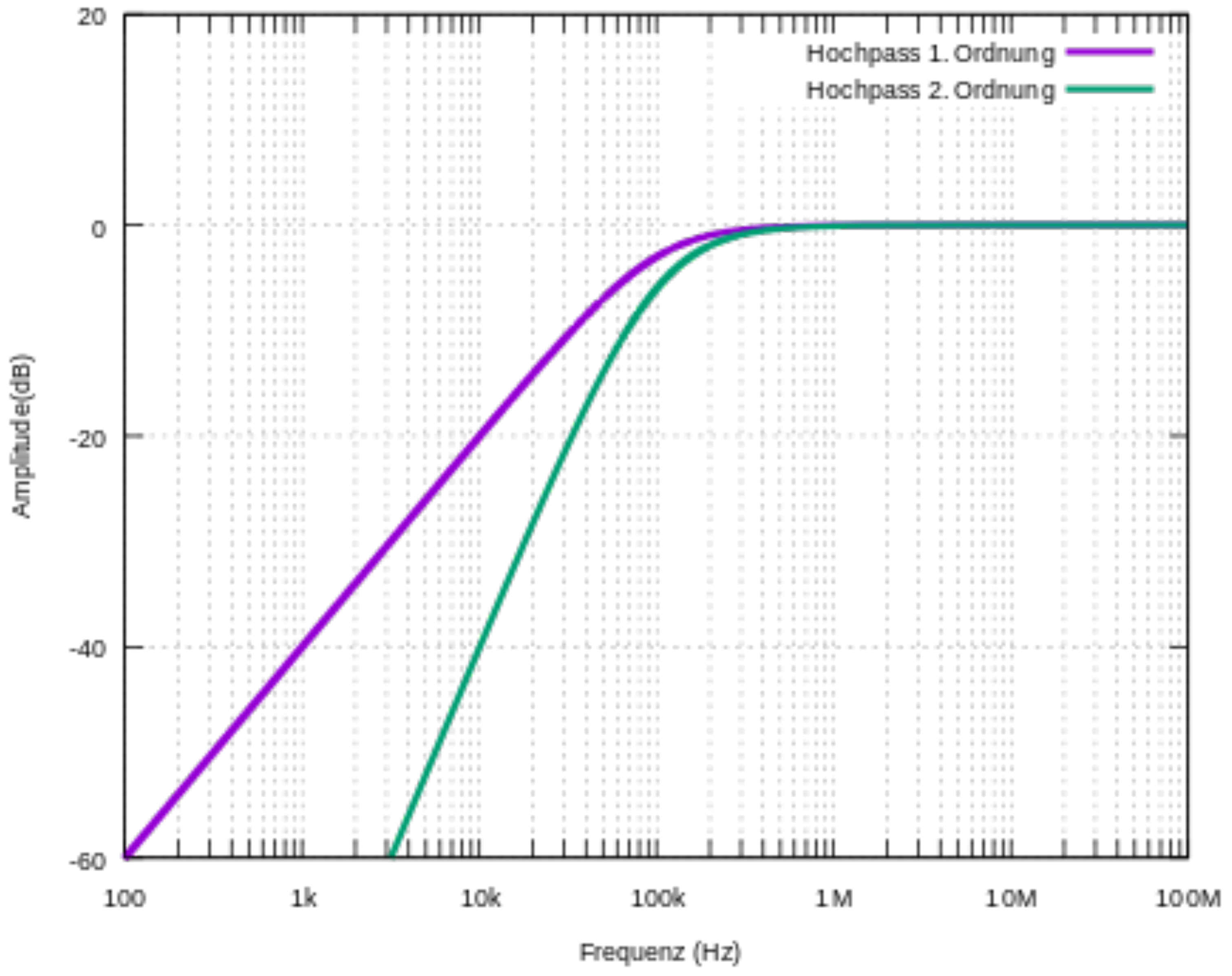
time-frequency analysis

$$\mathrm{PSD}(f) = |\tilde{H}(f)|^2 |\tilde{I}(f)|^2$$

Lowpass filter

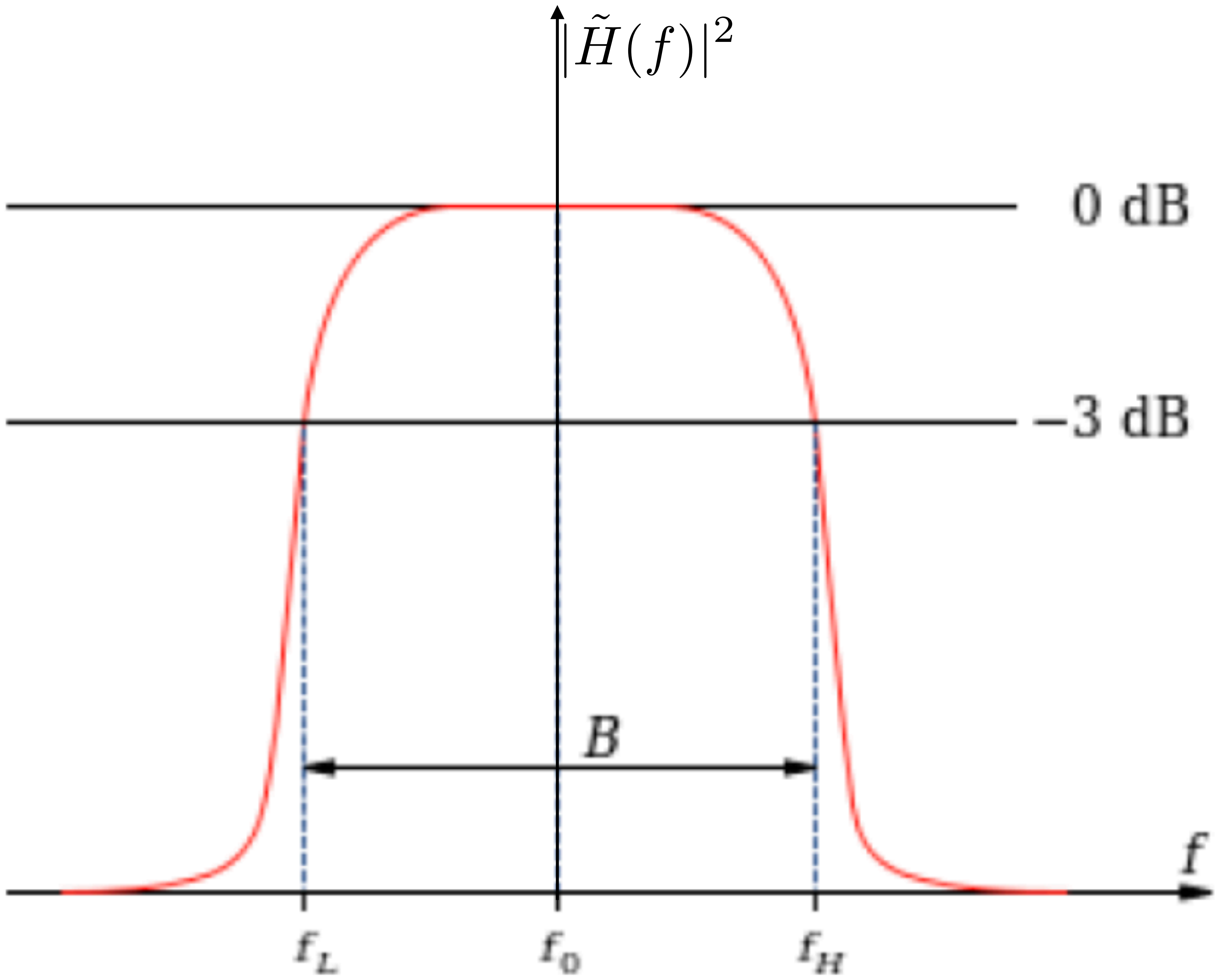


Highpass filter



$$\mathrm{PSD}(f) = |\tilde{H}(f)|^2 |\tilde{I}(f)|^2$$

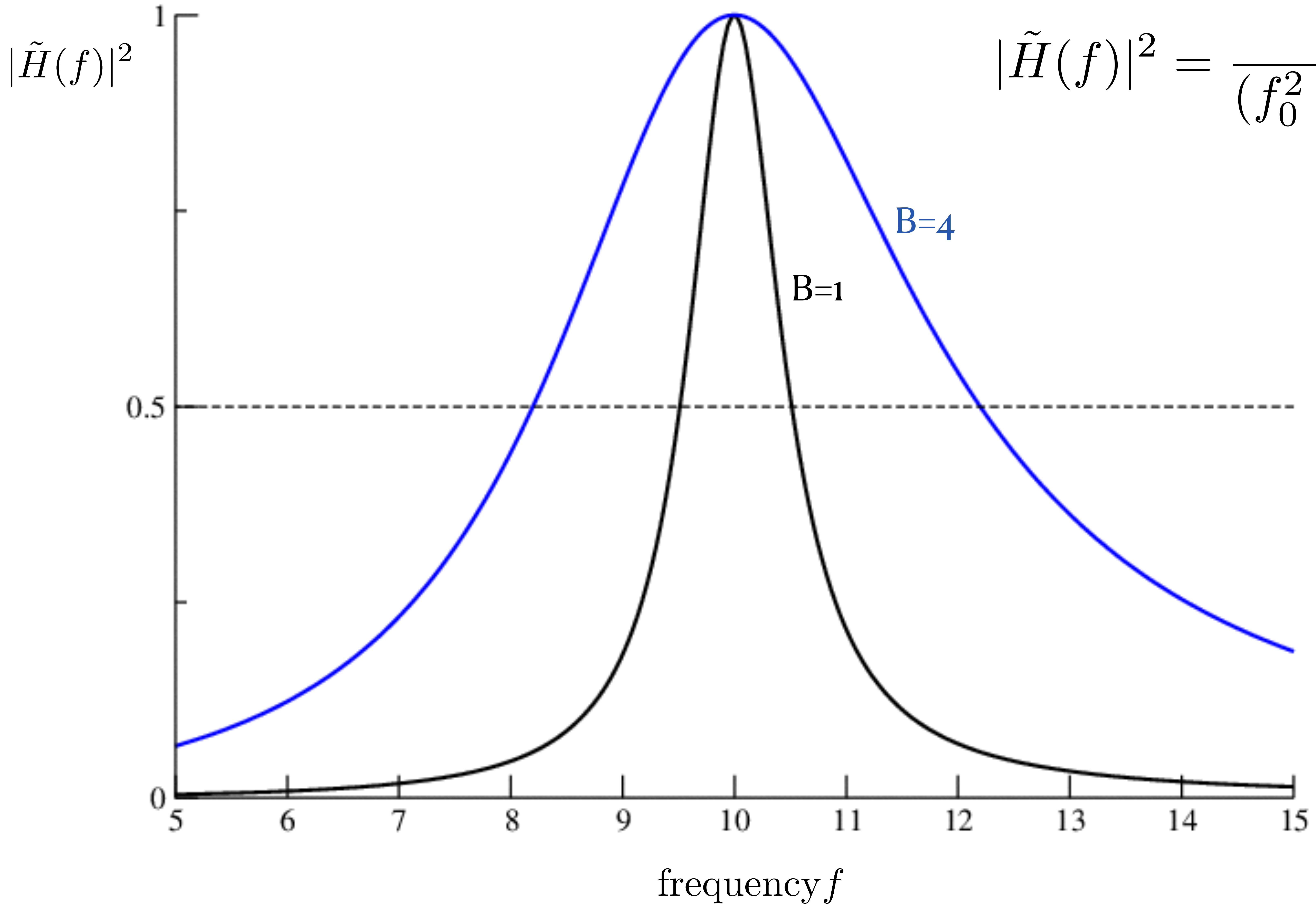
Bandpass filter



bandwidth B

Example: **2nd order bandpass filter**

$$|\tilde{H}(f)|^2 = \frac{B^2 f^2}{(f_0^2 - f^2)^2 + B^2 f^2}$$



Example: 2nd order bandpass filter

$$\tilde{s}(w) = \frac{iB'w}{-w^2 + iB'w + w_0^2}\tilde{I}(w) \qquad w = 2\pi f \ , \ B' = 2\pi B$$

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$$iw\tilde{s}(w) \leftrightarrow \frac{d}{dt}s(t)$$

$$-w^2\tilde{s}(w) \leftrightarrow \frac{d^2}{dt^2}s(t)$$

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$$-w^2\tilde{s}(w) \leftrightarrow \frac{d^2}{dt^2}s(t)$$

$$\left(\frac{d^2}{dt^2} + B'\frac{d}{dt} + w_0^2\right)s(t) = B'\frac{d}{dt}I(t)$$

continuous time

$$\dot{s} = u$$

$$\dot{u} = -B'u - w_0^2 s + B'\dot{I}$$

continuous time

$$\dot{s} = u$$

$$\dot{u} = -B'u - w_0^2 s + B'\dot{I}$$

discrete time

$$s_{n+1} = s_n + \Delta t u_n$$

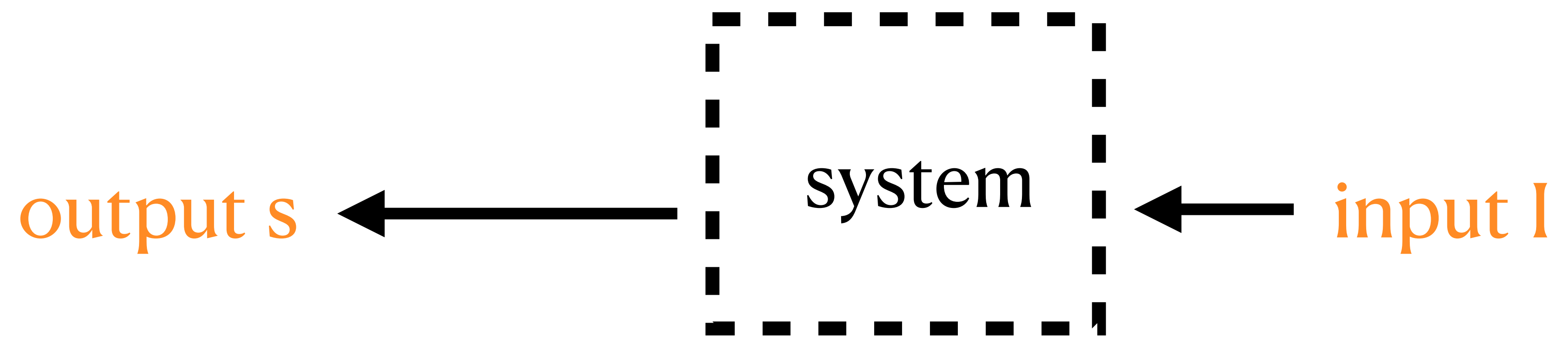
$$u_{n+1} = u_n + \Delta t(-B'u_n - w_0^2 s_n + B'\dot{I}_n)$$

continuous time

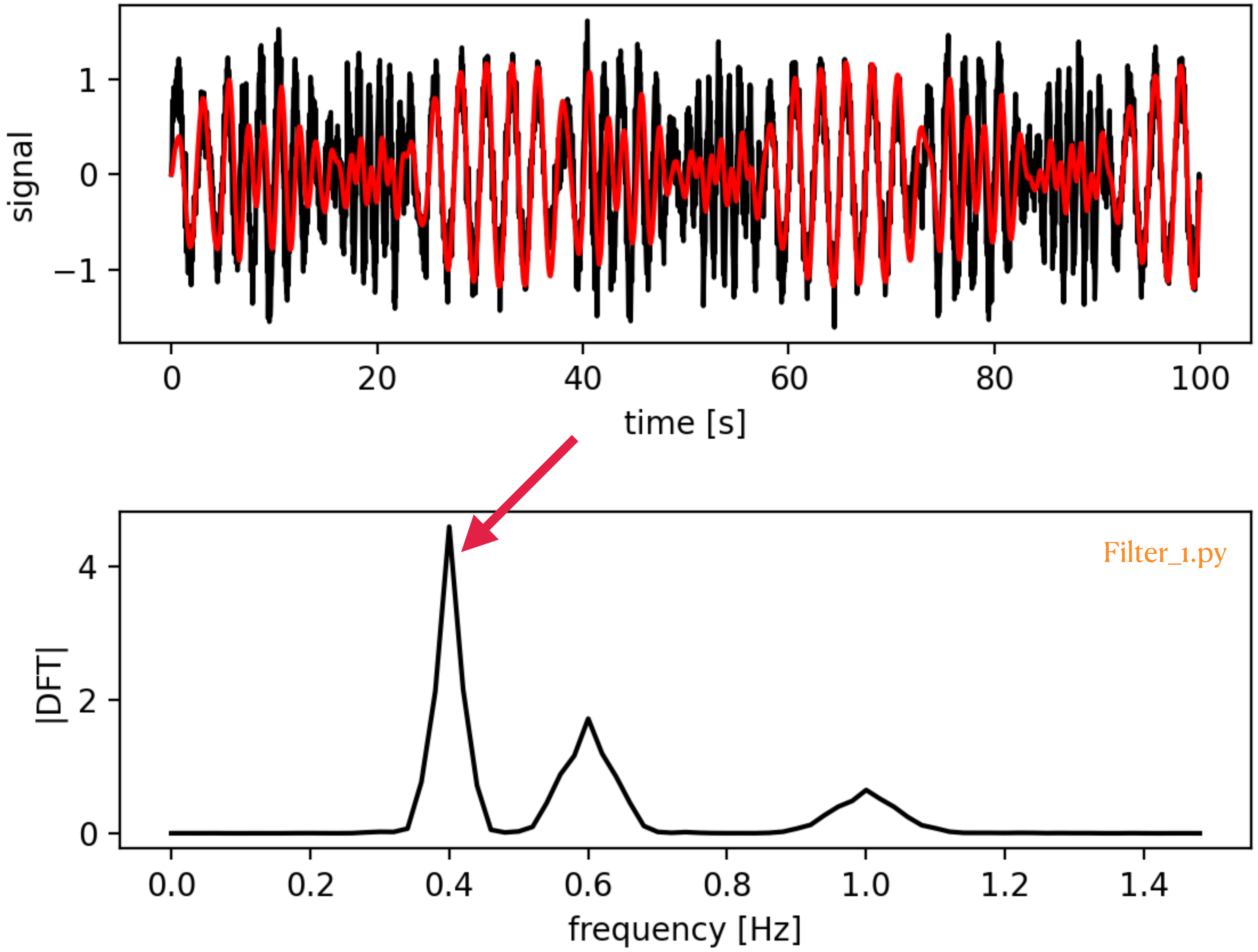
$$\dot{s} = u$$
$$\dot{u} = -B'u - w_0^2 s + B'\dot{I}$$

discrete time

$$s_{n+1} = s_n + \Delta t u_n$$
$$u_{n+1} = u_n + \Delta t(-B'u_n - w_0^2 s_n + B'\dot{I}_n)$$

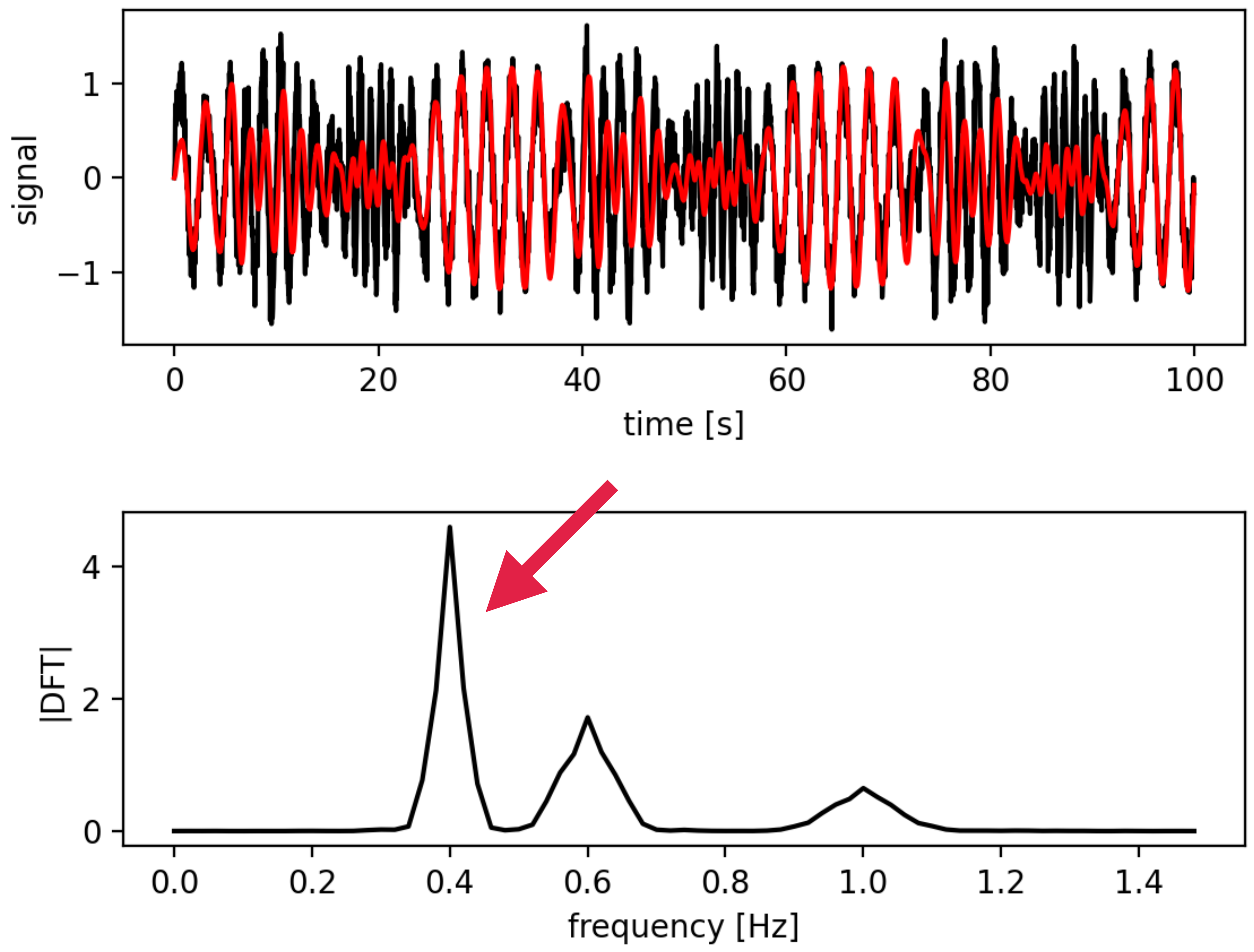


application to time series



f_0=0.4Hz , B=0.2Hz

application to time series

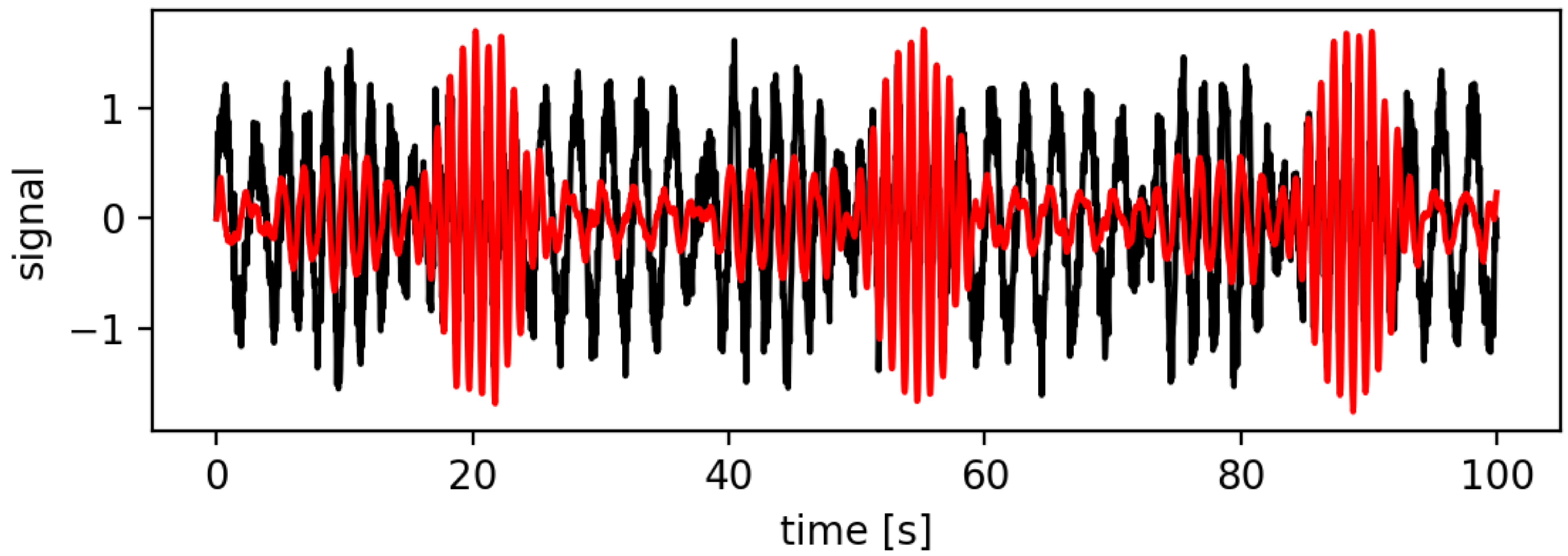


f_0=0.4Hz , B=0.2Hz

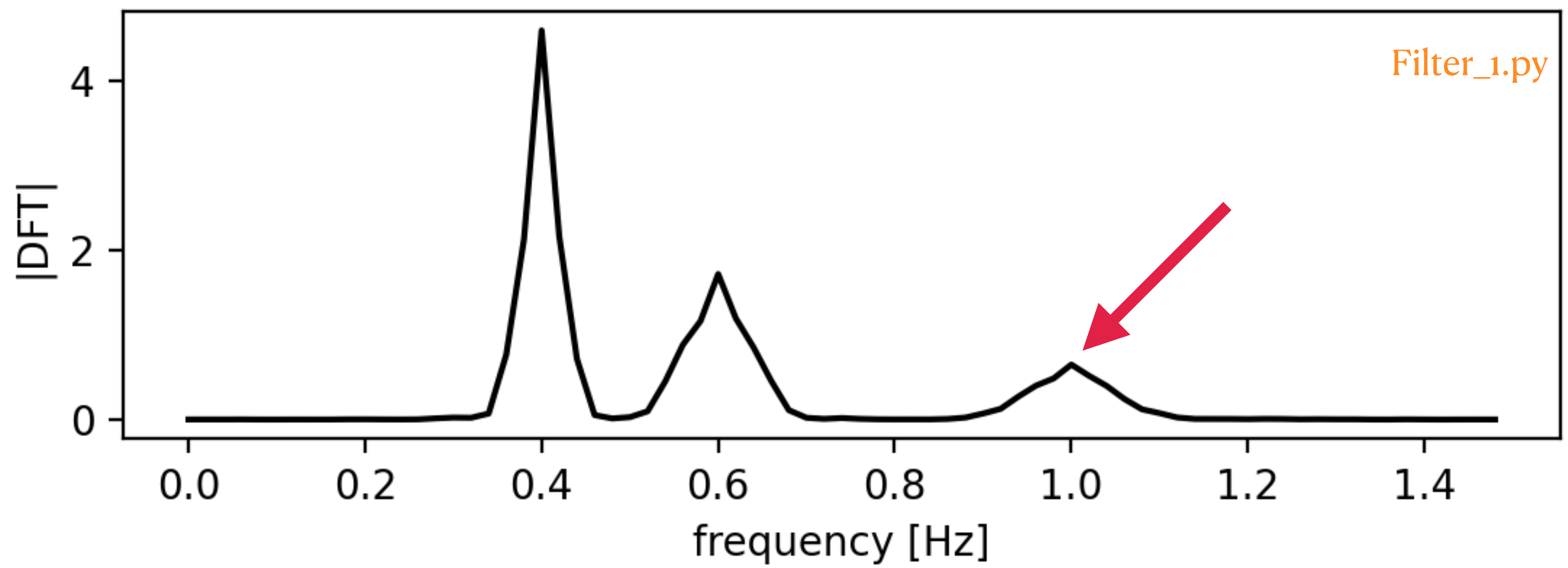
selection of 0.4Hz oscillation,

but contribution of neighbour 0.6Hz-band visible

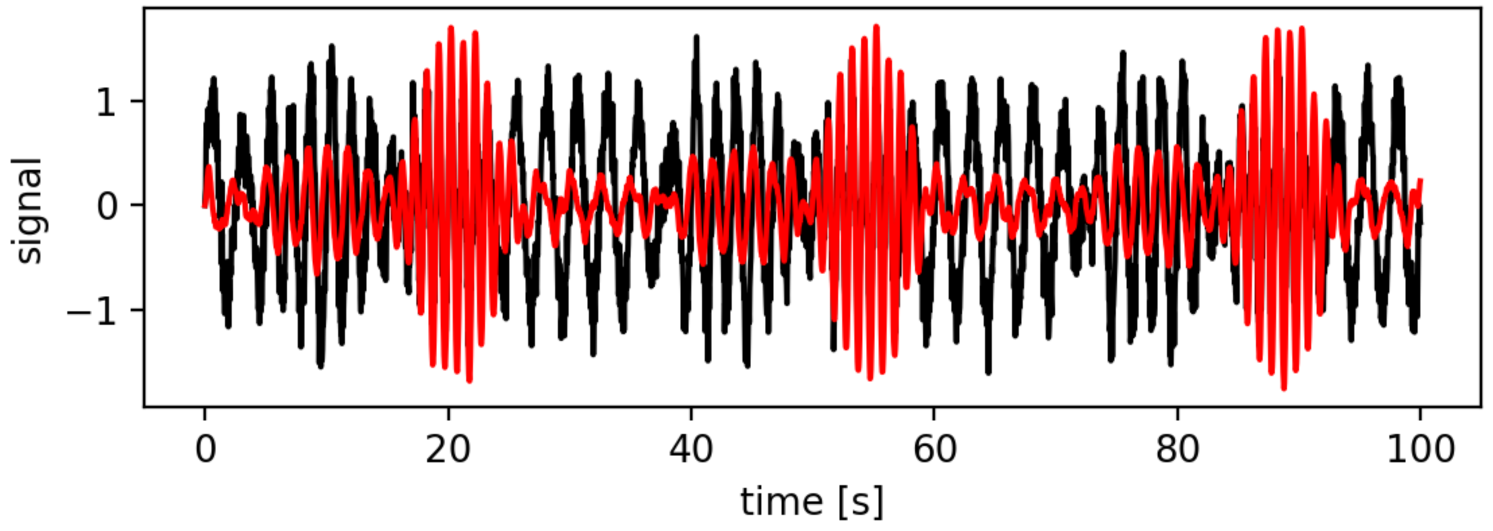
application to time series

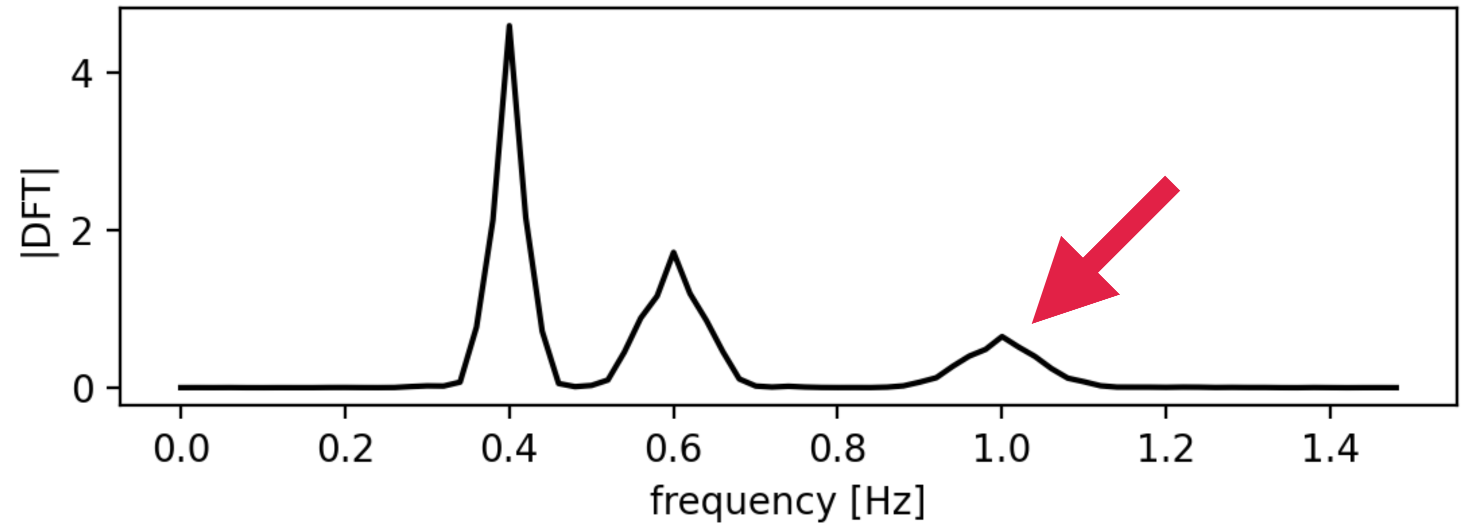


f_0=1.0Hz , B=0.5Hz



application to time series



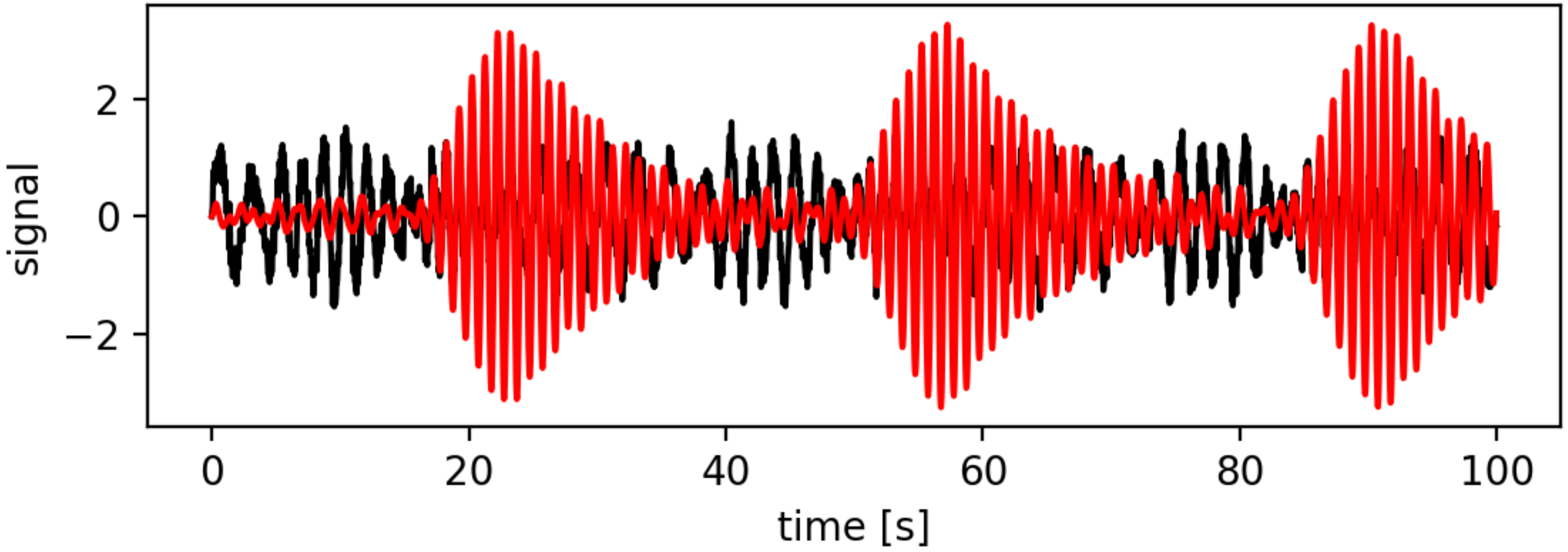


f_0=1.0Hz , B=0.5Hz

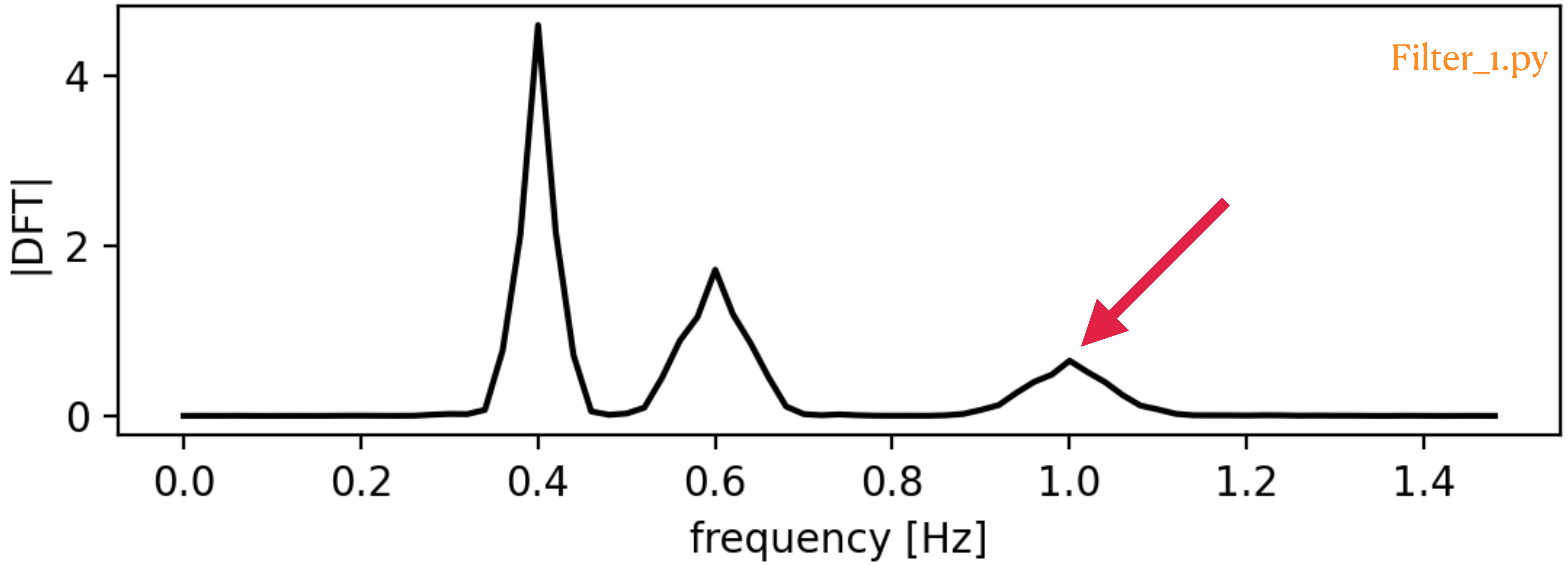
selection of 1.0Hz oscillation,

but contribution of neighbour 0.4 and 0.6Hz-band visible

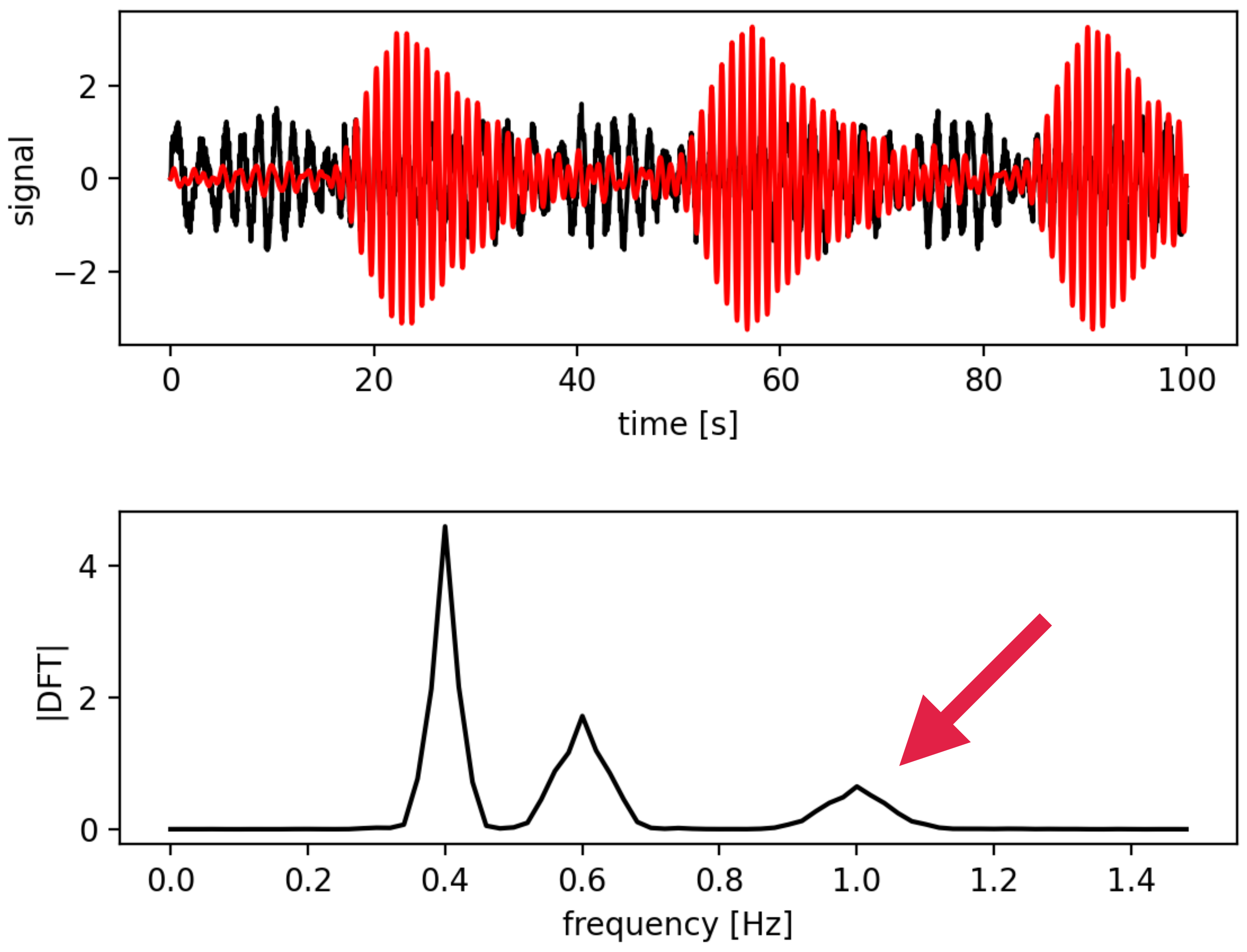
application to time series: **filter divergence**



f_0=1.0Hz , B=**0.25**Hz



application to time series: **filter divergence**

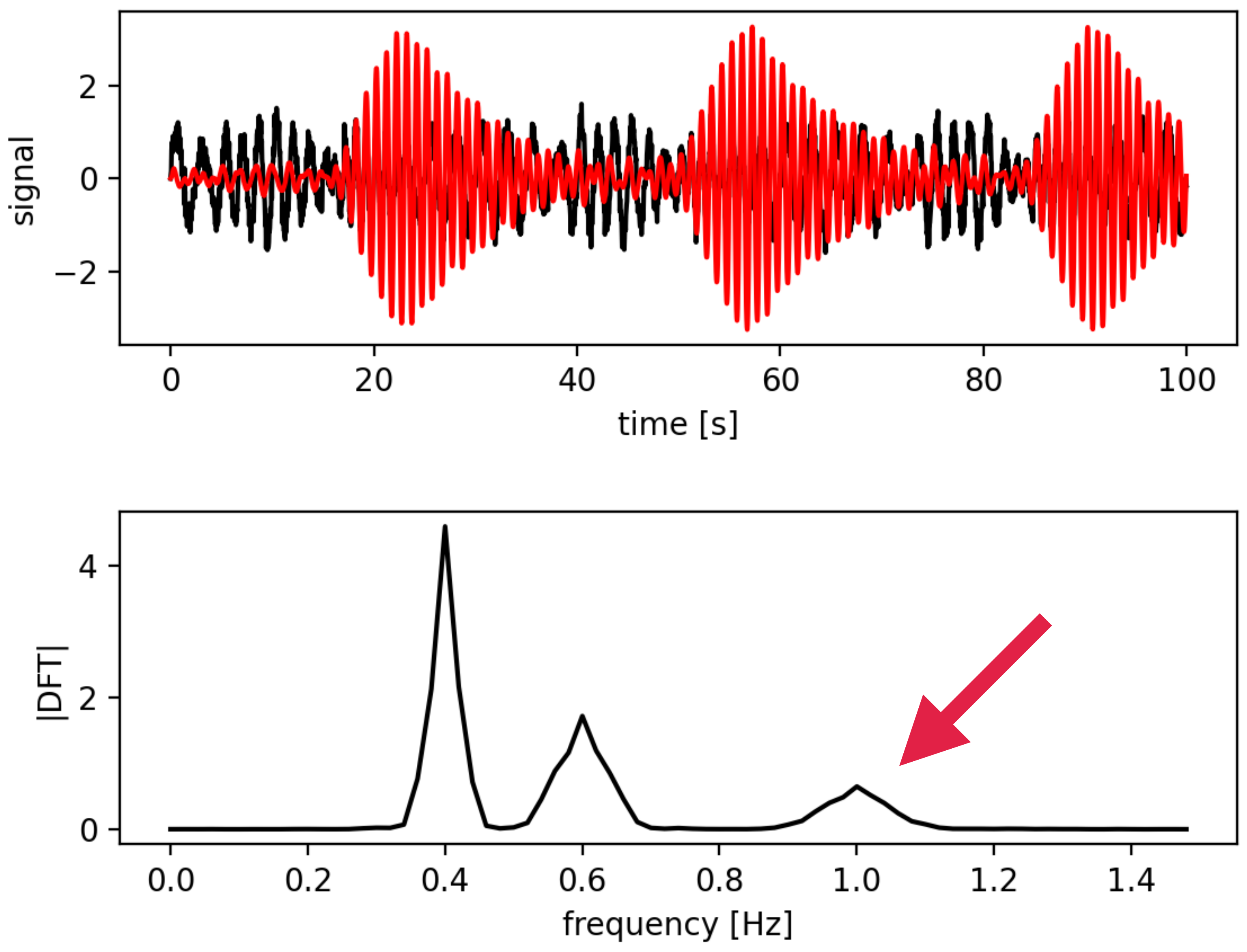


f_0=1.0Hz , B=**0.25**Hz

enhancement of 1.0Hz oscillation by filter divergence

due to too small bandwidth B

application to time series: **filter divergence**



f_0=1.0Hz , B=**0.25**Hz

enhancement of 1.0Hz oscillation by filter divergence

due to too small bandwidth B

solution: larger bandwidth or filter of higher order

bandpass filter of higher order:

e.g. 4th order

$$\tilde{s}(w) = \frac{a_1 w^2}{b_0 + b_1 w + b_2 w^2 + b_3 w^3 + b_4 w^4} \tilde{I}(w)$$

with parameters a_1, b_1,..,b_4

bandpass filter of higher order:

e.g. 4th order

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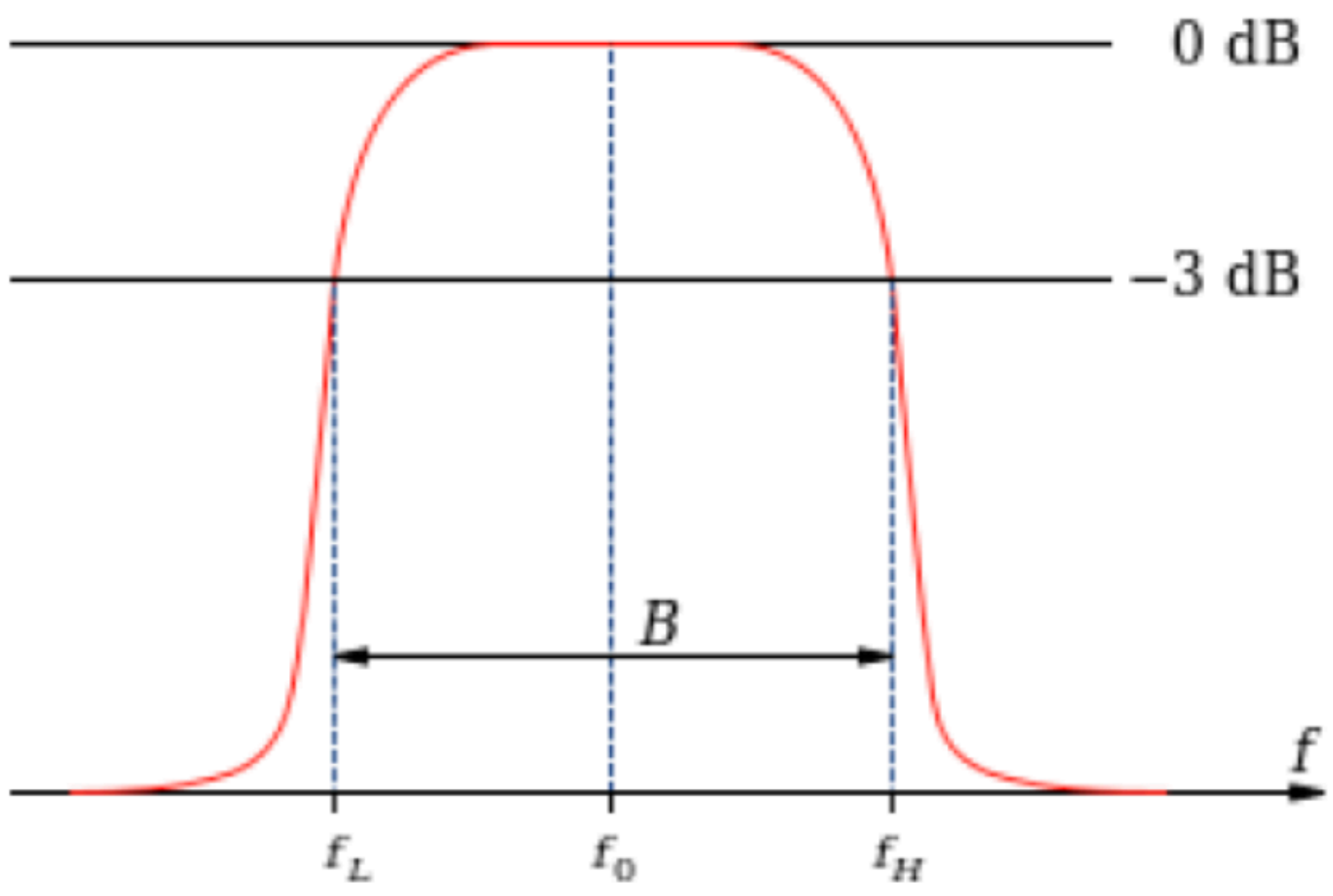
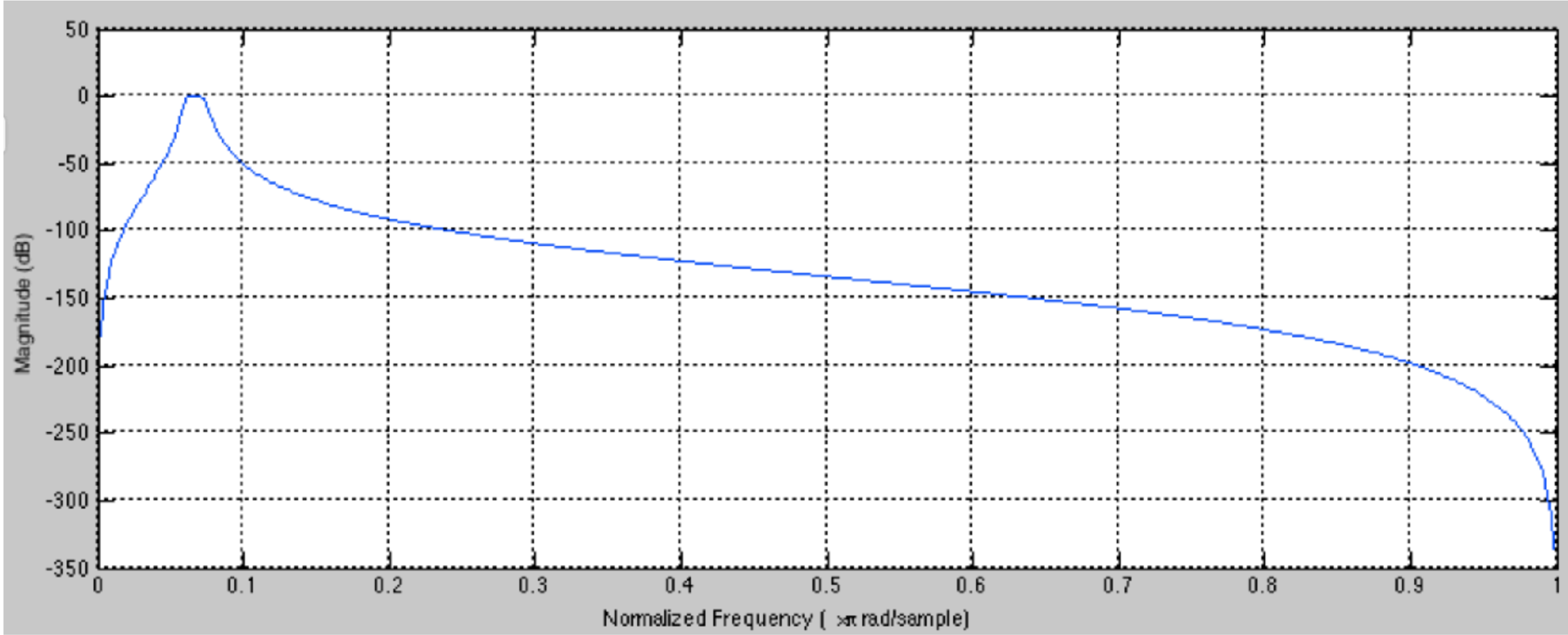


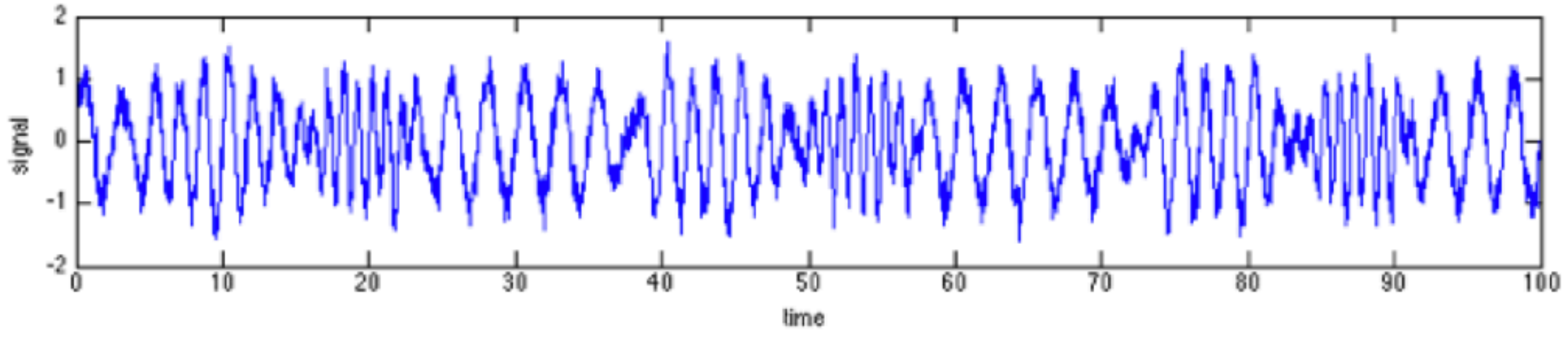
advantage:

faster decay at borders

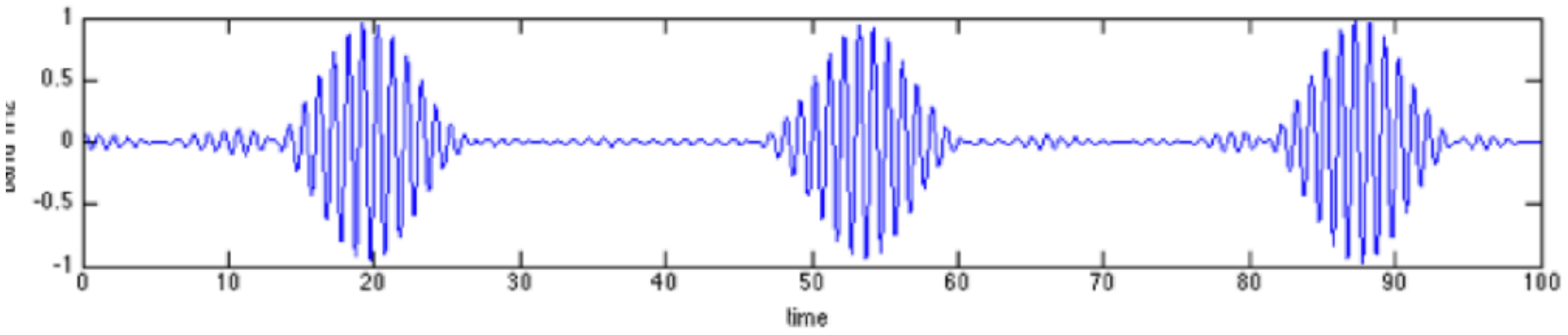
bandpass filter of higher order: 4th order Butterworth filter



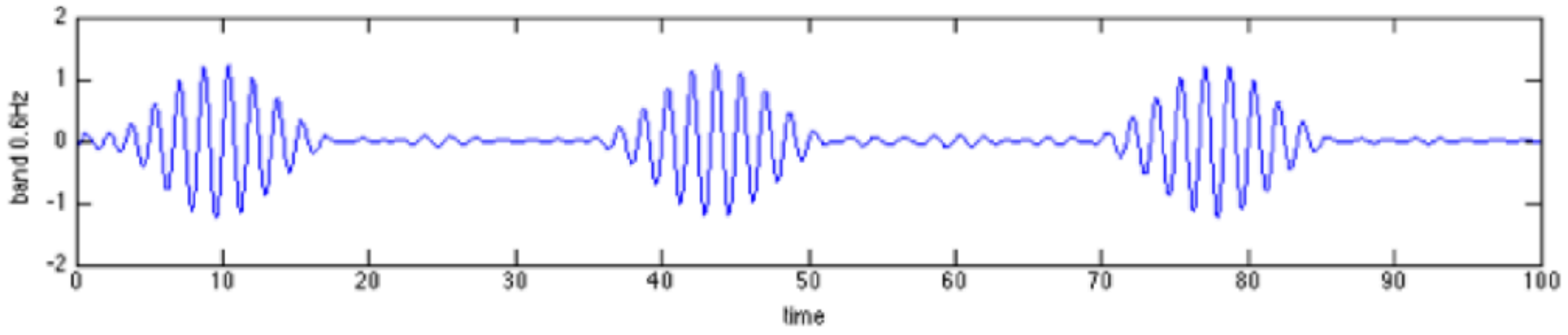
bandpass filter of higher order: 4th order Butterworth filter



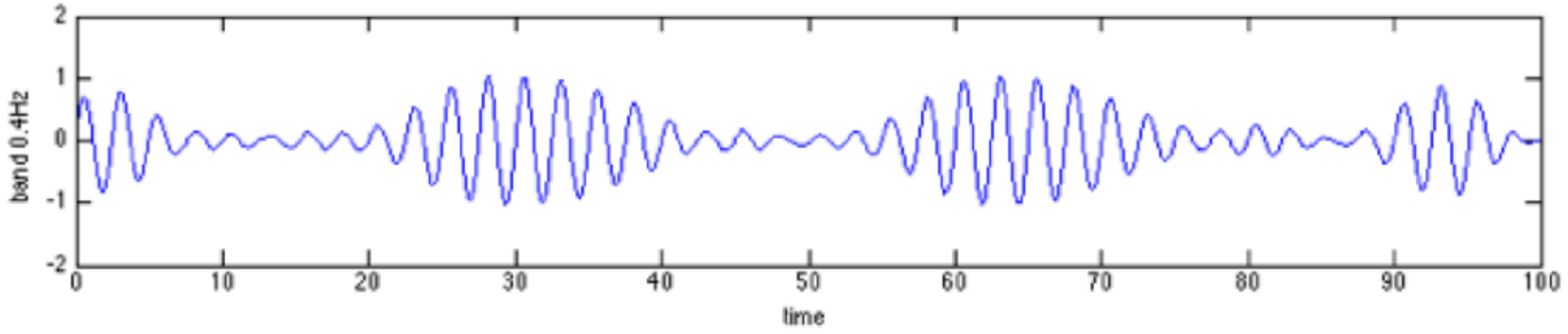
bandpass
with center frequency 1.0Hz



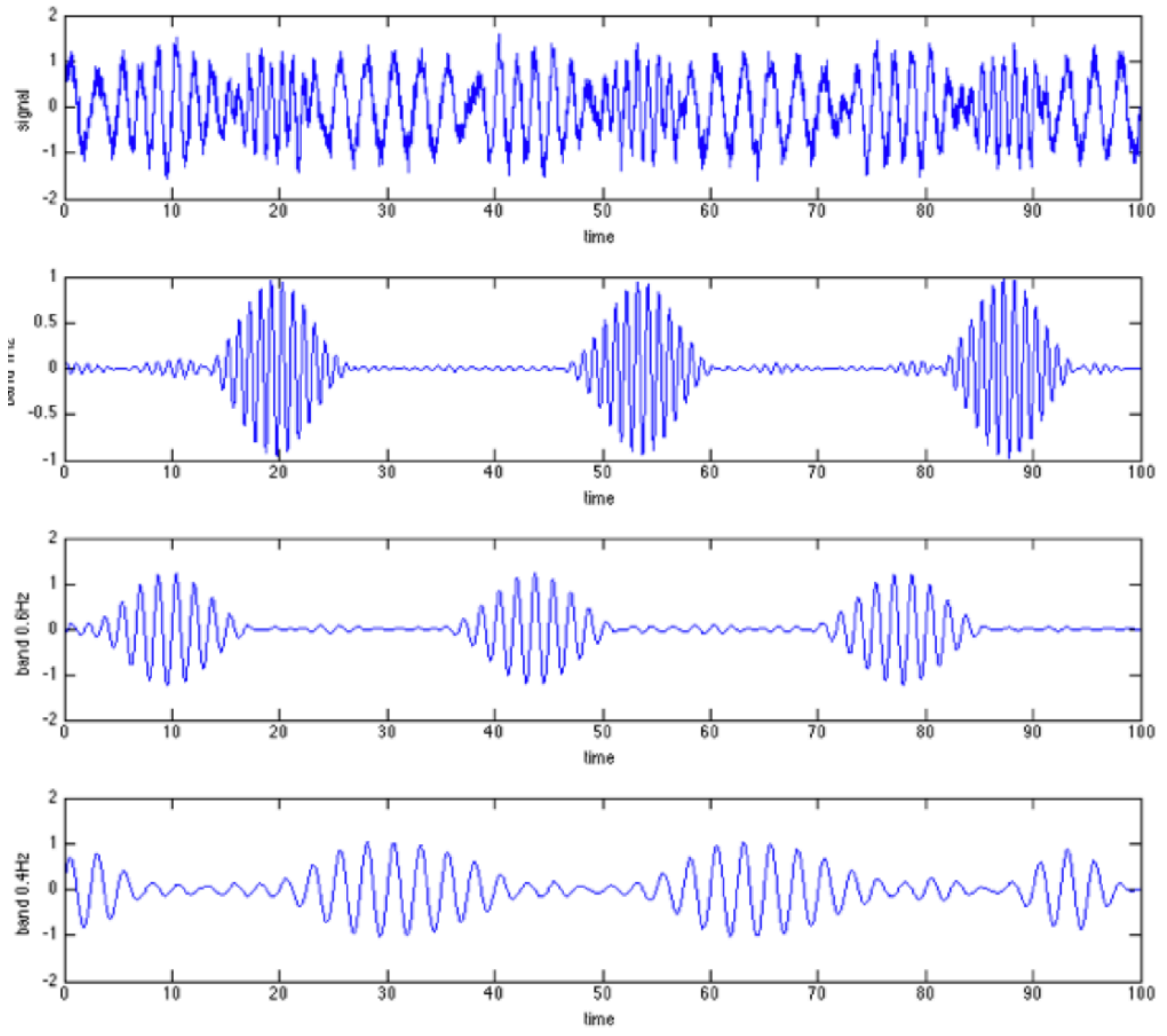
bandpass
with center frequency 0.6Hz



bandpass
with center frequency 0.4Hz

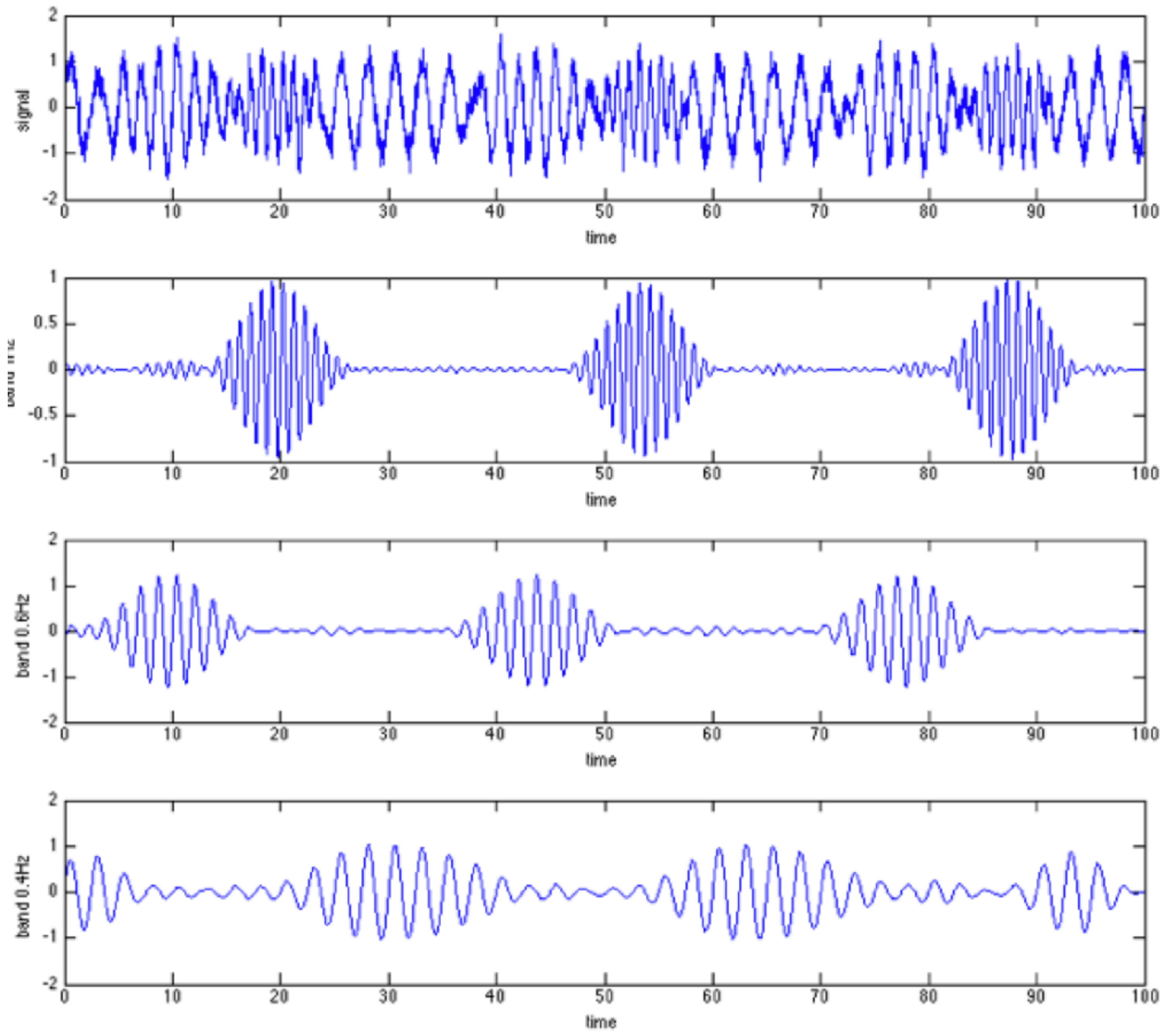


bandpass filter of higher order: 4th order Butterworth filter



4th order Butterworth bandpass filter is more stable than previous 2nd order

bandpass filter of higher order: 4th order Butterworth filter



4th order Butterworth bandpass filter is more stable than previous 2nd order

Take home message: try different orders and check stability

data sampling

Fourier analysis

errors in analysis

linear filters frequency pass filter

time-dependent filters

time-frequency analysis

Example:

$$\tilde{s}(w) = \frac{1}{iw + 1/\tau_0} \tilde{I}(w)$$

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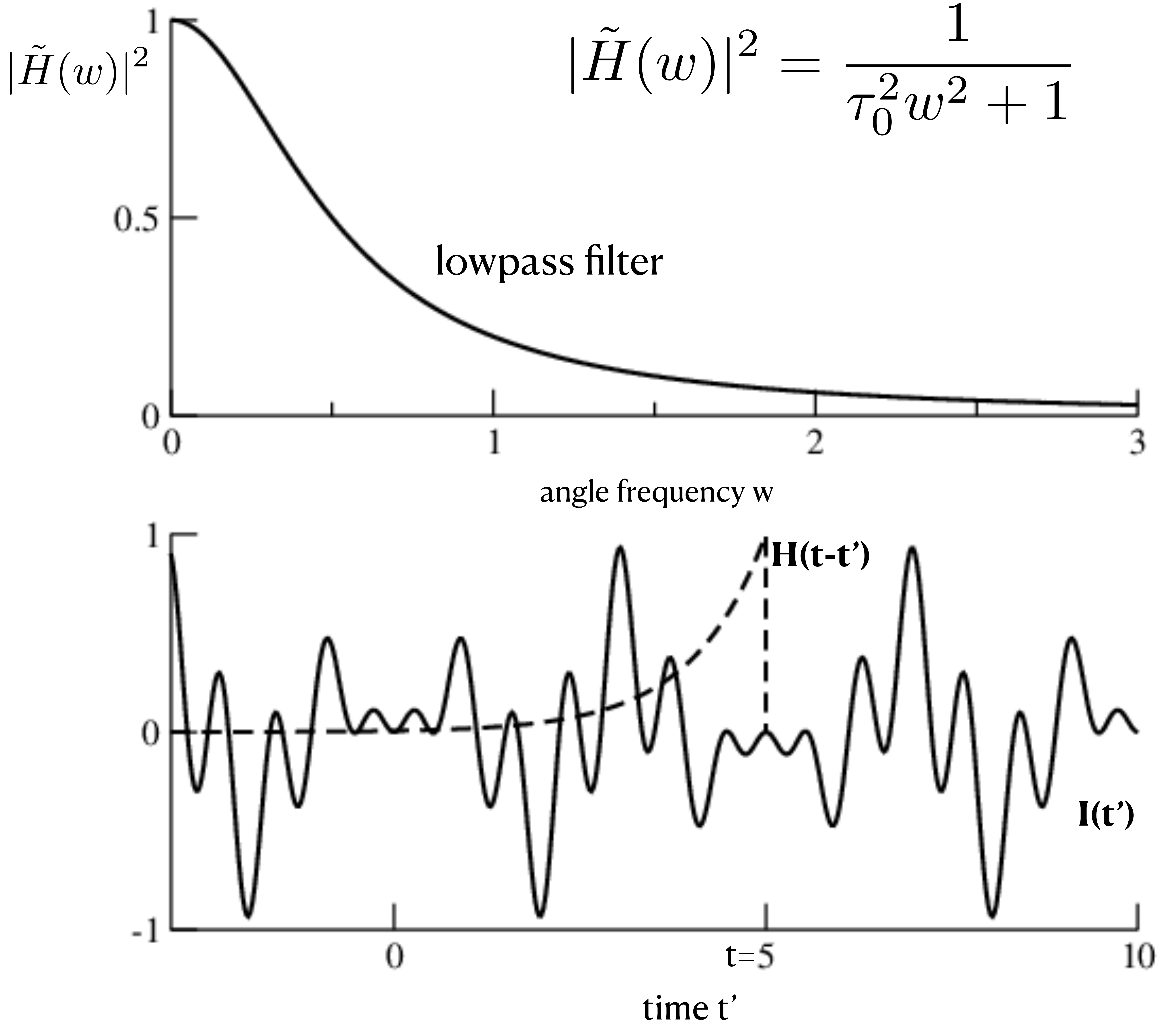
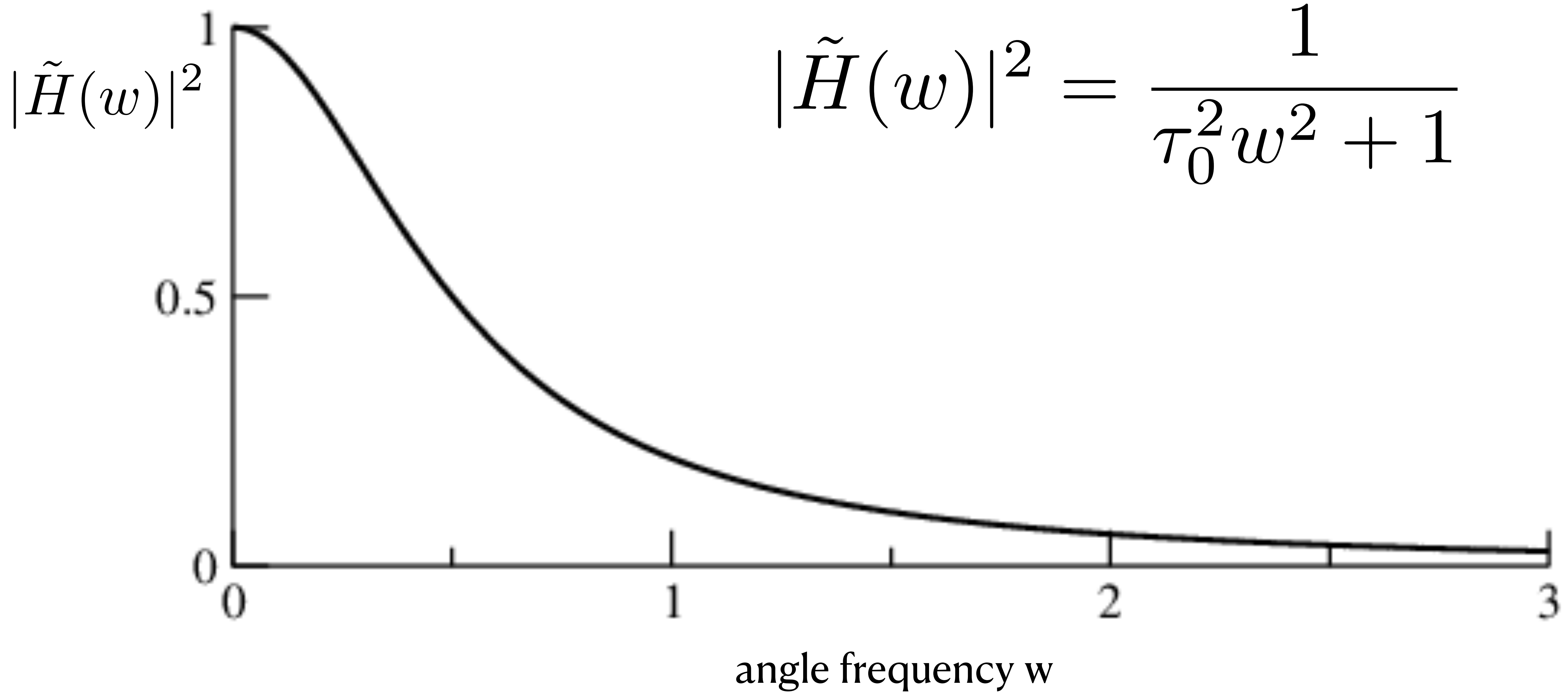
$$\tilde{s}(w) = \frac{1}{iw + 1/\tau_0} \tilde{I}(w)$$

$$|\tilde{H}(w)|^2 = \frac{1}{\tau_0^2 w^2 + 1}$$



Example:

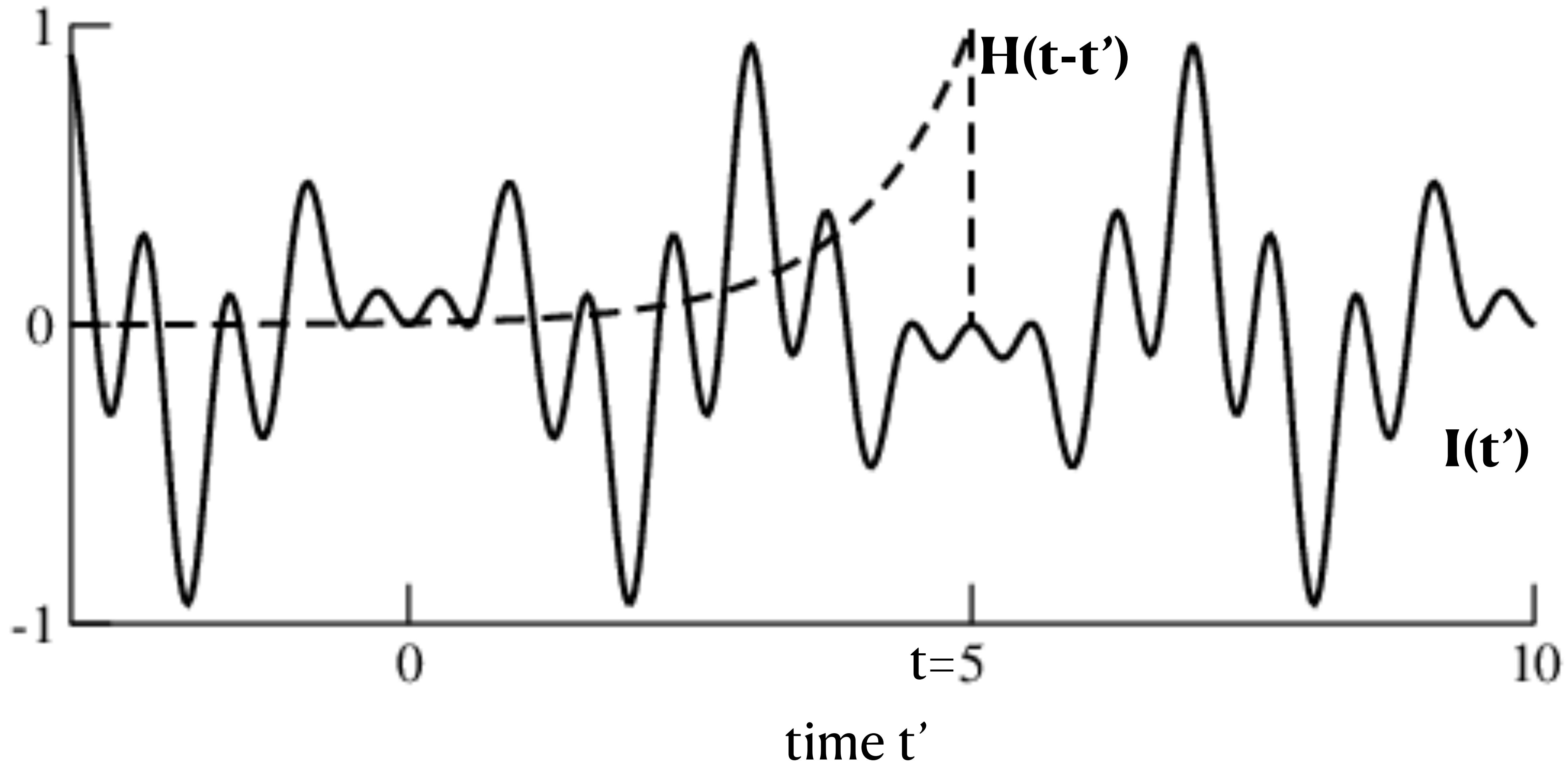
$$\tilde{s}(w) = \frac{1}{iw + 1/\tau_0}\tilde{I}(w)$$



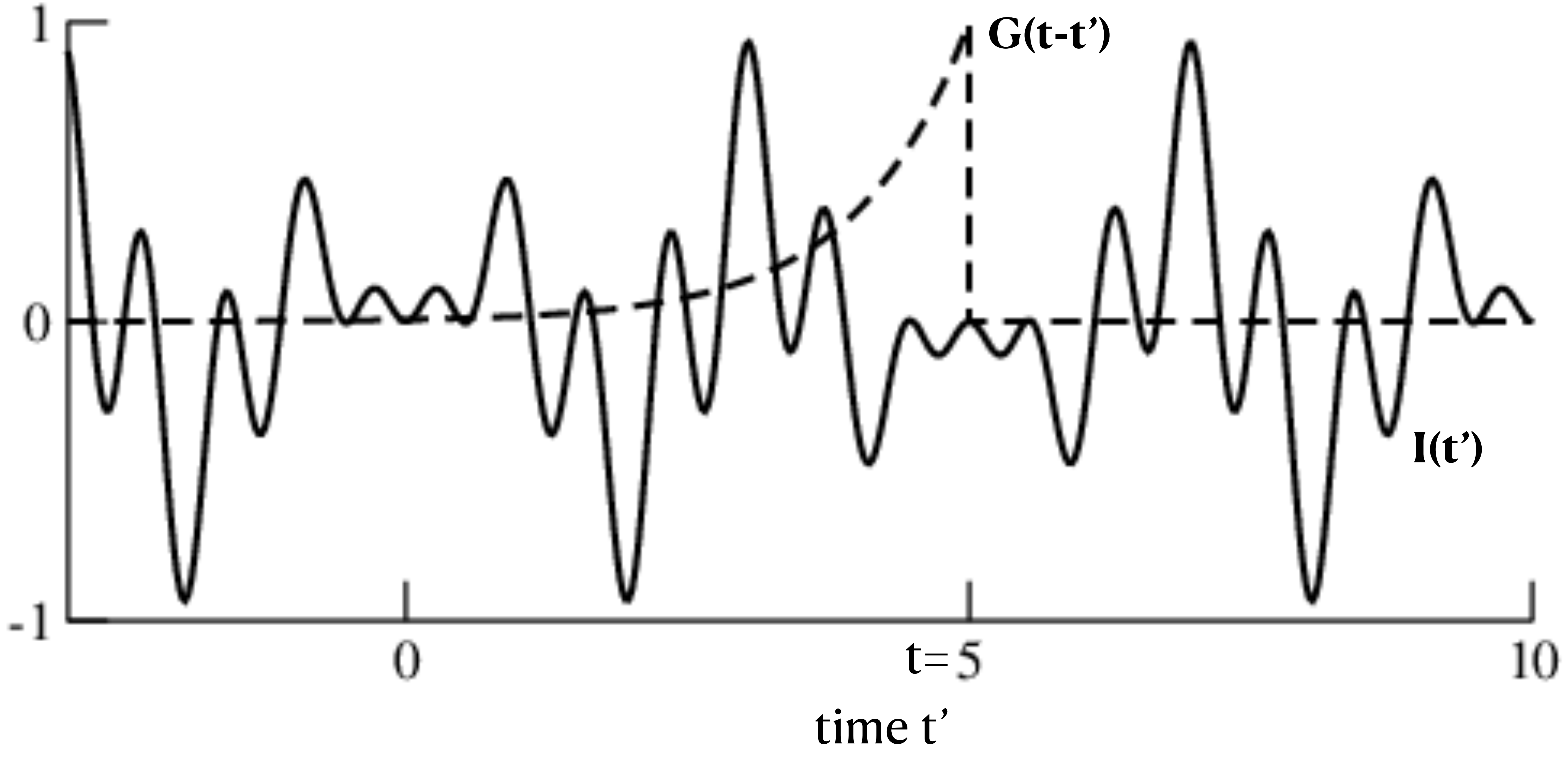
$$|\tilde{H}(w)|^2 = \frac{1}{\tau_0^2 w^2 + 1}$$

$$H(t) = e^{-t/\tau_0}$$

$$s(t) = \int_{-\infty}^{t} e^{-(t-t')/\tau_0} I(t')dt'$$



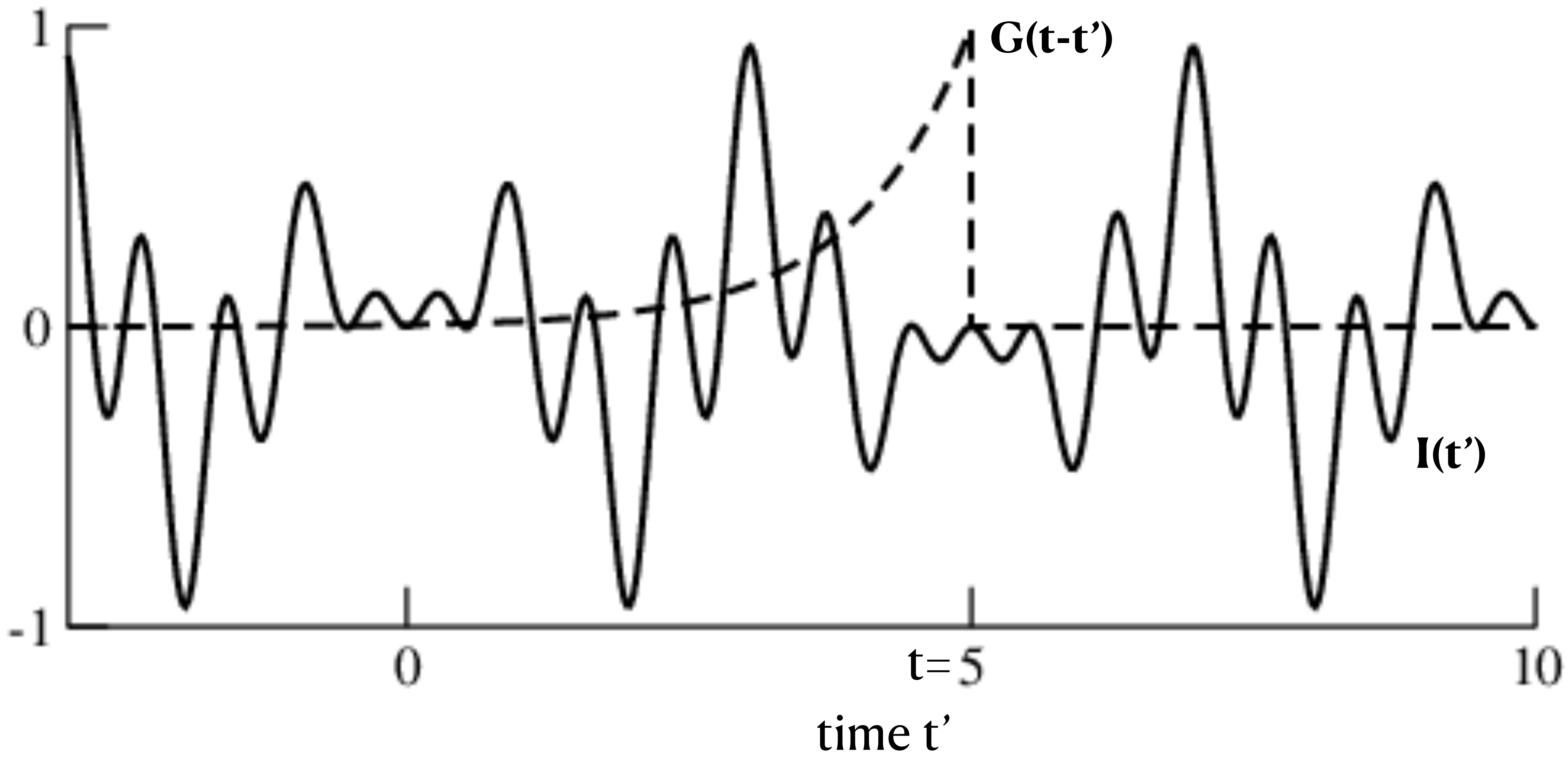
$$s(t) = \int_{-\infty}^{\infty} G(t-t')I(t')dt'$$



$$s(t) = \int_{-\infty}^{\infty} G(t-t')I(t')dt'$$

G(t): filter window = sliding window

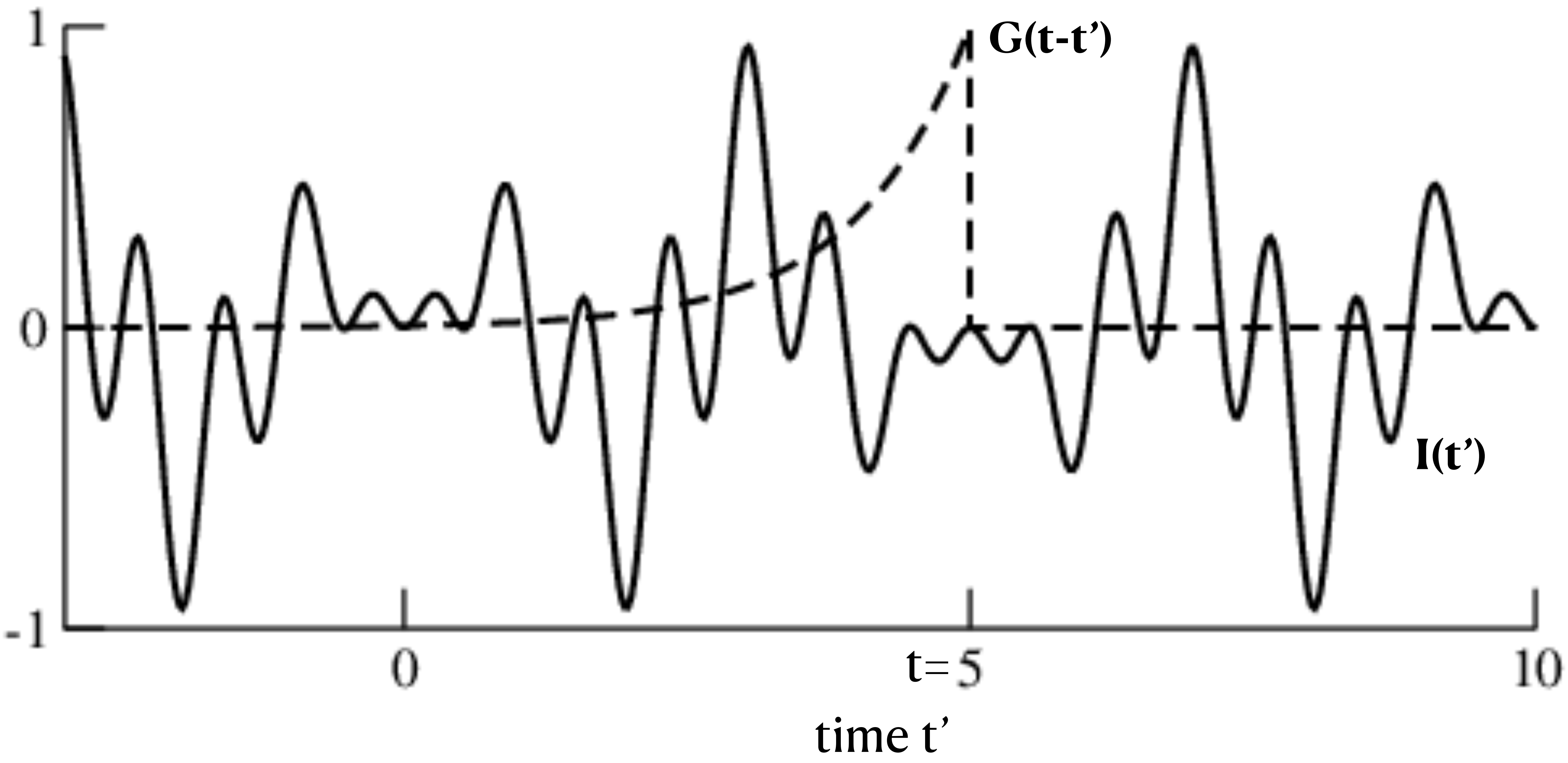
s(t): correlation function of G and I



$$s(t) = \int_{-\infty}^{\infty} G(t - t')I(t')dt'$$

G(t): filter window = sliding window

s(t): correlation function of G and I



linear filters

can be seen as a time-dependent correlation function

of a signal with a sliding window

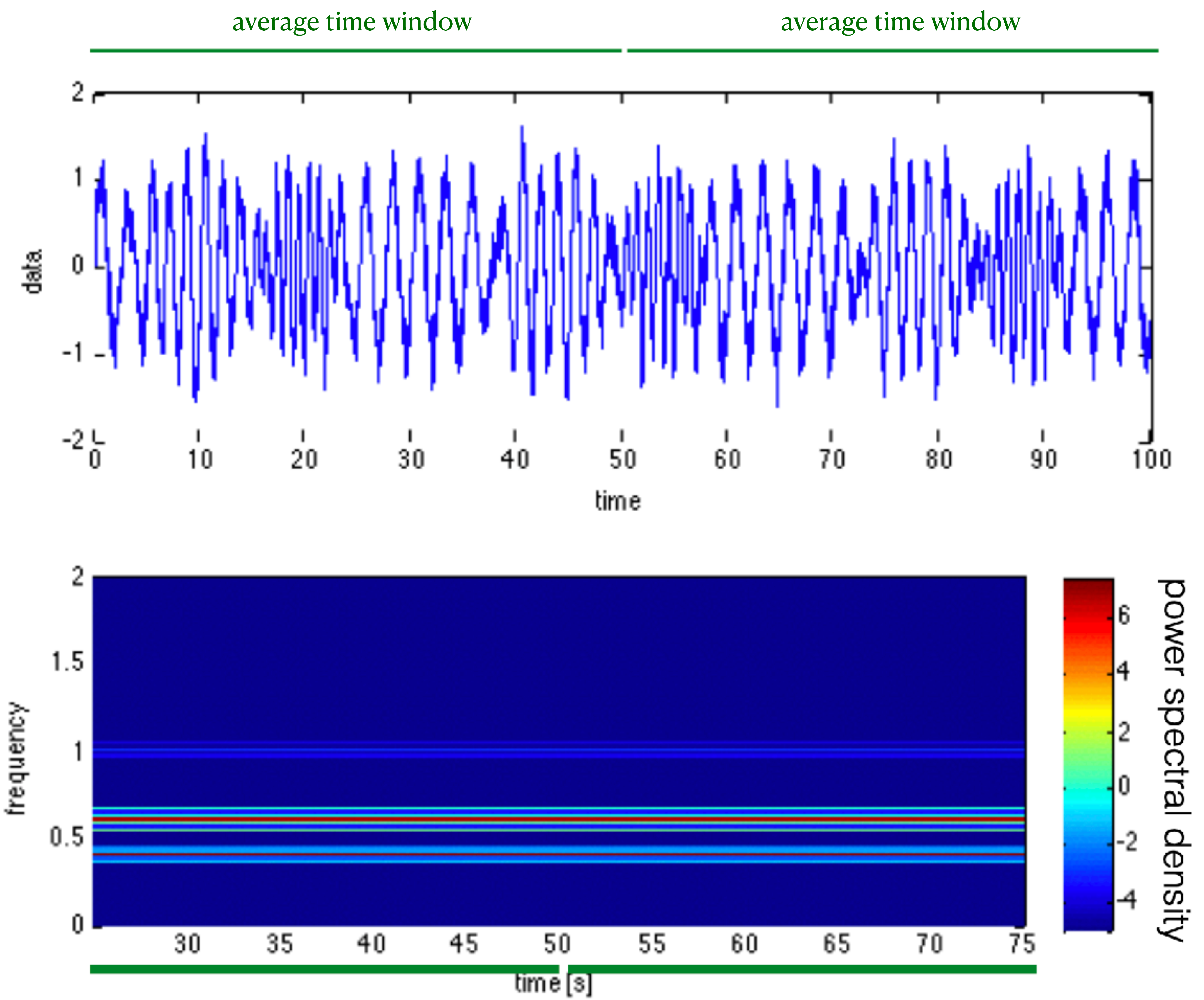
data sampling

Fourier analysis

errors in analysis

linear filters

time-frequency analysis



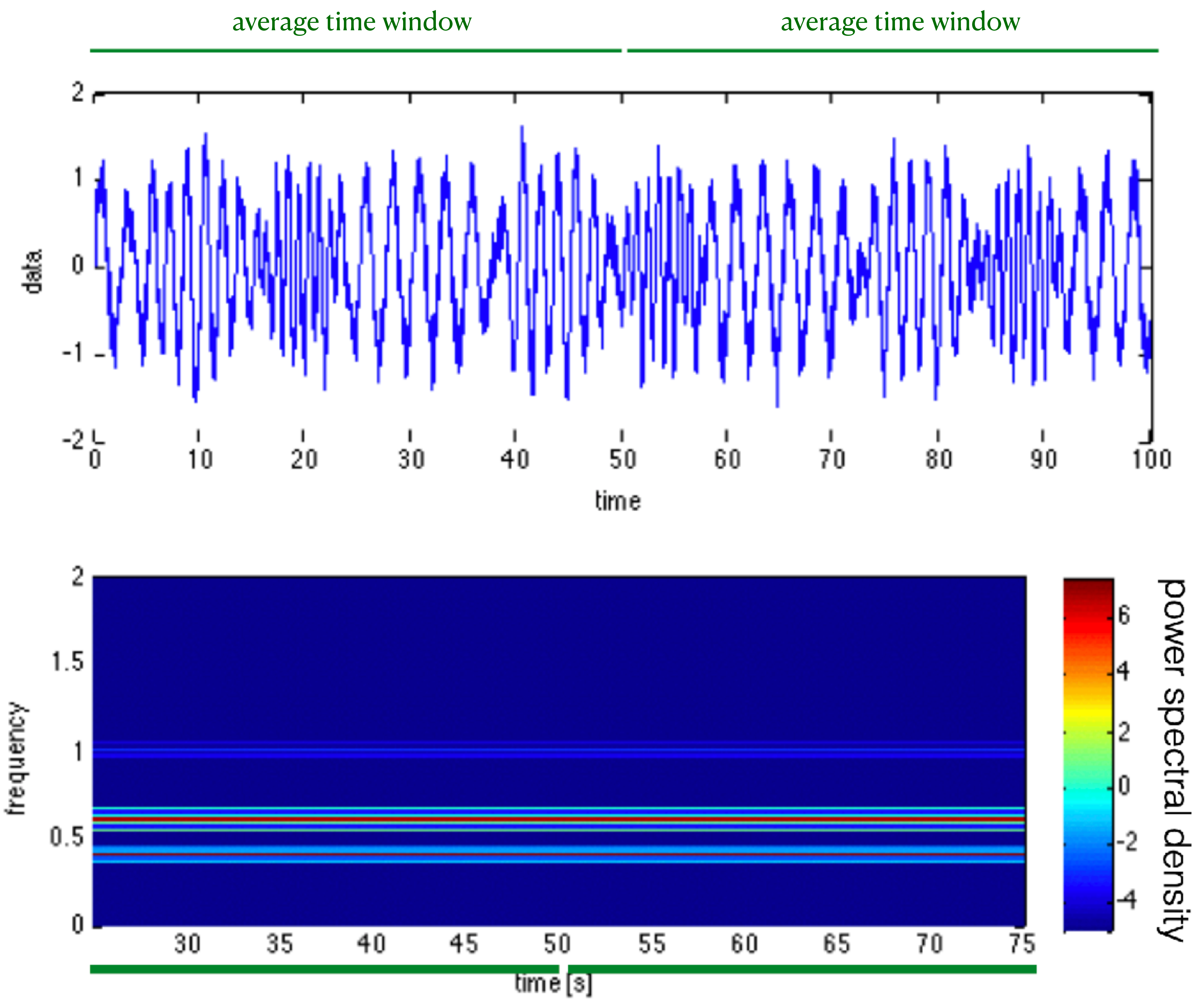
time window:

ΔT=50s

frequency resolution:

$\Delta f = 0.02$Hz

very good frequency resolution, very bad time resolution



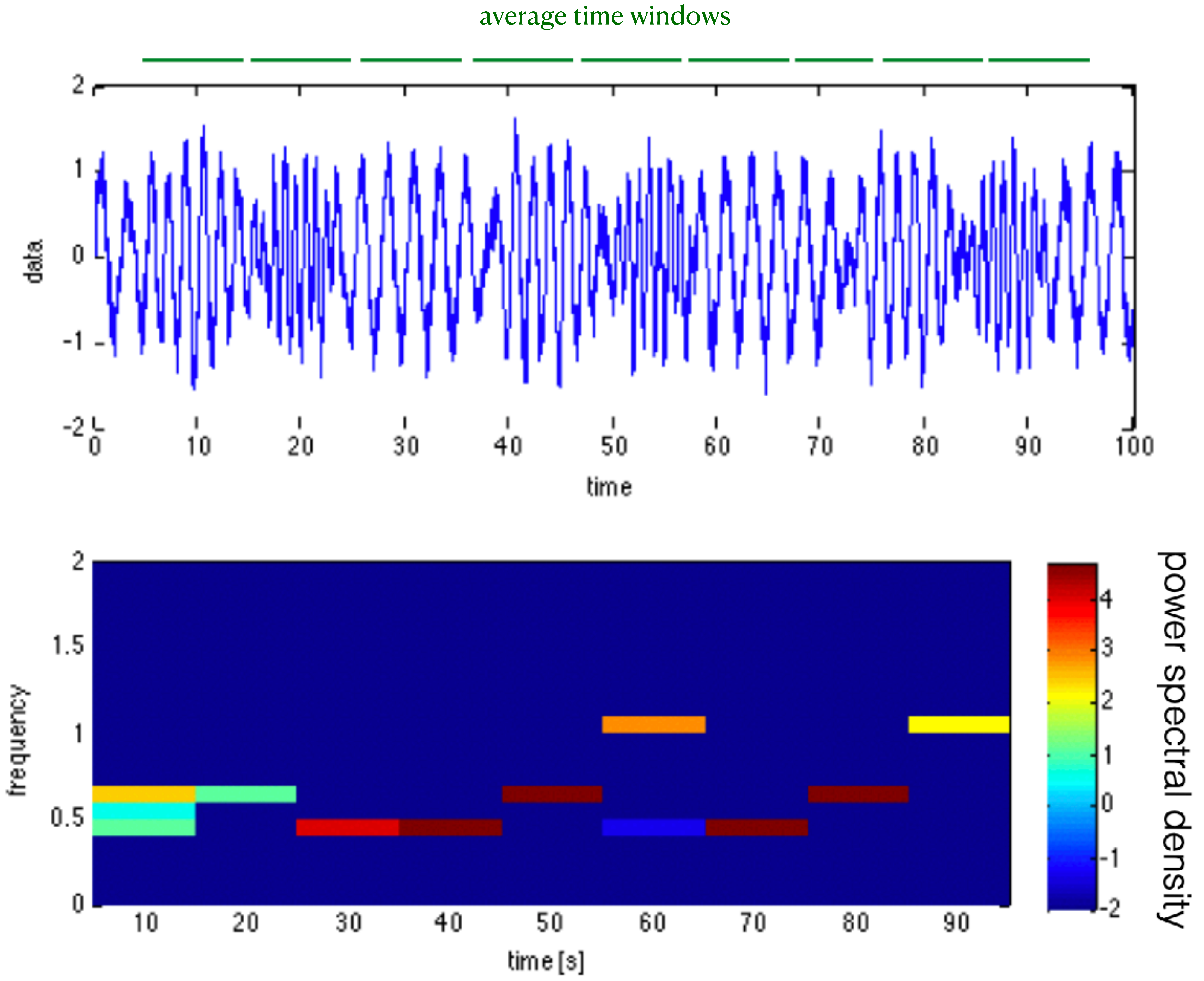
time window:

ΔT=50s

frequency resolution:

$\Delta f = 0.02$Hz

time-independent frequency on large time scale



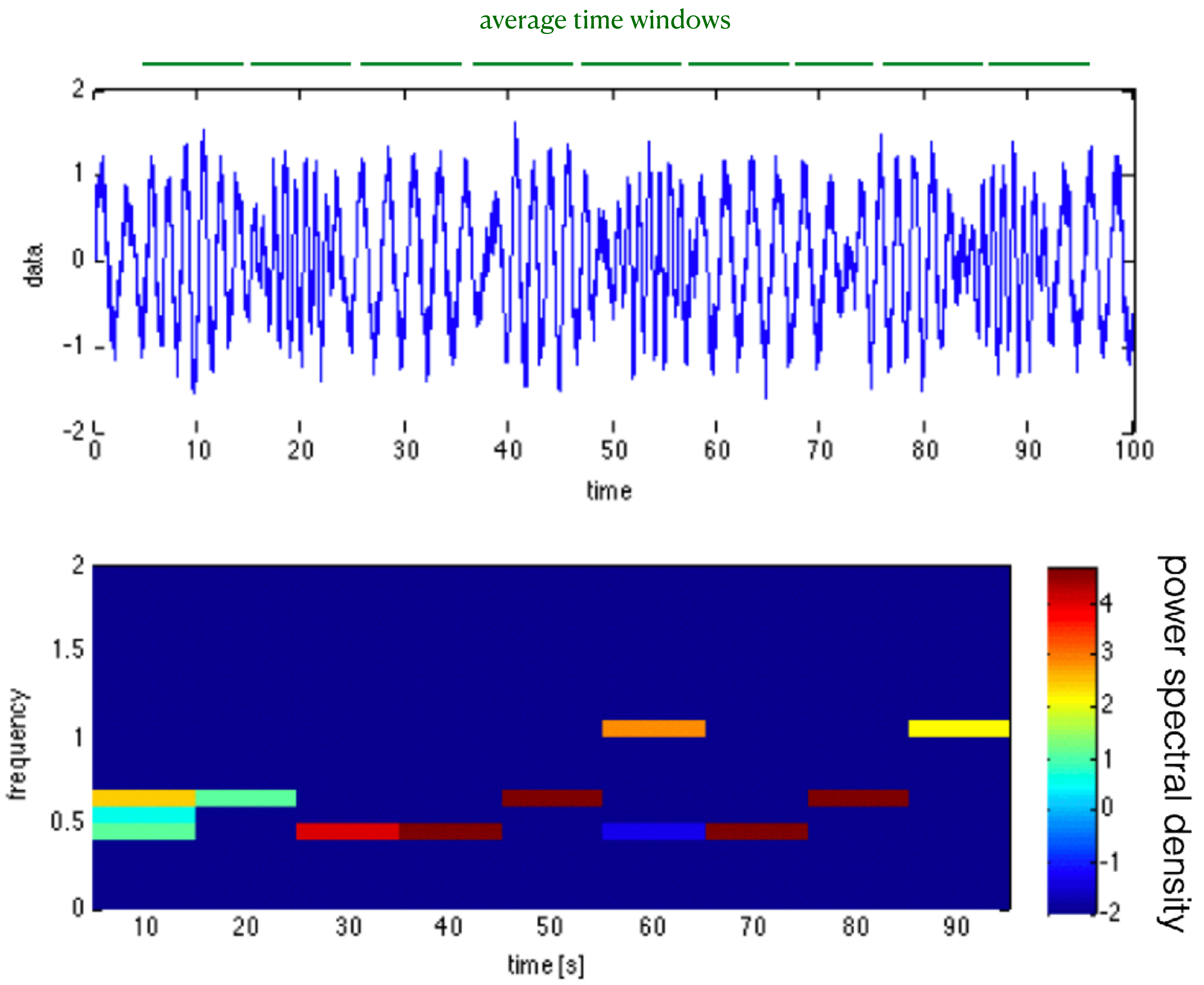
time window:

ΔT=10s

frequency resolution:

$\Delta f = 0.1\text{Hz}$

good frequency resolution — bad time resolution



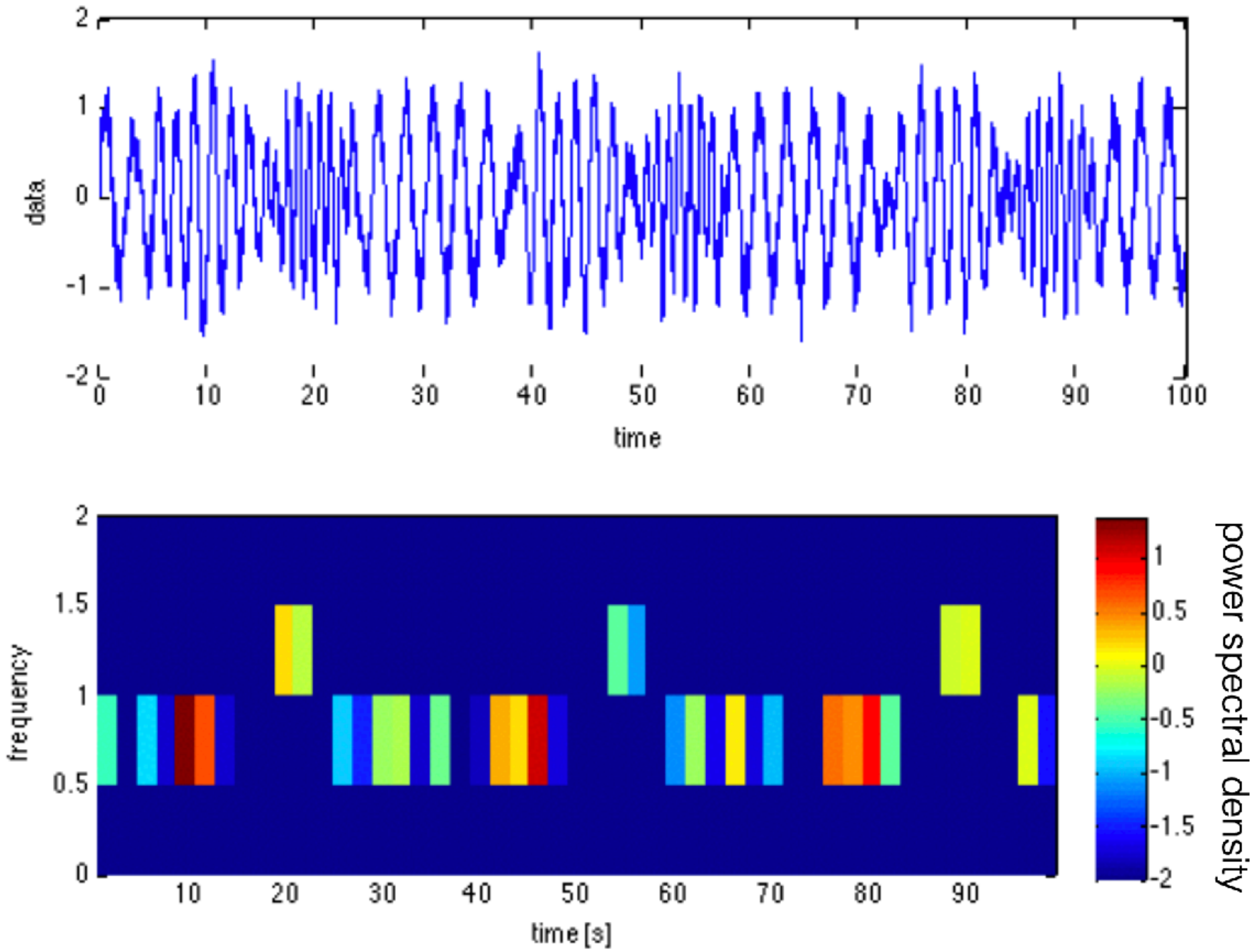
time window:

ΔT=10s

frequency resolution:

$\Delta f = 0.1\text{Hz}$

time-dependent frequency on shorter time scale



time window:

Δ**T**=**2s**

frequency resolution:

$\Delta f = 0.5$Hz

bad frequency resolution — good time resolution

it is necessary to balance time and frequency resolution since

$$\Delta T \sim \frac{1}{\Delta f}$$

it is necessary to balance time and frequency resolution since

$$\Delta T \sim \frac{1}{\Delta f}$$

$$\Rightarrow \quad \boxed{\Delta T \Delta f = \mathrm{const}}$$

Heisenberg uncertainty relation

it is necessary to balance time and frequency resolution since

$$\Delta T \sim \frac{1}{\Delta f}$$

$$\rightarrow \quad \boxed{\Delta T \Delta f = \mathrm{const}}$$

Heisenberg uncertainty relation

Question: **is there an instantaneous frequency ?**

data sampling

Fourier analysis

errors in analysis

linear filters

time-frequency analysis

uni-resolution analysis

multi-resolution analysis

non-Fourier analysis

Short-time Fourier Transform (STFT)

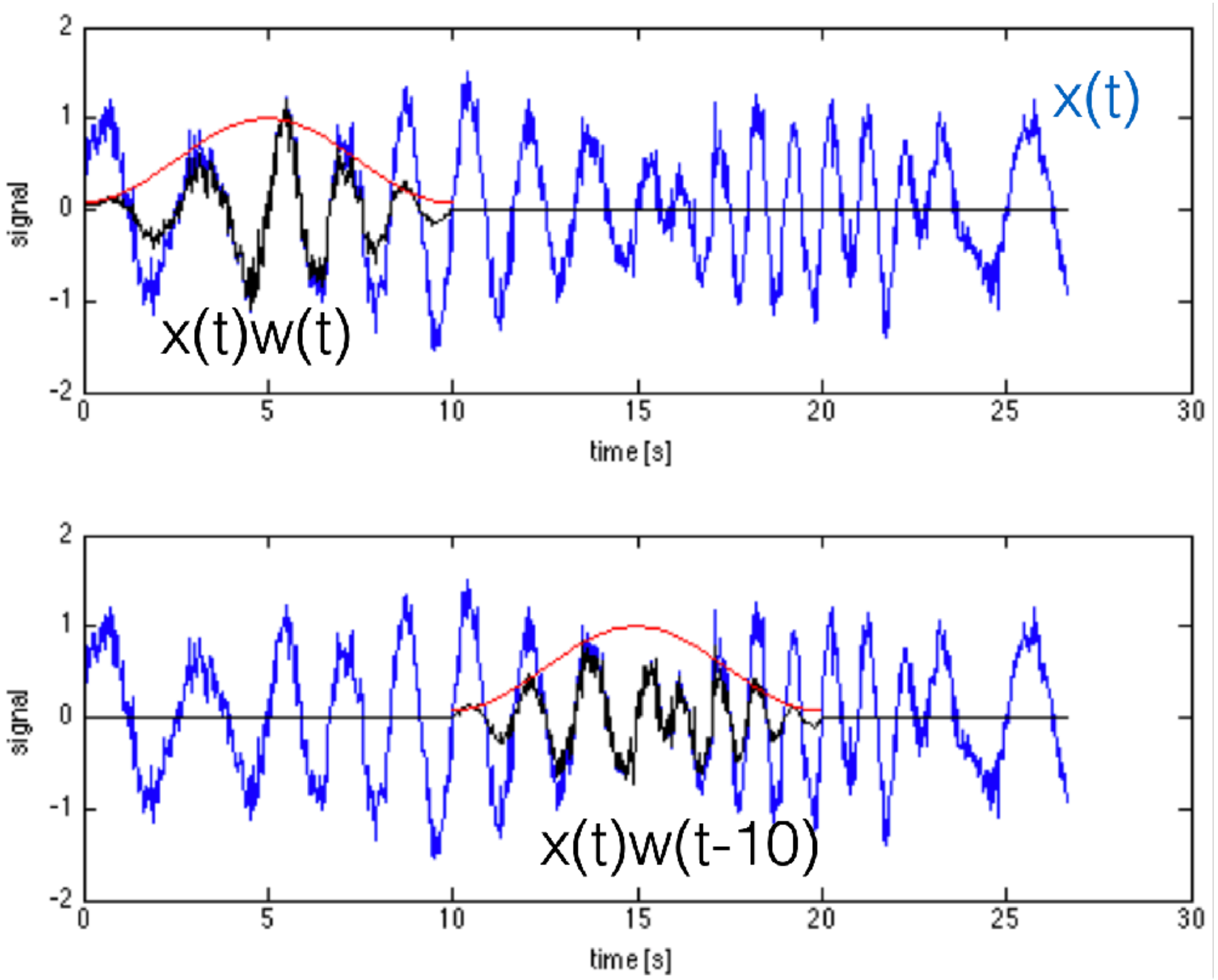
$$s(t,f) = \int_{-\infty}^{\infty} w(t-t')x(t')e^{-i2\pi f t'}dt'$$

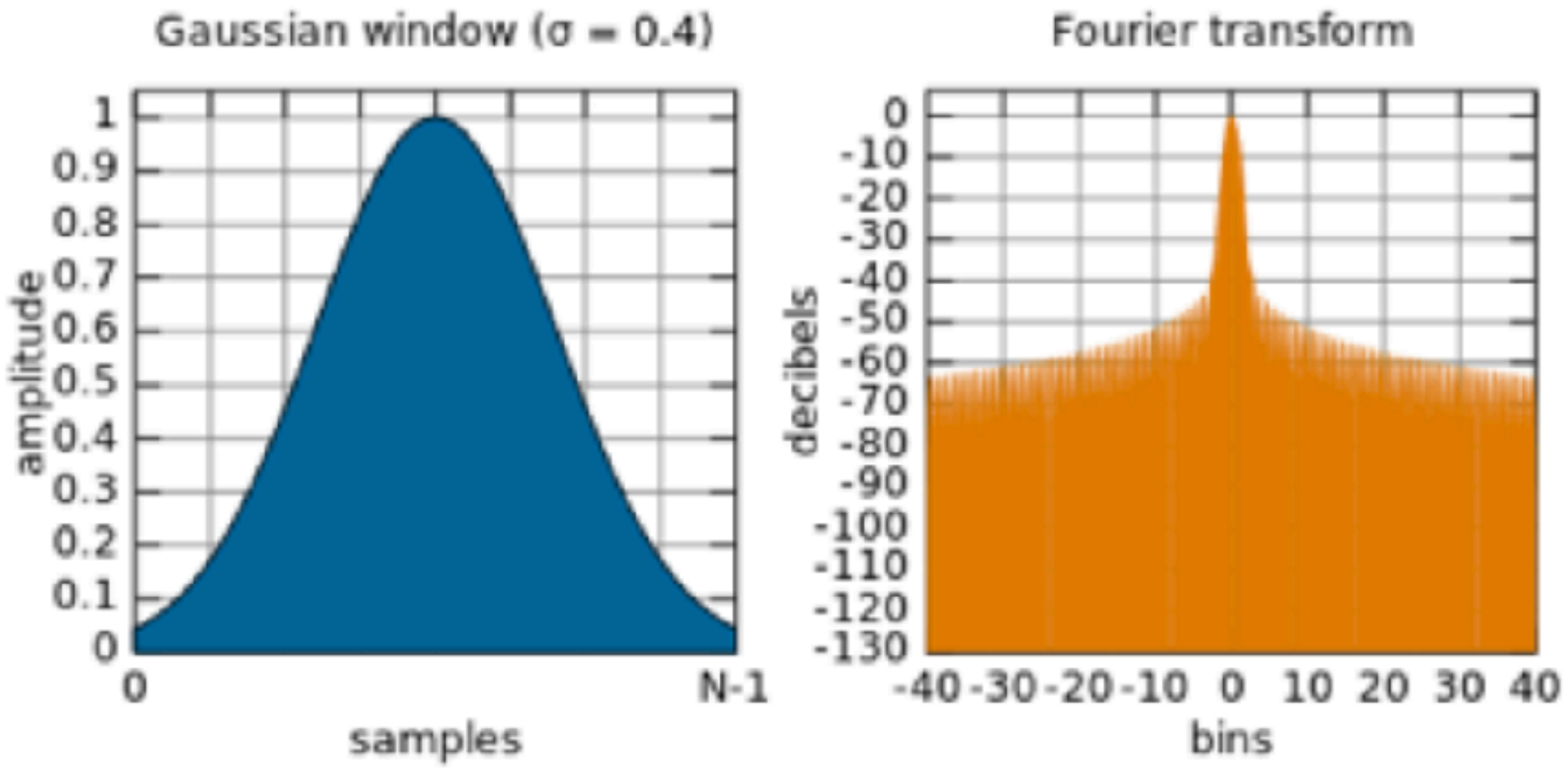
filter window   input to filter

finite size !!

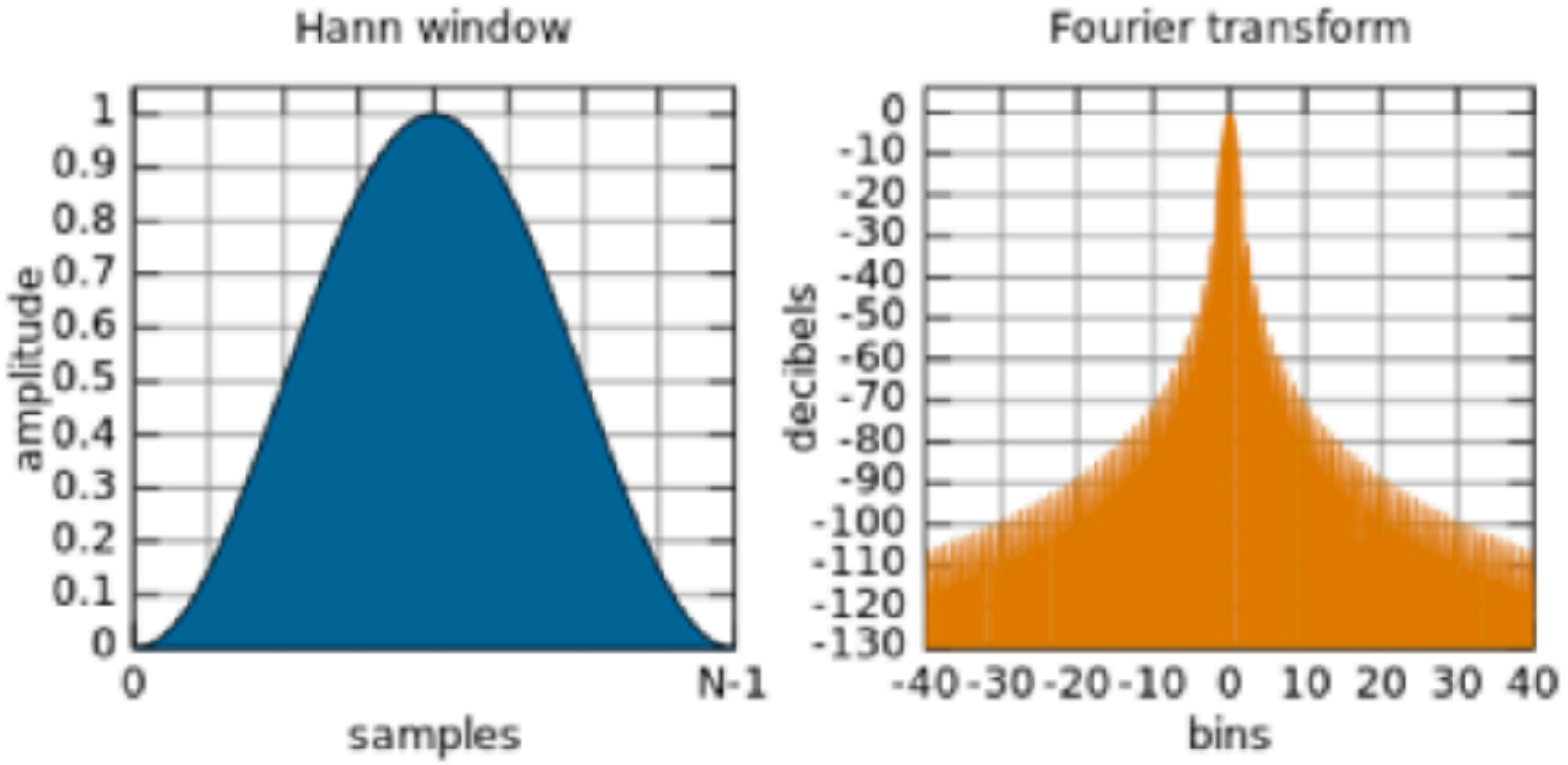
Short-time Fourier Transform (STFT)

$$s(t, f) = \int_{-\infty}^{\infty} w(t - t')x(t')e^{-i2\pi f t'}dt'$$

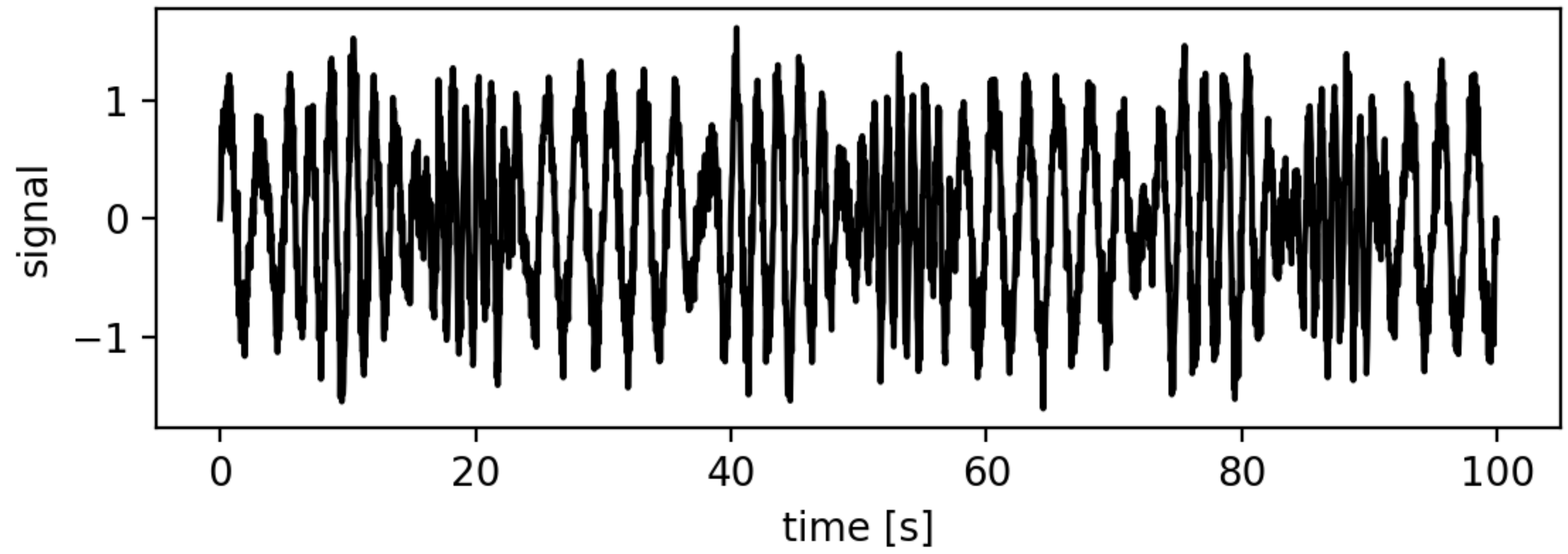




typical **finite size** windows w(t)



choose Hanning window in STFT



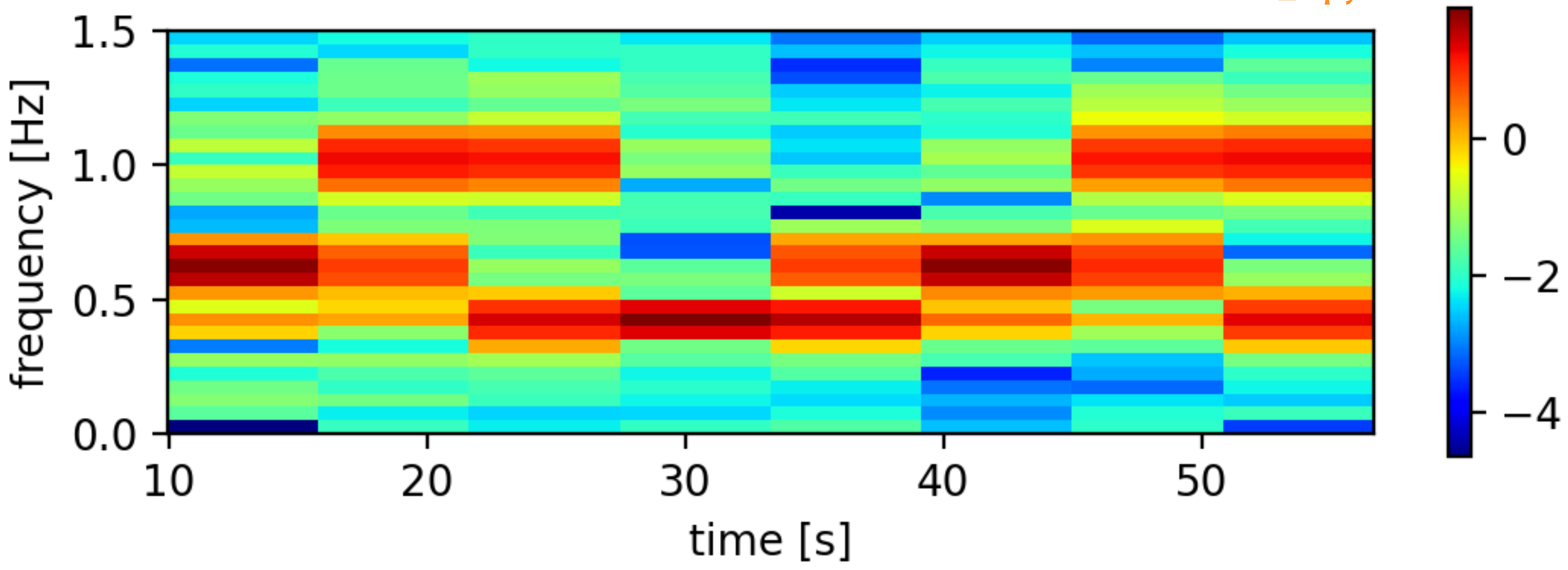
time window:

ΔT=20s

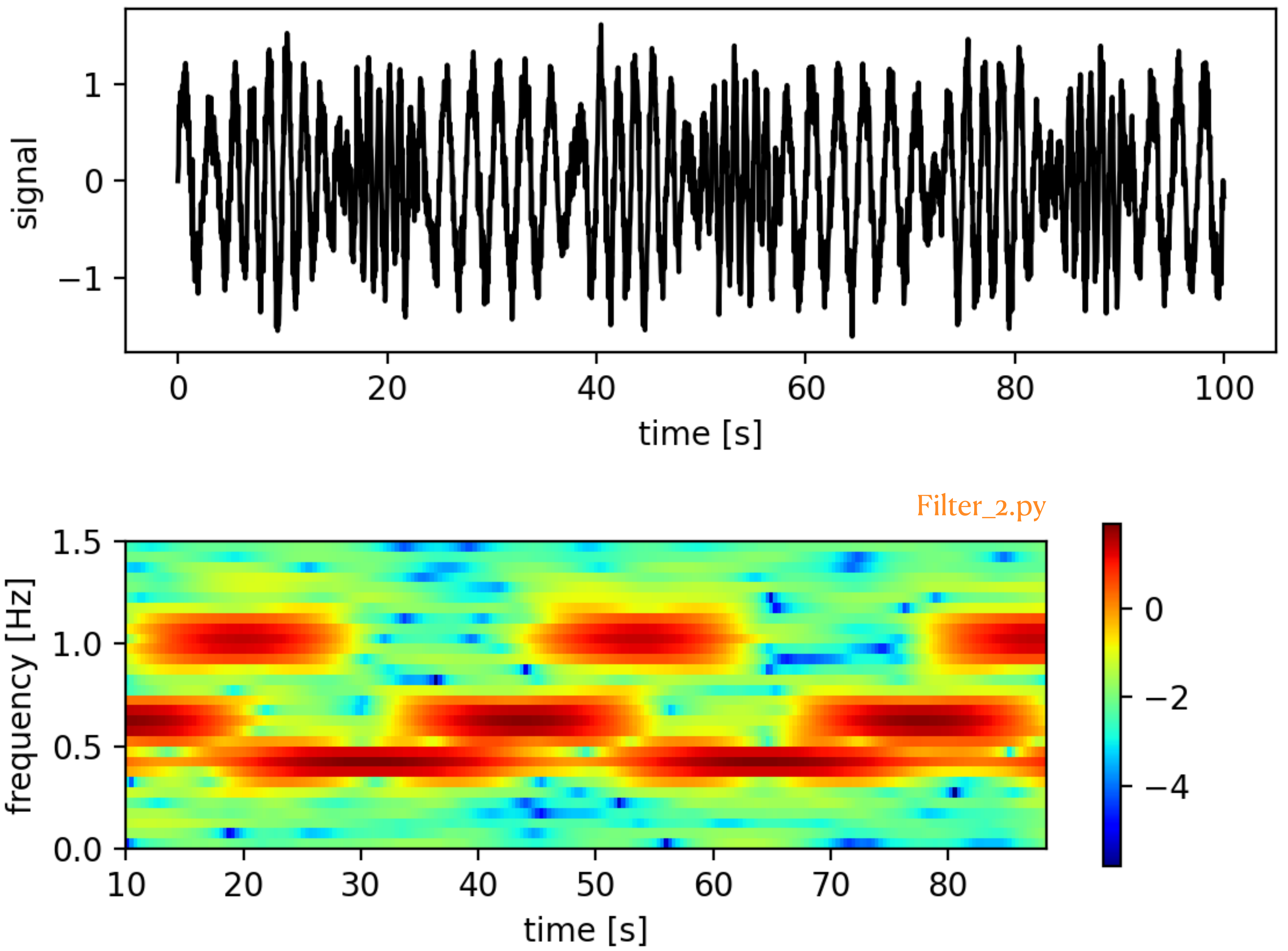
frequency resolution:

$\Delta f = 0.05\text{Hz}$

Filter_2.py



overlap of 13.4s



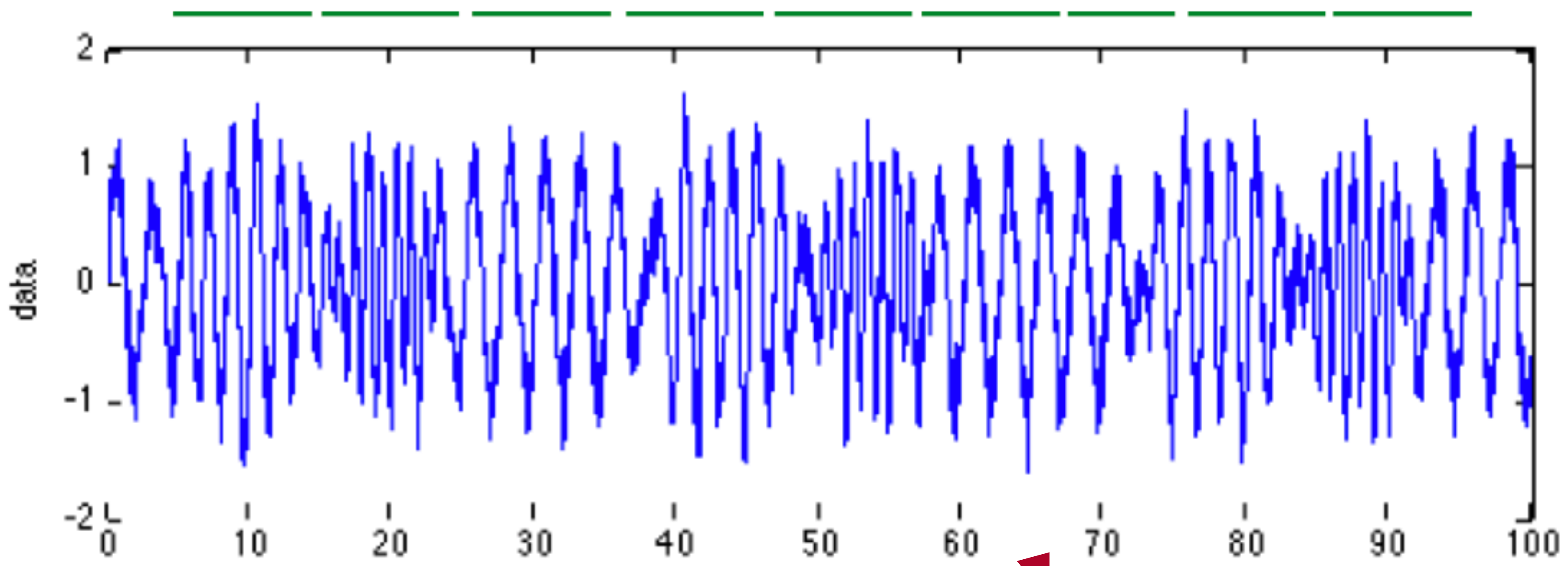
time window:

ΔT=20s

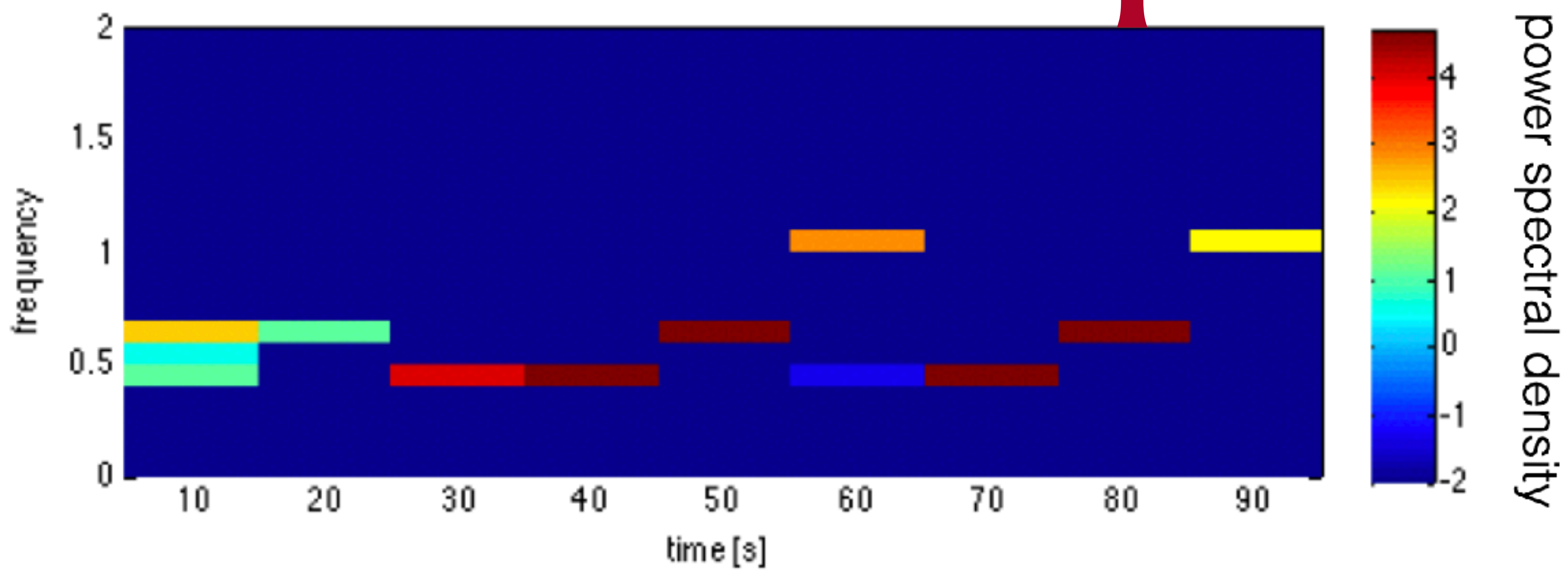
frequency resolution:

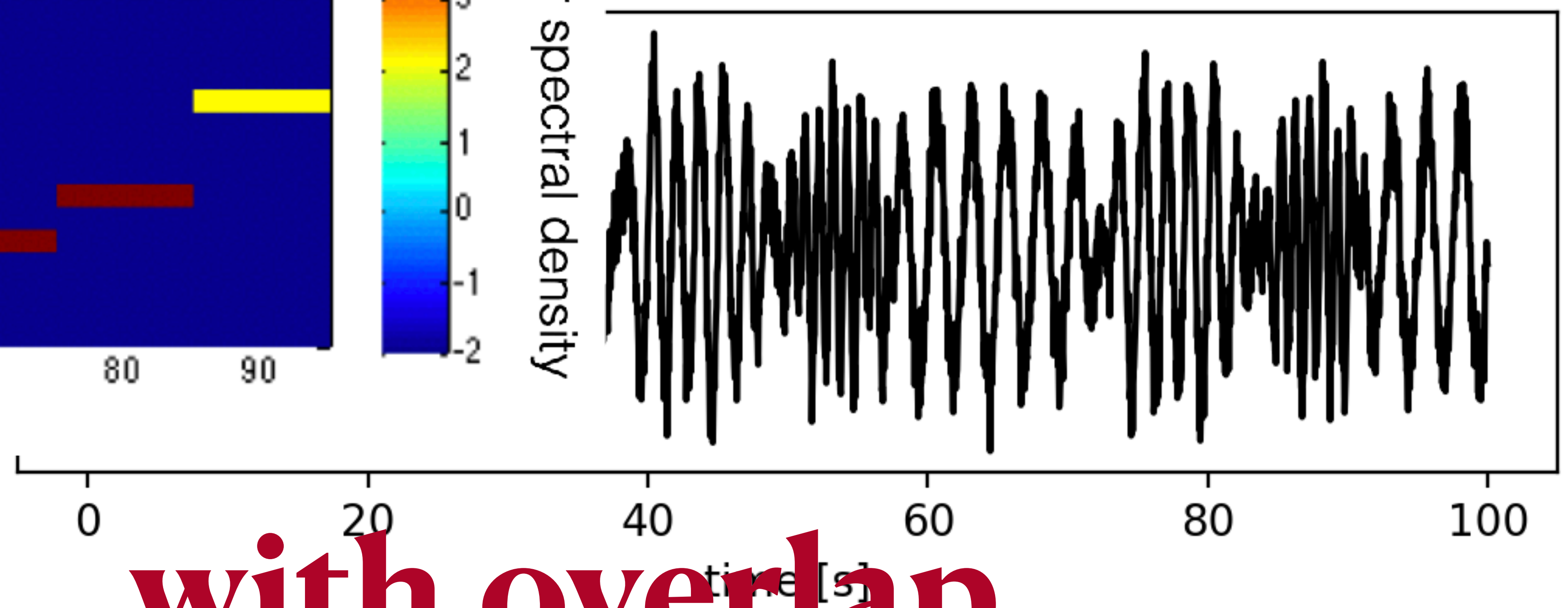
$\Delta f = 0.05\text{Hz}$

overlap of 19.7s

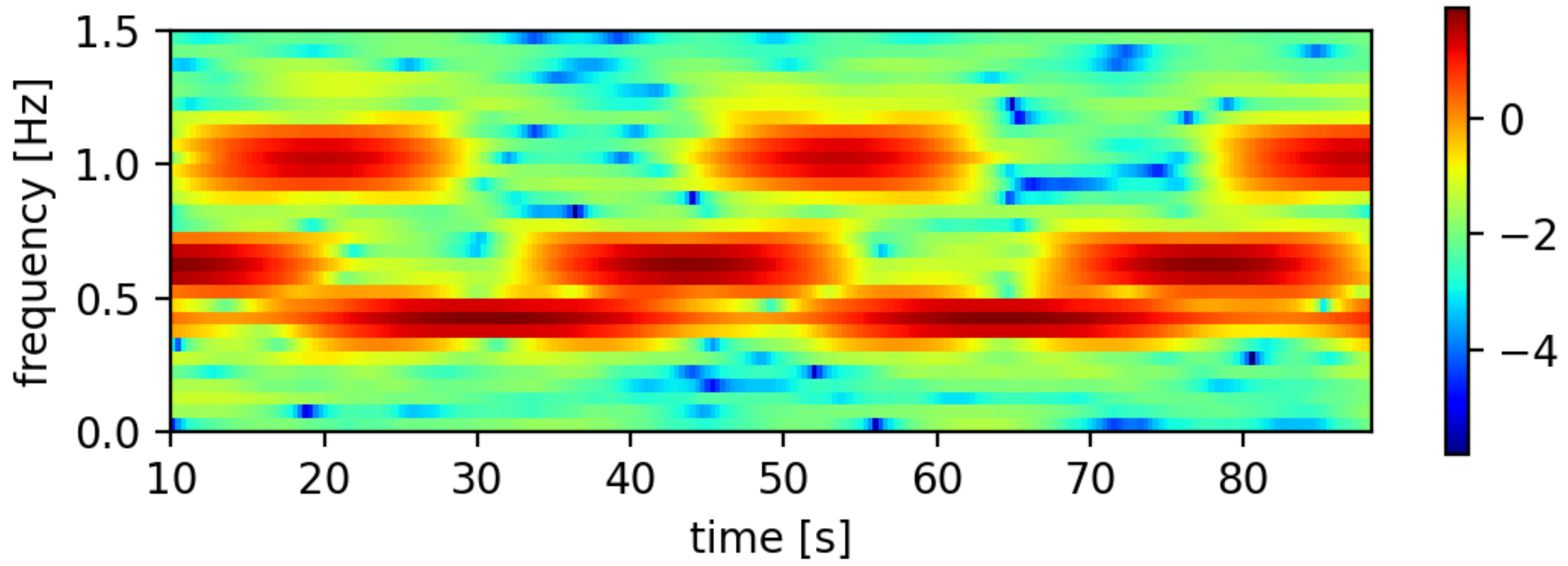


no overlap





with overlap



time window:

ΔT=20s

frequency resolution:

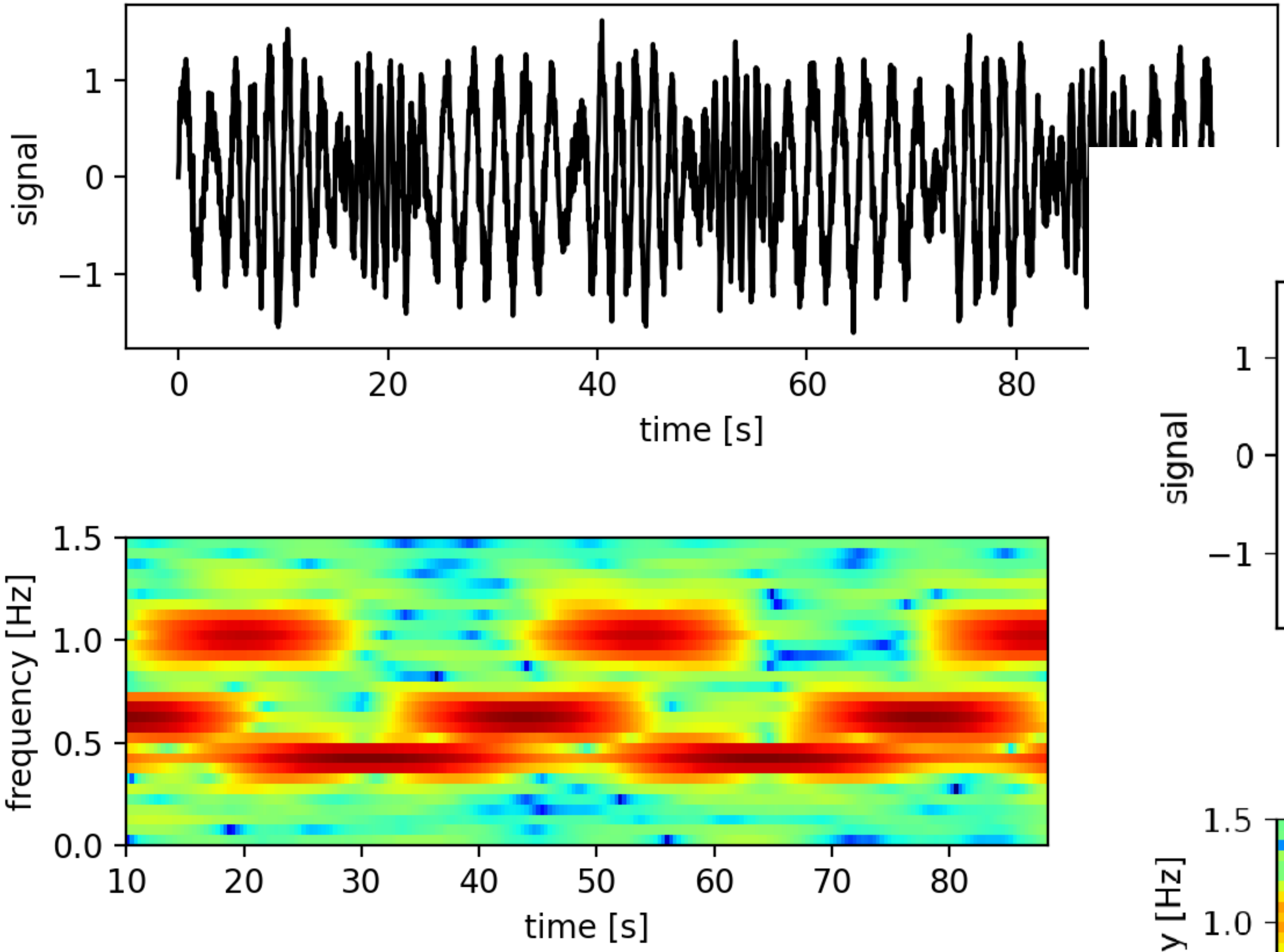
$\Delta f = 0.05\text{Hz}$

overlapping sliding windows

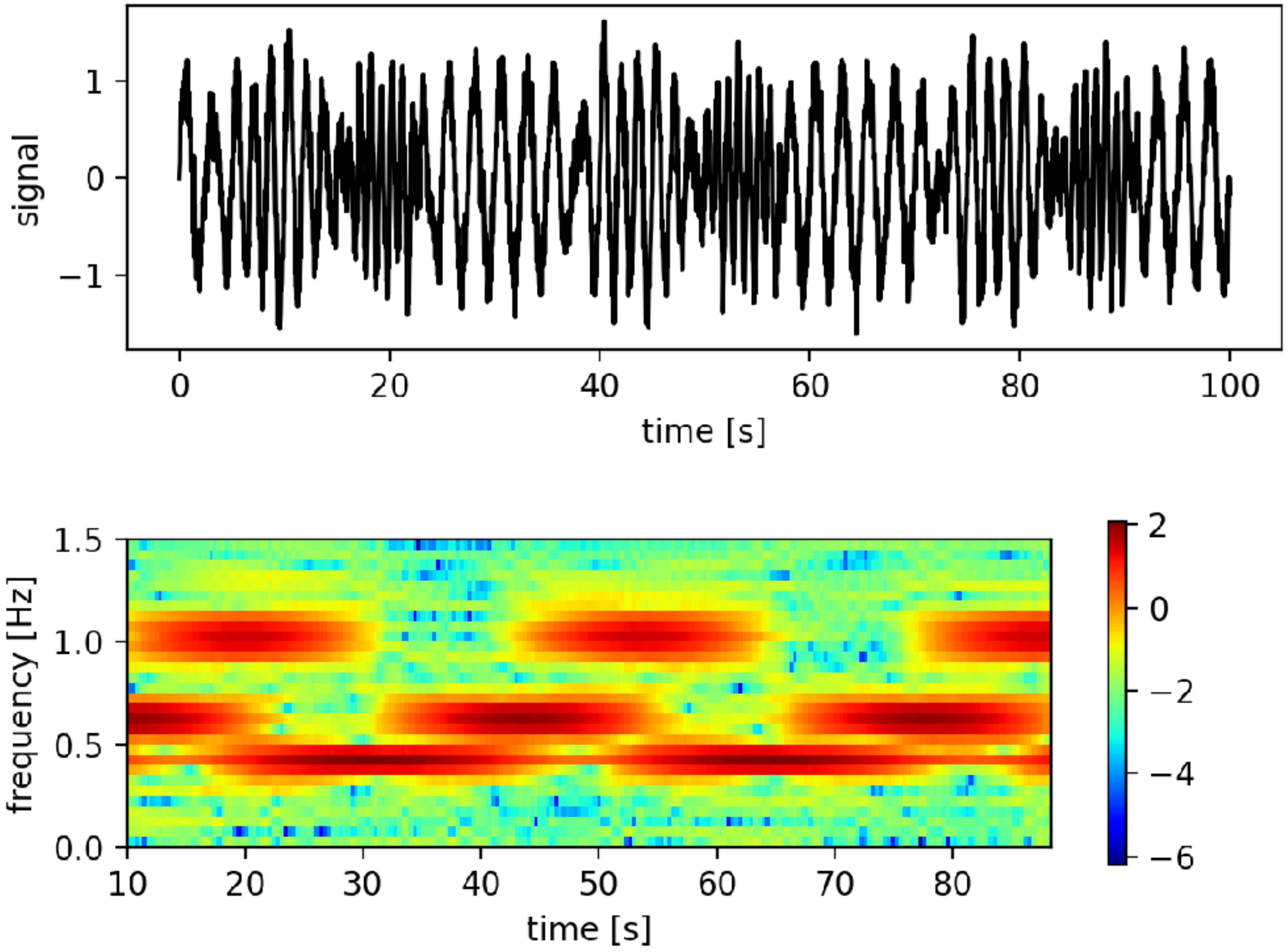
improve

time and frequency resolution

Hanning window



Gaussian window



Hanning and Gaussian results are similar

comment: if

$$(\mu_2)_{|f|^2} = \sigma^2_{|f|^2} = \int_{\mathbb{R}} (x - x_m)^2 \left|f(x)\right|^2 dx$$

temporal variance

$$(\mu_2)_{\left|\hat{f}\right|^2} = \sigma^2_{\left|\hat{f}\right|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^2 \left|\hat{f}(\xi)\right|^2 d\xi$$

frequency variance

and

$$\|f\|_2^2 = \int_{\mathbb{R}} \left|f(x)\right|^2 dx$$

comment: if

$$(\mu_2)_{|f|^2} = \sigma^2_{|f|^2} = \int_{\mathbb{R}} (x - x_m)^2 \, |f(x)|^2 \, dx$$

temporal variance

$$(\mu_2)_{|\hat{f}|^2} = \sigma^2_{|\hat{f}|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^2 \left|\hat{f}(\xi)\right|^2 d\xi$$

frequency variance

and

$$\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx$$

then

$$\boxed{\sigma^2_{|f|^2} \cdot \sigma^2_{|\hat{f}|^2} \geq \frac{\|f\|_2^4}{16\pi^2}}$$

Heisenberg-Weyl inequality

minimum uncertainty

$$\sigma^2_{|f|^2}\sigma^2_{|\hat{f}|^2} = \frac{||f||_2^4}{16\pi^2}$$

if

$$f(t) = c_0 e^{i2\pi f t} e^{-c_1(x-x_m)^2}$$

minimum uncertainty

$$\sigma^2_{|f|^2}\sigma^2_{|\hat{f}|^2} = \frac{||f||_2^4}{16\pi^2}$$

if

$$f(t) = c_0 e^{i2\pi f t} e^{-c_1(x-x_m)^2}$$

➡ best time-frequency resolution if window is of Gaussian shape

➡ Gabor transformation

data sampling

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Short-time Fourier Tranform:

$$X(\tau, f) = \int_{-\infty}^{\infty} x(t)w(t-\tau)e^{-i2\pi ft}dt$$

window function

Short-time Fourier Tranform:

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Linear frequency filter:

$$X(\tau, f) = \int_{-\infty}^{\infty} x(t)w(t-\tau)e^{-i2\pi f(t-\tau)}dt$$

$$= \int_{-\infty}^{\infty} x(t)h(t-\tau)dt$$

impulse response function

Short-time Fourier Tranform:

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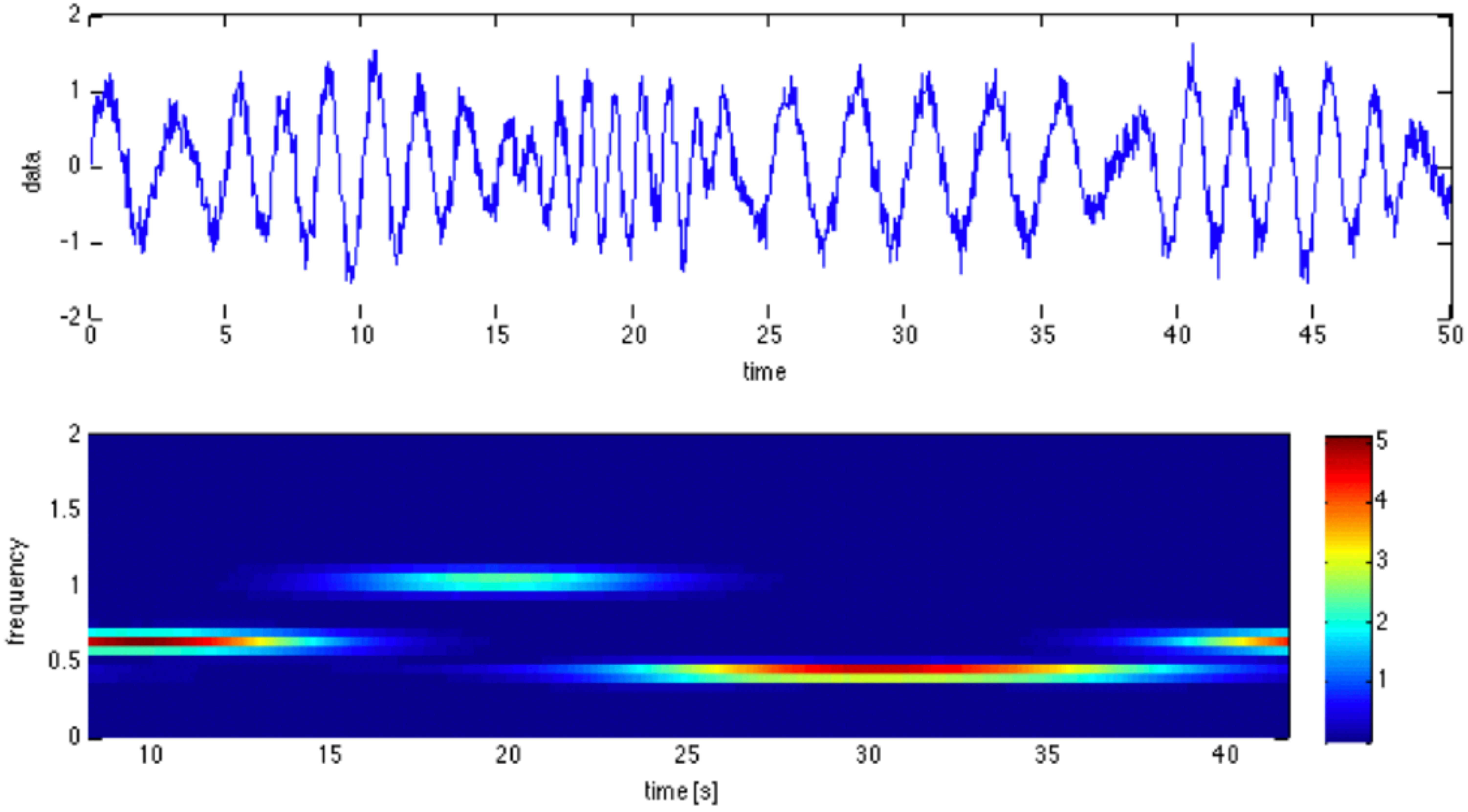
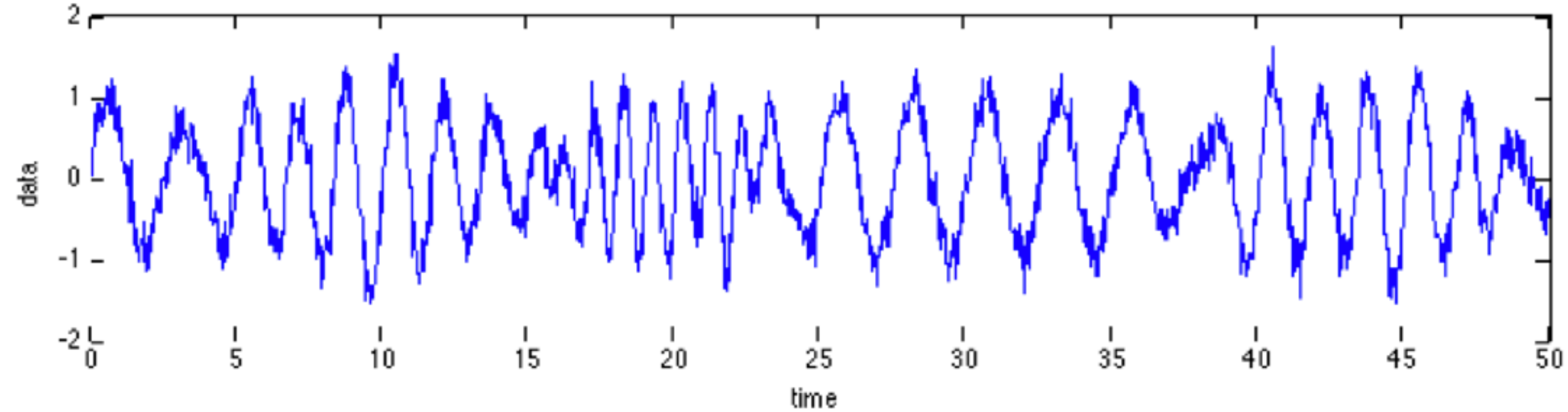
impulse response function

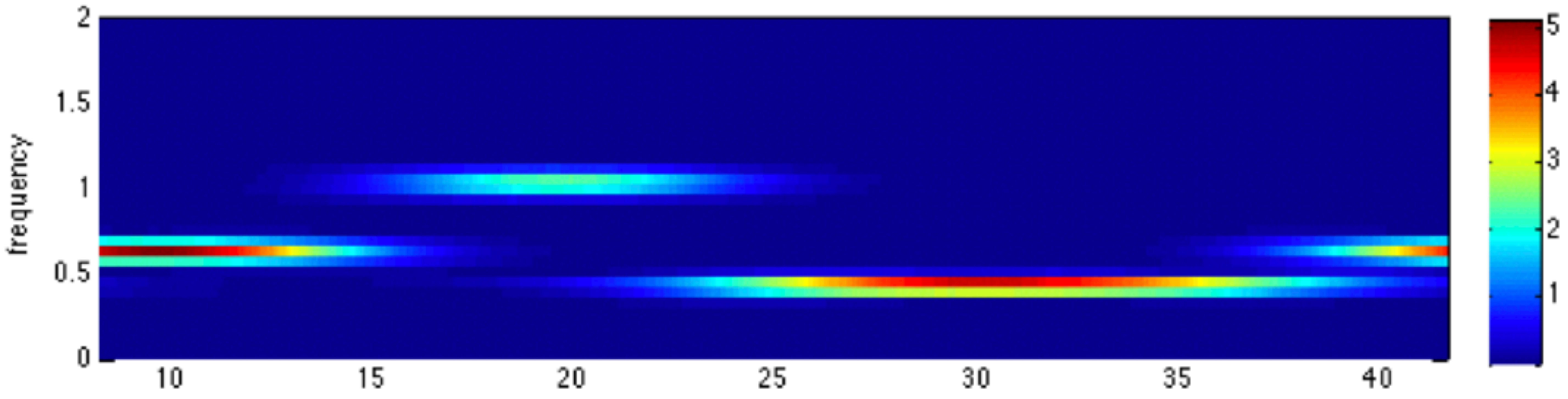
correlation between signal x and impulse response function h



STFT has constant time-frequency resolution,

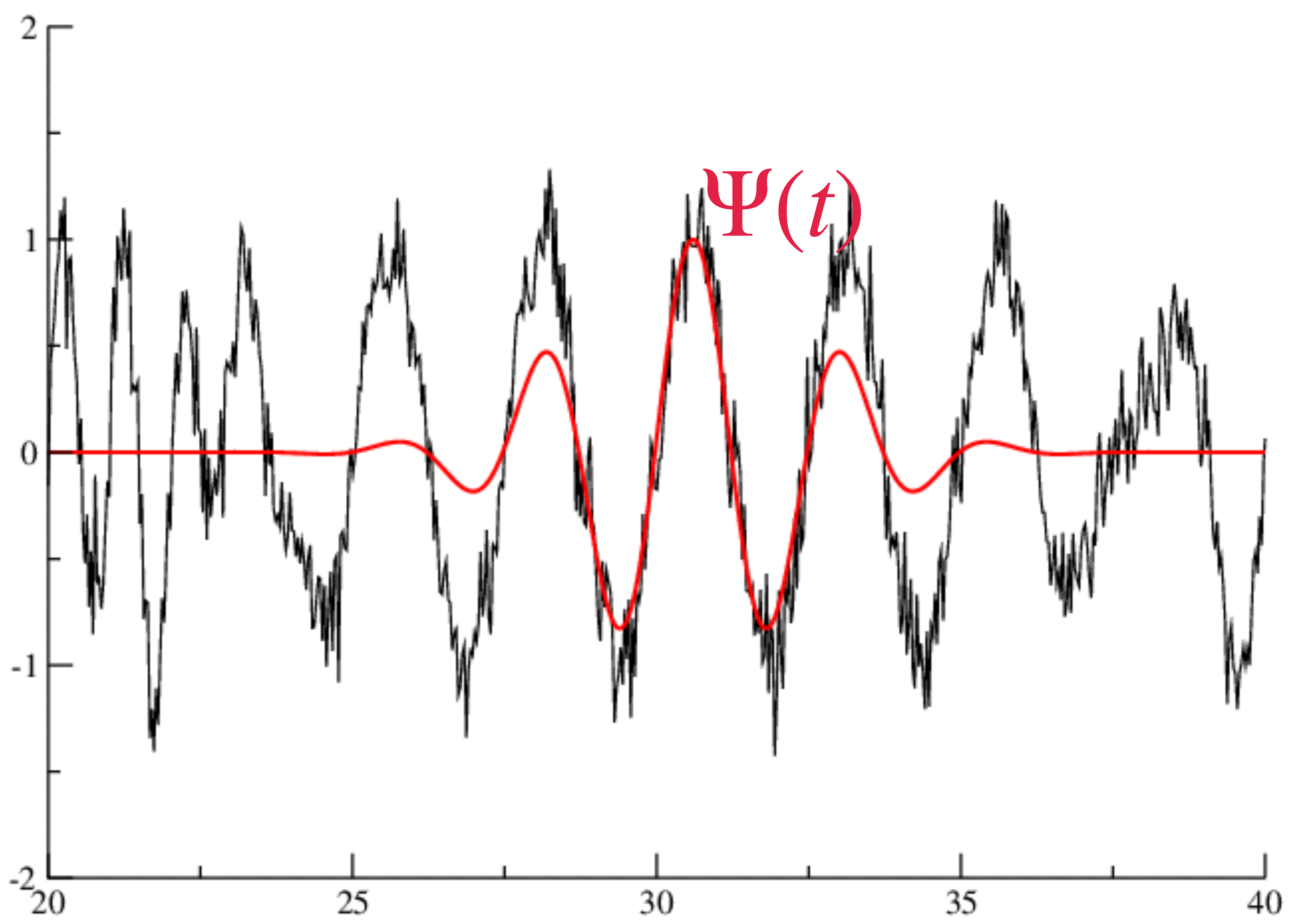
but transient signals need different time resolutions





$$X(\tau, a) \sim \int_{-\infty}^{\infty} x(t)\,\bar{\Psi}\left(\frac{t-\tau}{a}\right) dt$$

temporal scale factor a

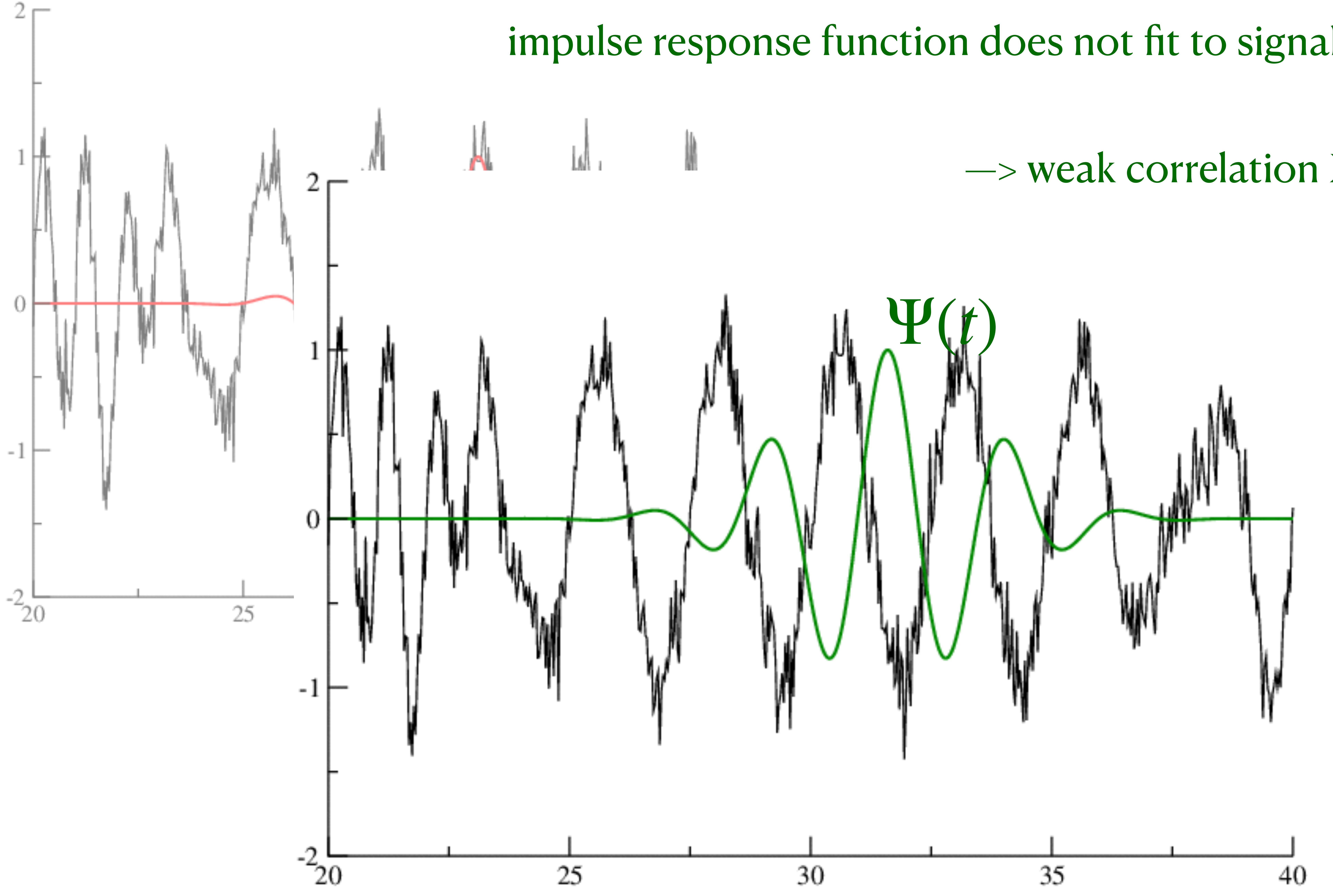


impulse response function fits to signal at that time instance

—> strong correlation X

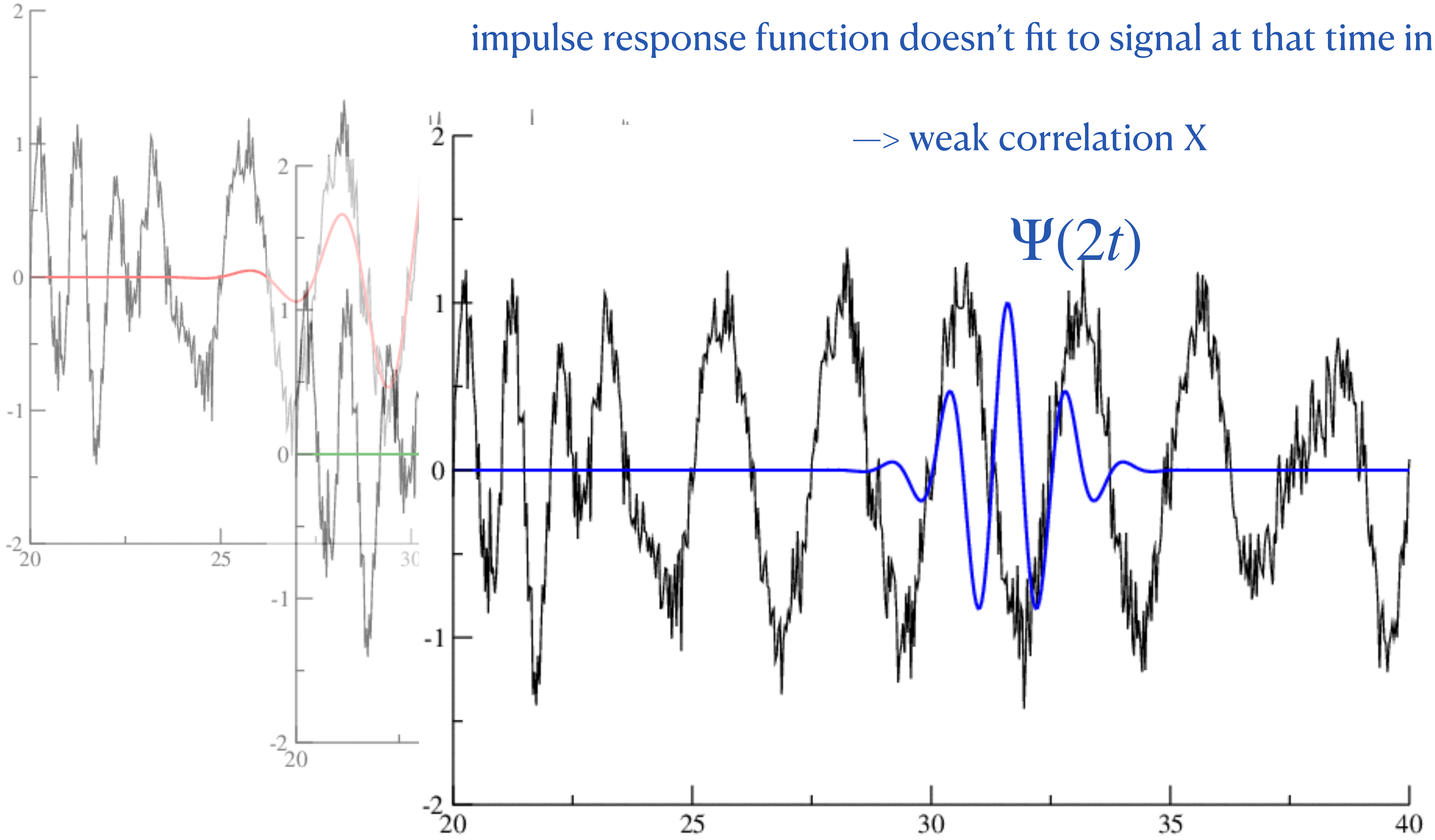
impulse response function does not fit to signal at that time instance

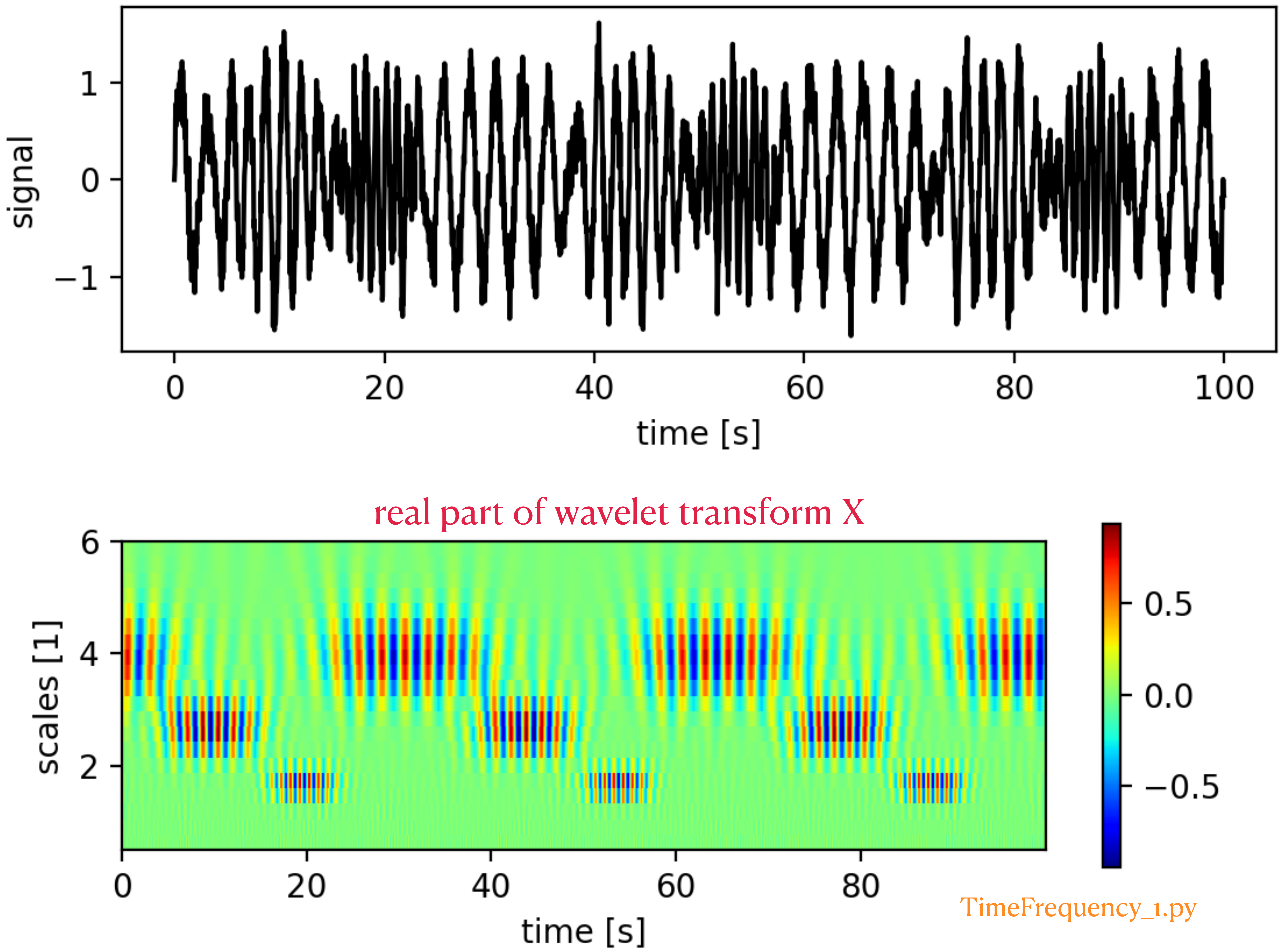
—> weak correlation X



impulse response function doesn't fit to signal at that time instance

—> weak correlation X

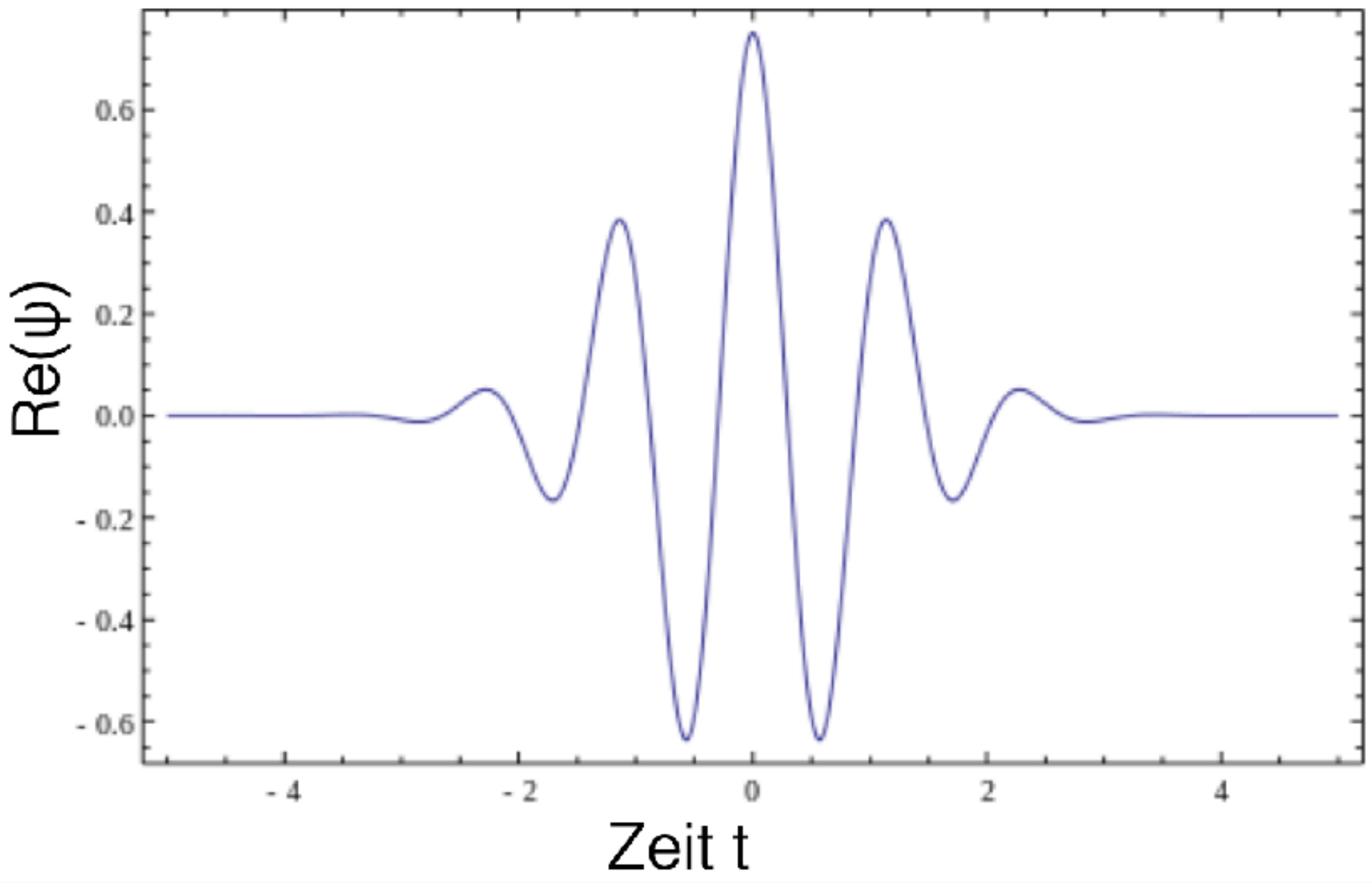




TimeFrequency_1.py

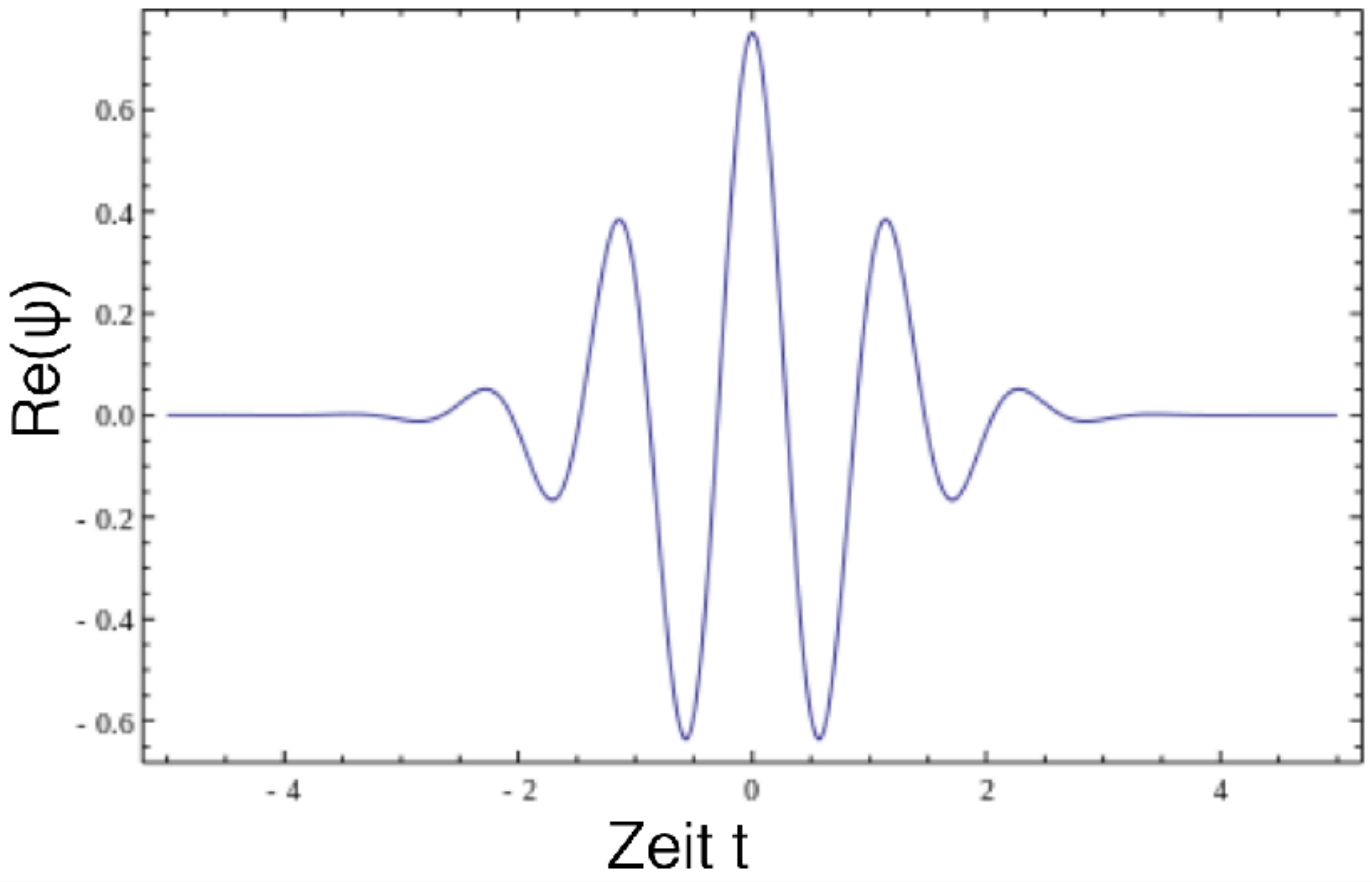
e.g.

$$\Psi_a(t) = k_1 e^{-t^2}(e^{-i\frac{t}{a}} - k_2)$$



e.g.

$$\Psi_a(t) = k_1 e^{-t^2}(e^{-i\frac{t}{a}} - k_2)$$



$$X(\tau, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t)\bar{\Psi}\left(\frac{t-\tau}{a}\right) dt$$

continuous wavelet transform

$\Psi(t)$: mother wavelet

properties:

administrative

$$\int_{-\infty}^{\infty} \Psi(t)\, dt = 0$$

mother wavelet has to be oscillatory

properties:

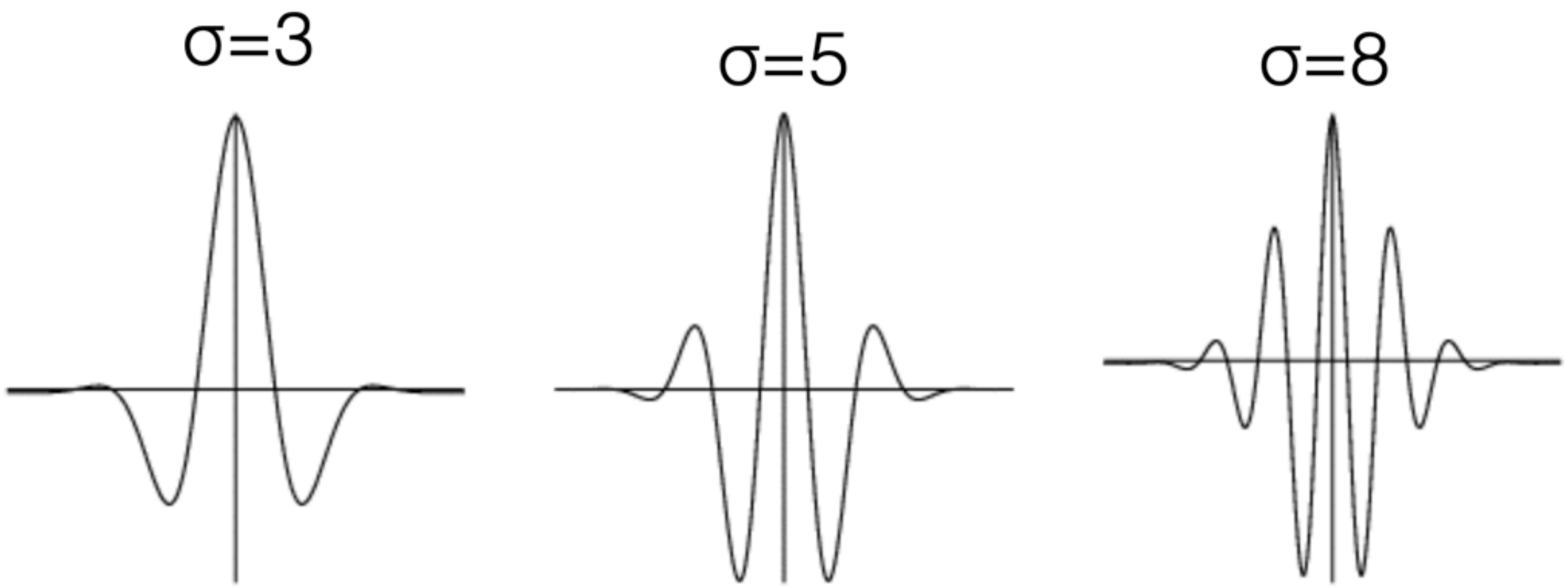
administsibility

$$\int_{-\infty}^{\infty} \Psi(t)\, dt = 0$$

mother wavelet has to be oscillatory

example: **complex Morlet wavelet**

$$\Psi(t) = k_1 e^{-t^2}(e^{-i\sigma t} - k_2)$$



in neuroscience: $5 \leq \sigma \leq 8$ recommended

$$X(\tau, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t) \bar{\Psi} \left(\frac{t - \tau}{a} \right) dt$$

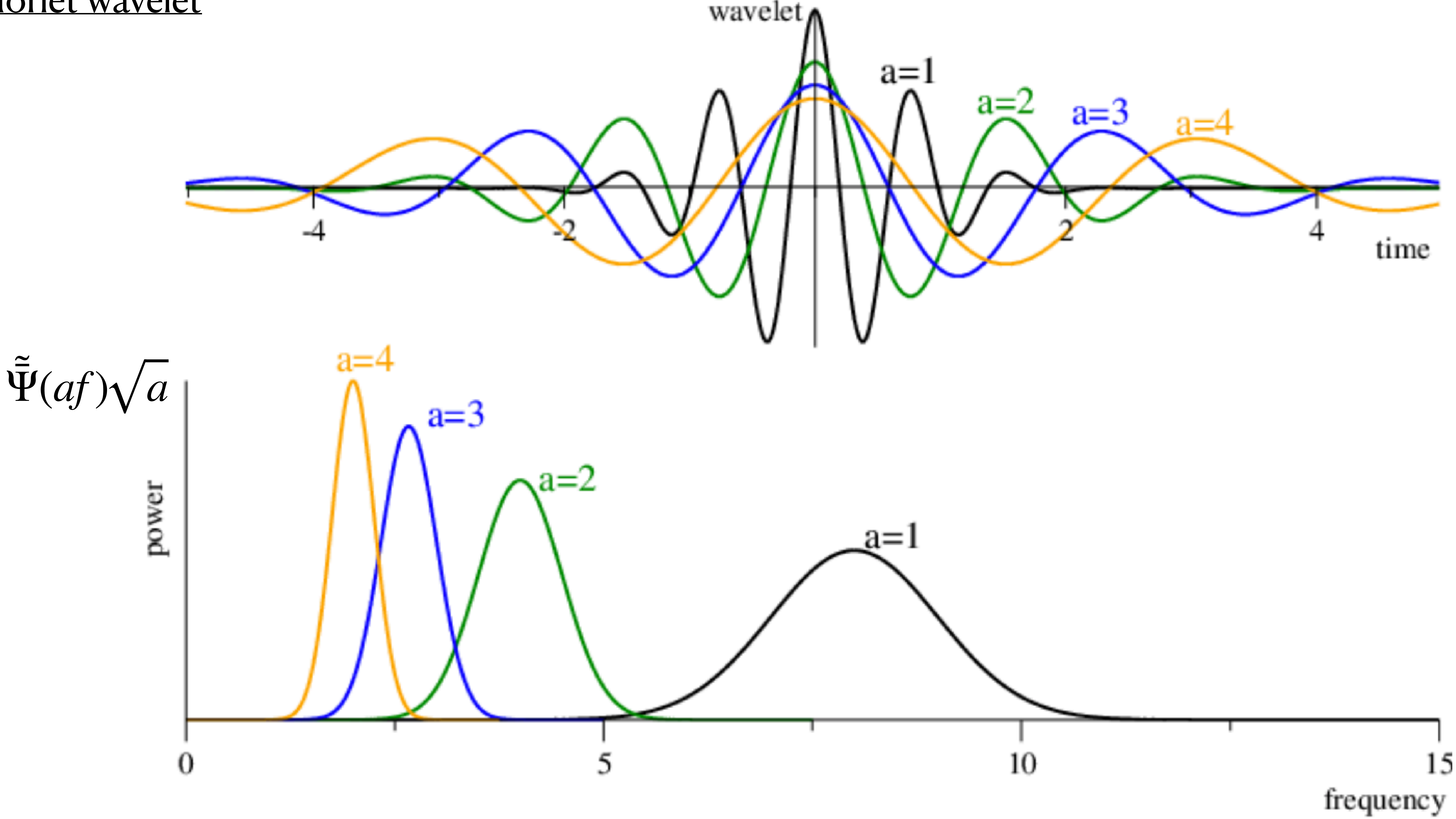
$$X(\tau, a) = \mathrm{IFT} \left[\ \tilde{x}(f) \ \cdot \ \tilde{\bar{\bar{\Psi}}}(af)\sqrt{a} \ \right] (\tau)$$

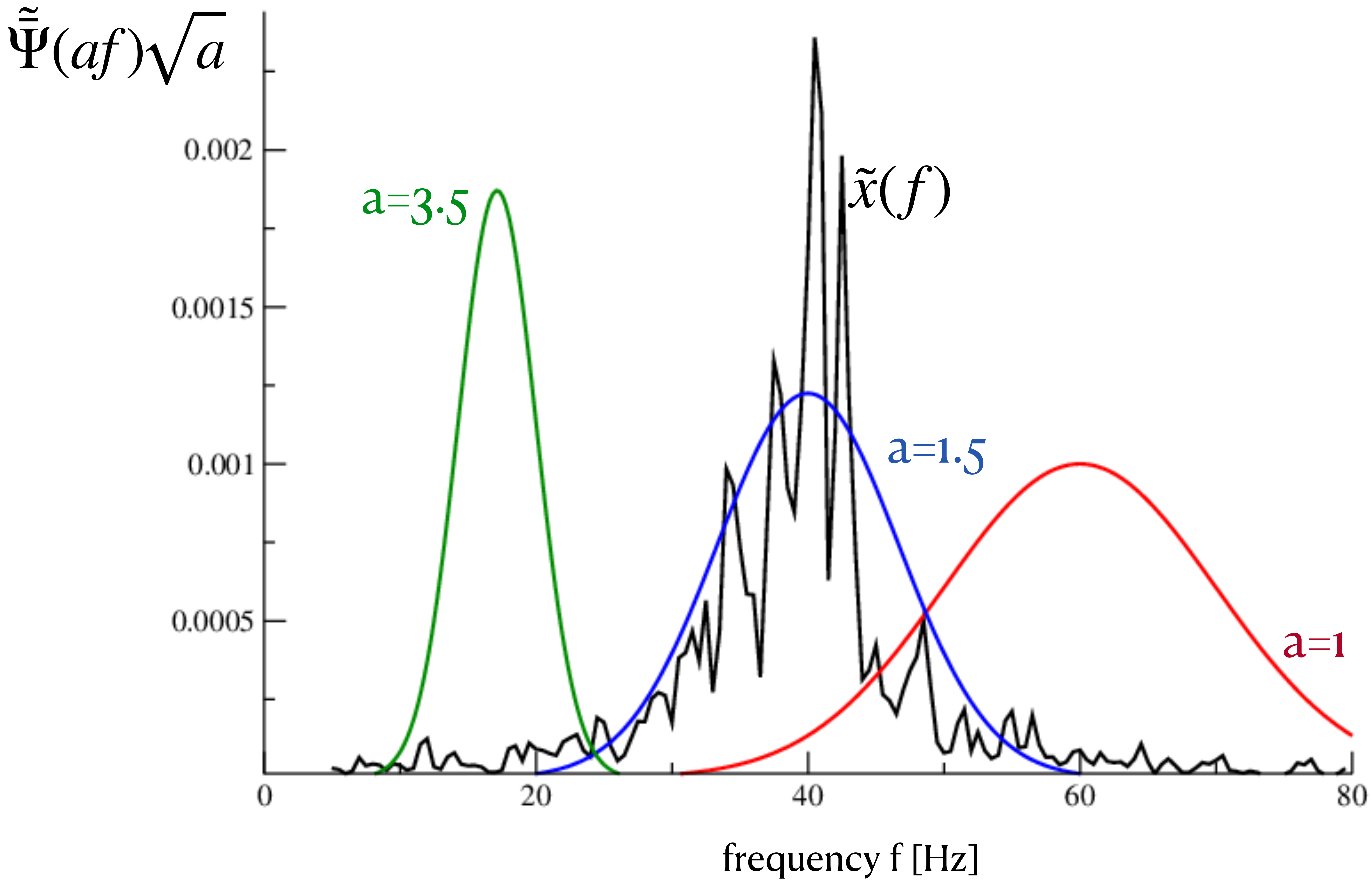
$$X(\tau, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t) \bar{\Psi} \left(\frac{t - \tau}{a} \right) dt$$

$$X(\tau, a) = \text{IFT} \left[\tilde{x}(f) \cdot \boxed{\tilde{\bar{\Psi}}(af)\sqrt{a}} \right] (\tau)$$

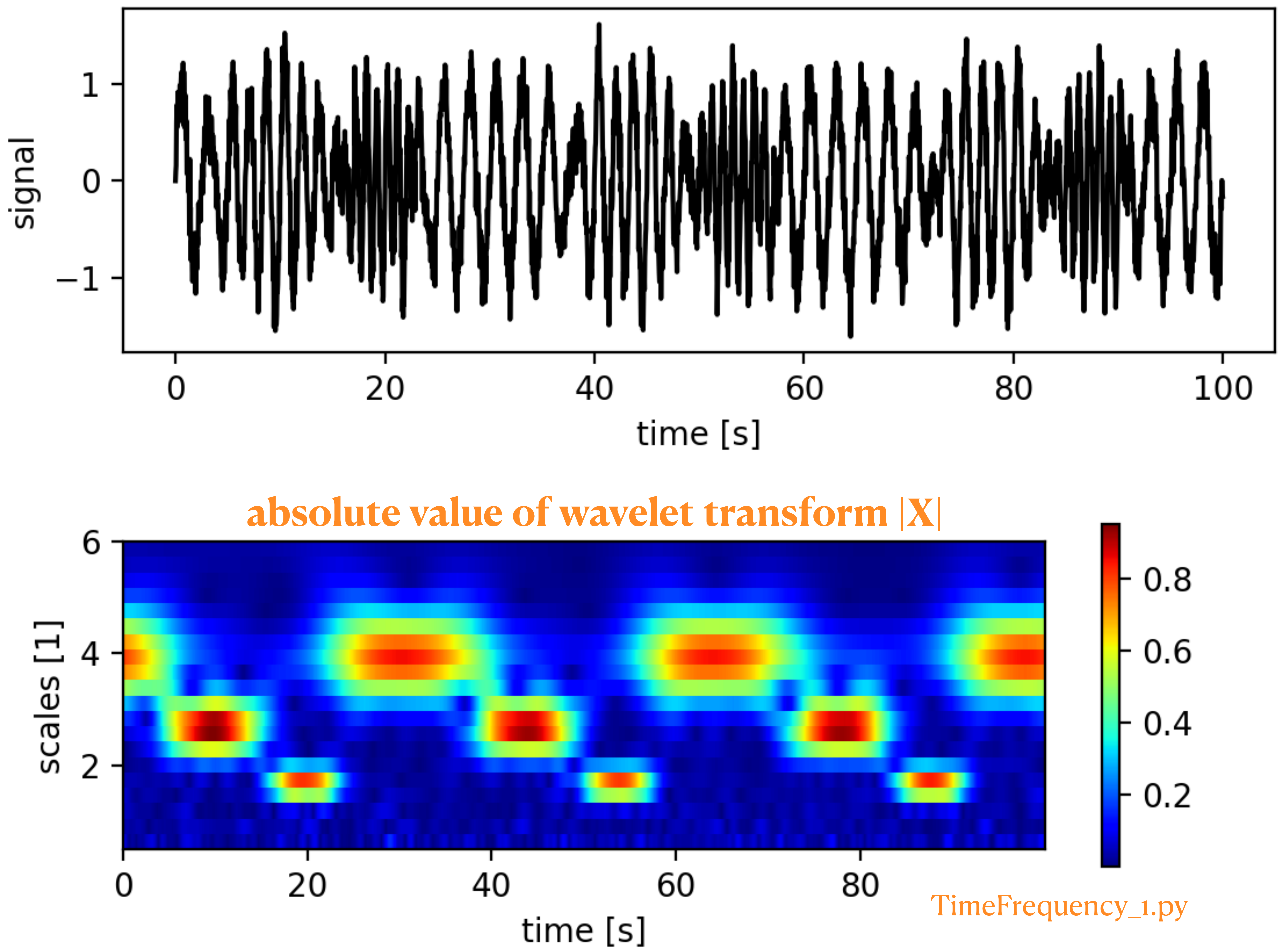
$$X(\tau, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t) \bar{\Psi}\left(\frac{t-\tau}{a}\right) dt \qquad X(\tau, a) = \mathrm{IFT}\left[\ \tilde{x}(f) \ \cdot \ \tilde{\bar{\Psi}}(af)\sqrt{a}\ \right](\tau)$$

Morlet wavelet



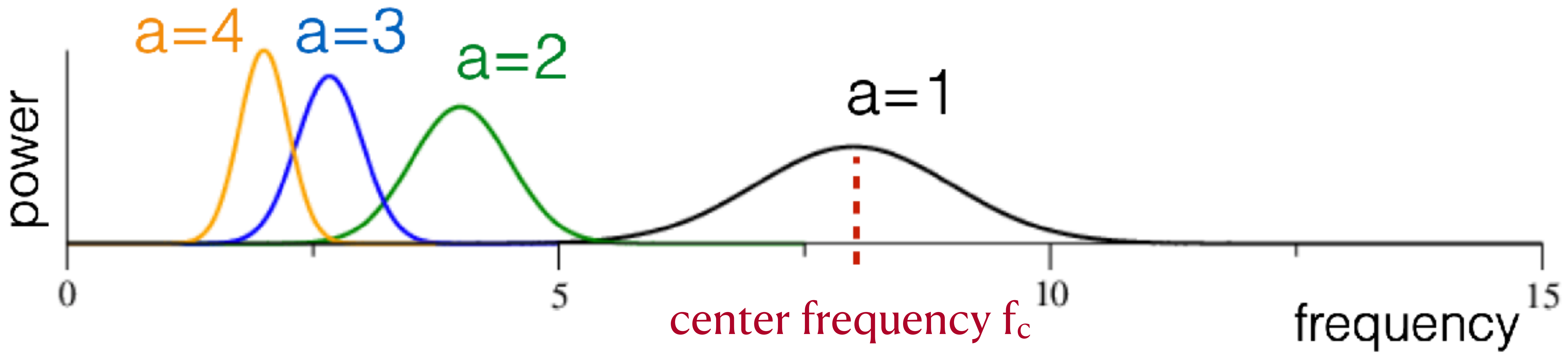


$\tilde{\tilde{\Psi}}(af)\sqrt{a}$
0.002
0.0015
0.001
0.0005
a=3.5
$\tilde{x}(f)$
a=1.5
a=1
0
20
40
60
80
frequency f [Hz]

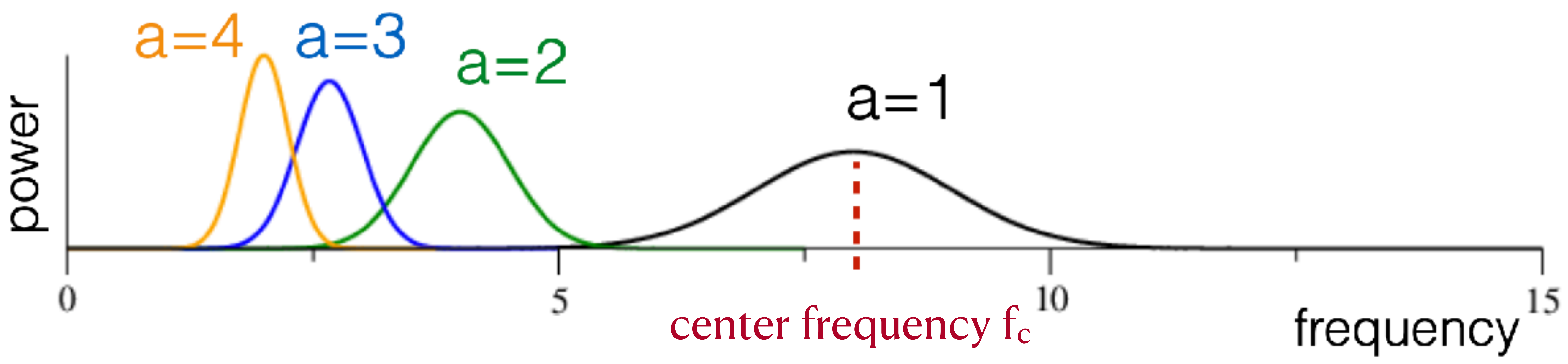


scalogram with Morlet wavelet

relation between scales and frequency



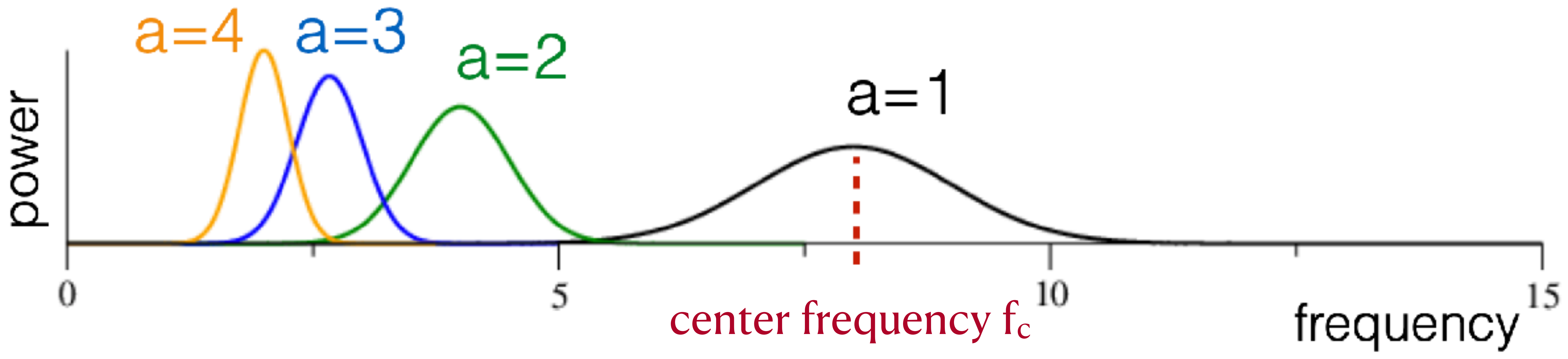
relation between scales and frequency



pseudo frequency $f_p = \frac{f_c}{a}$

pseudo frequency is the frequency of maximum mother wavelet power

relation between scales and frequency

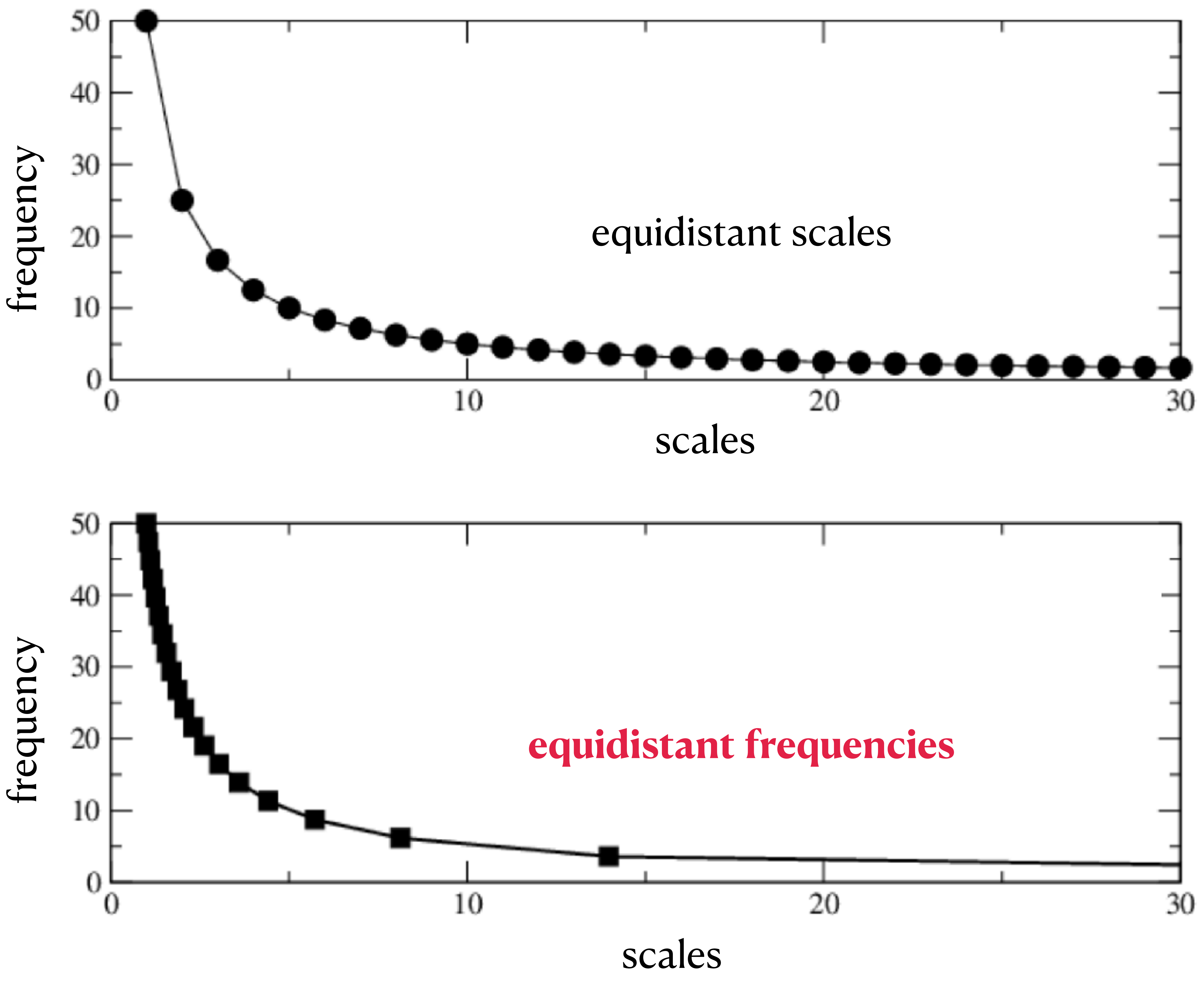


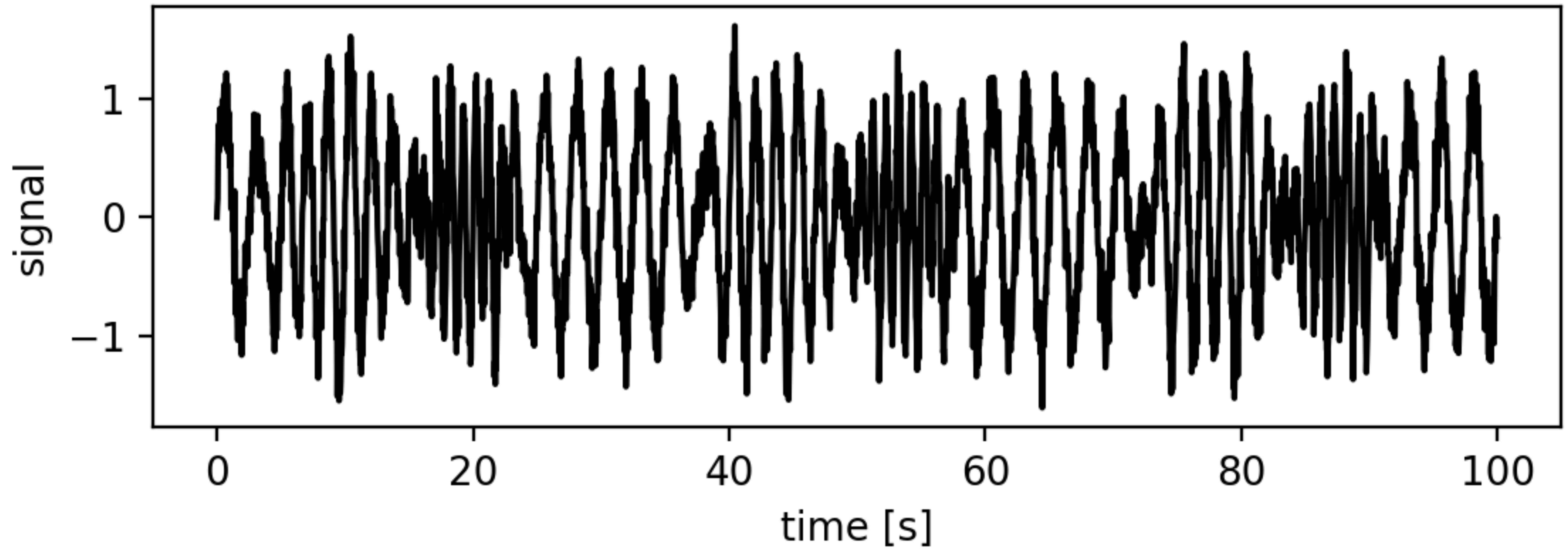
pseudo frequency $f_p = \frac{f_c}{a}$

pseudo frequency is the frequency of maximum mother wavelet power

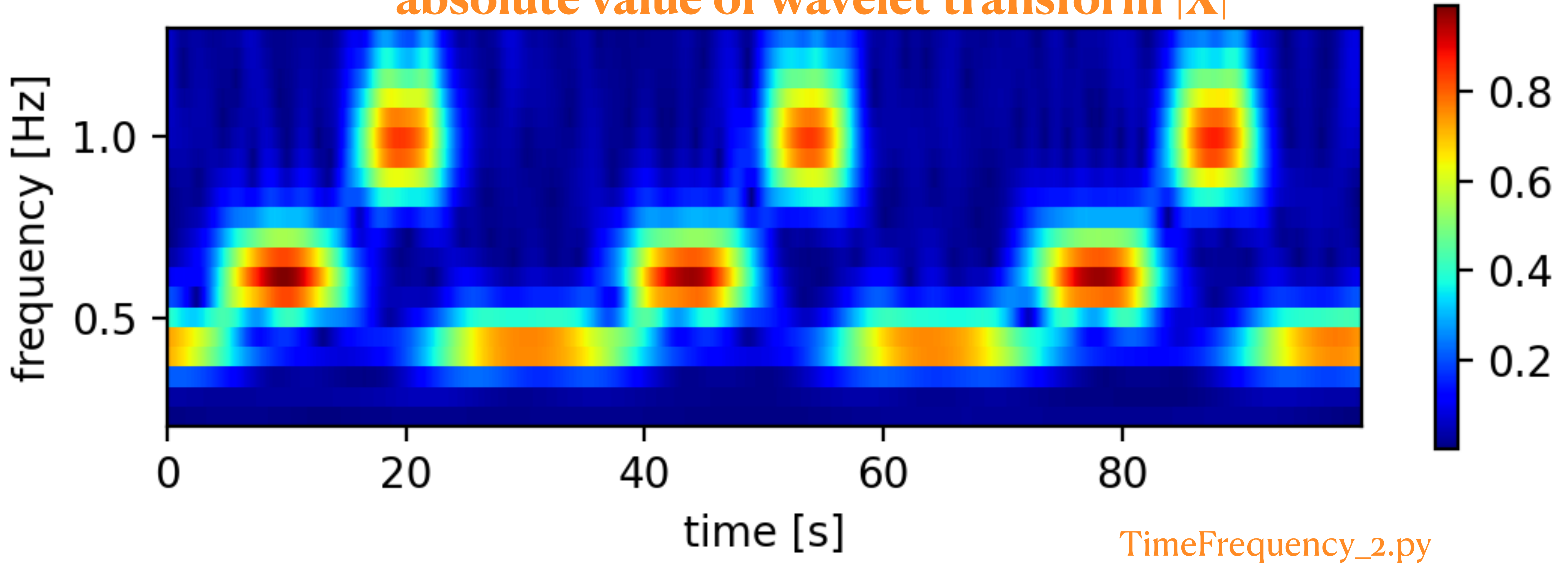
"pseudo" : not a unique frequency, but represents a distribution

relation between scales and frequency





absolute value of wavelet transform |X|

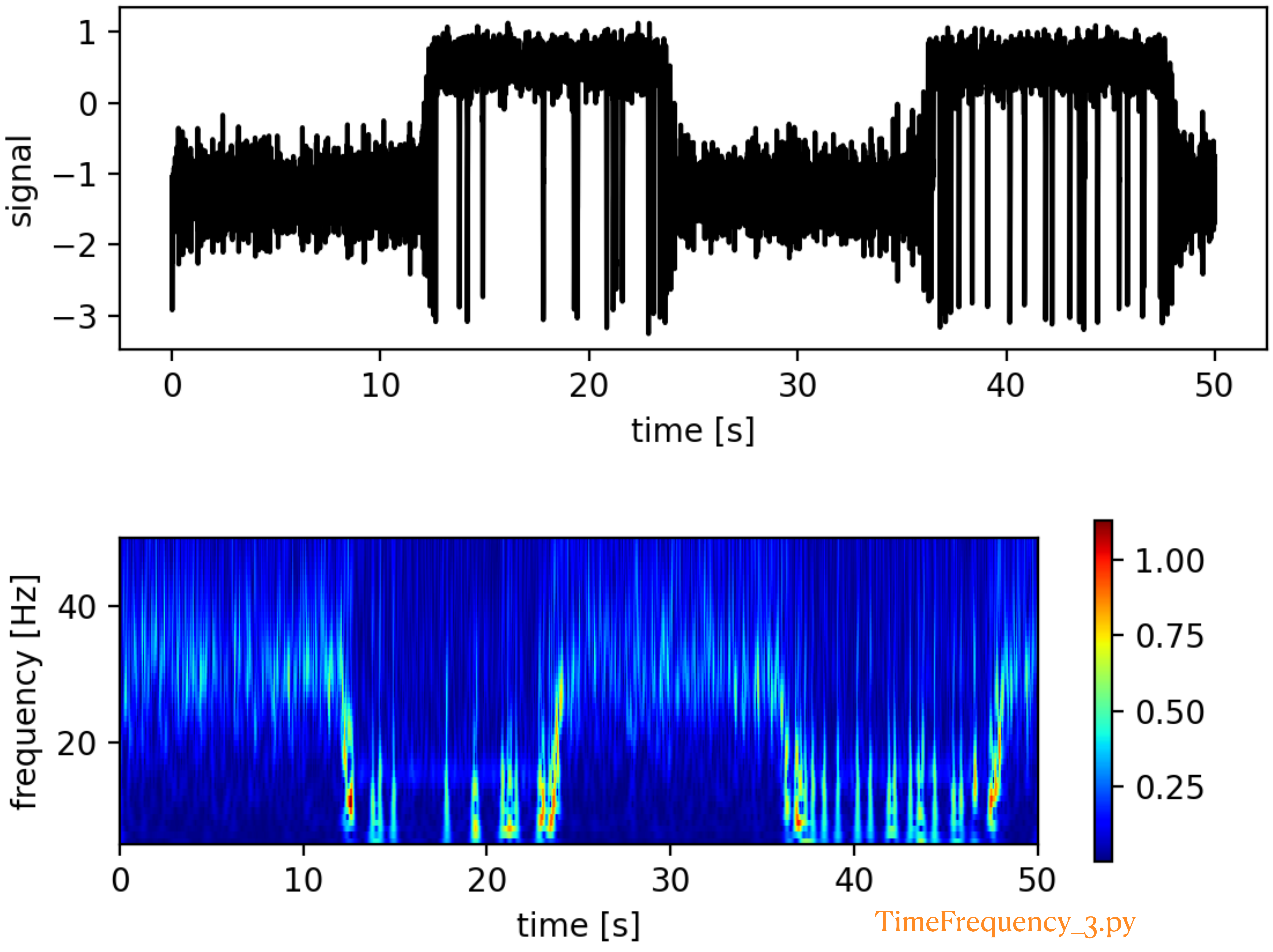


time-frequency plot

TimeFrequency_2.py

Example:

simulated cortical activity

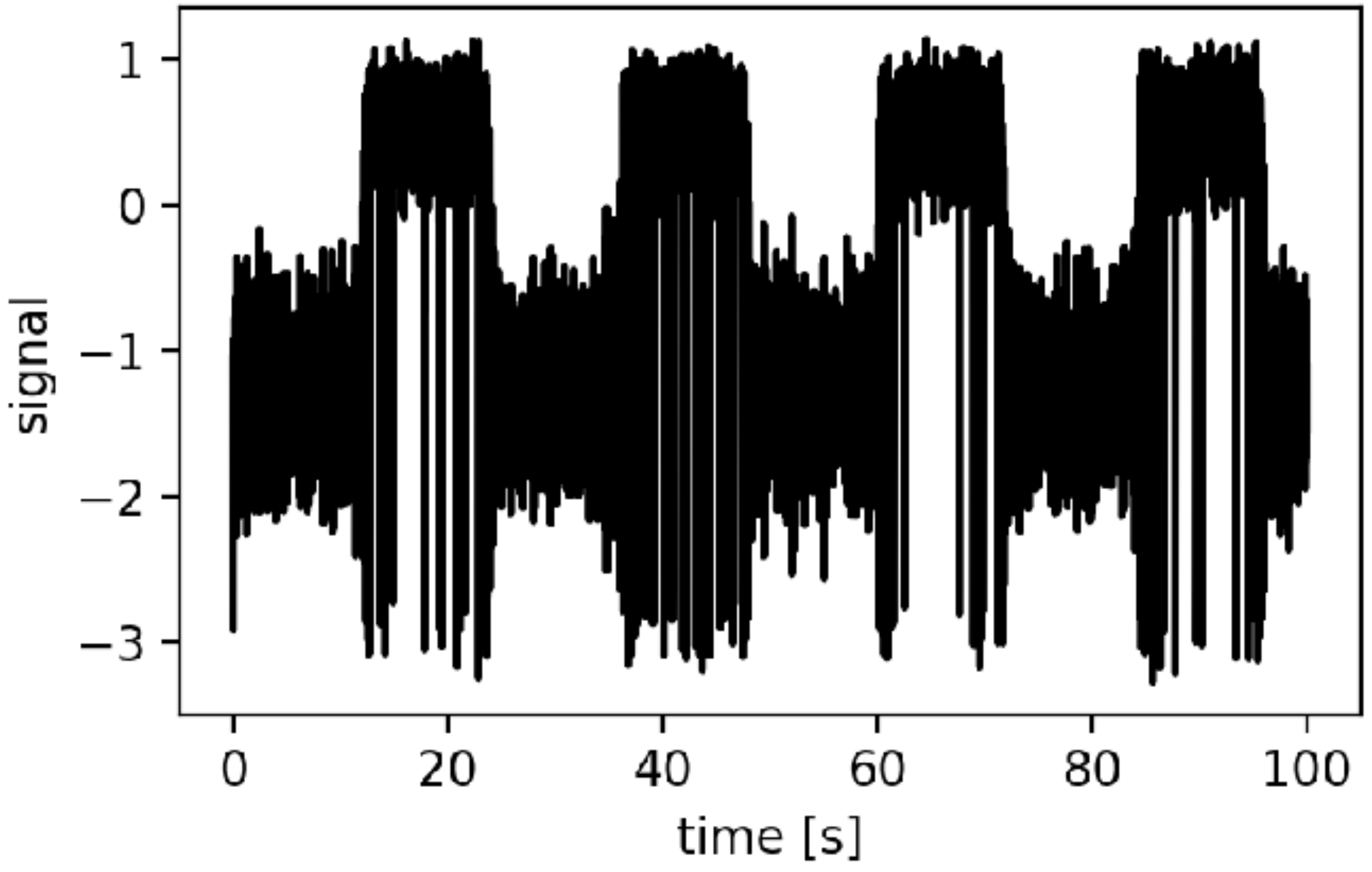


TimeFrequency_3.py

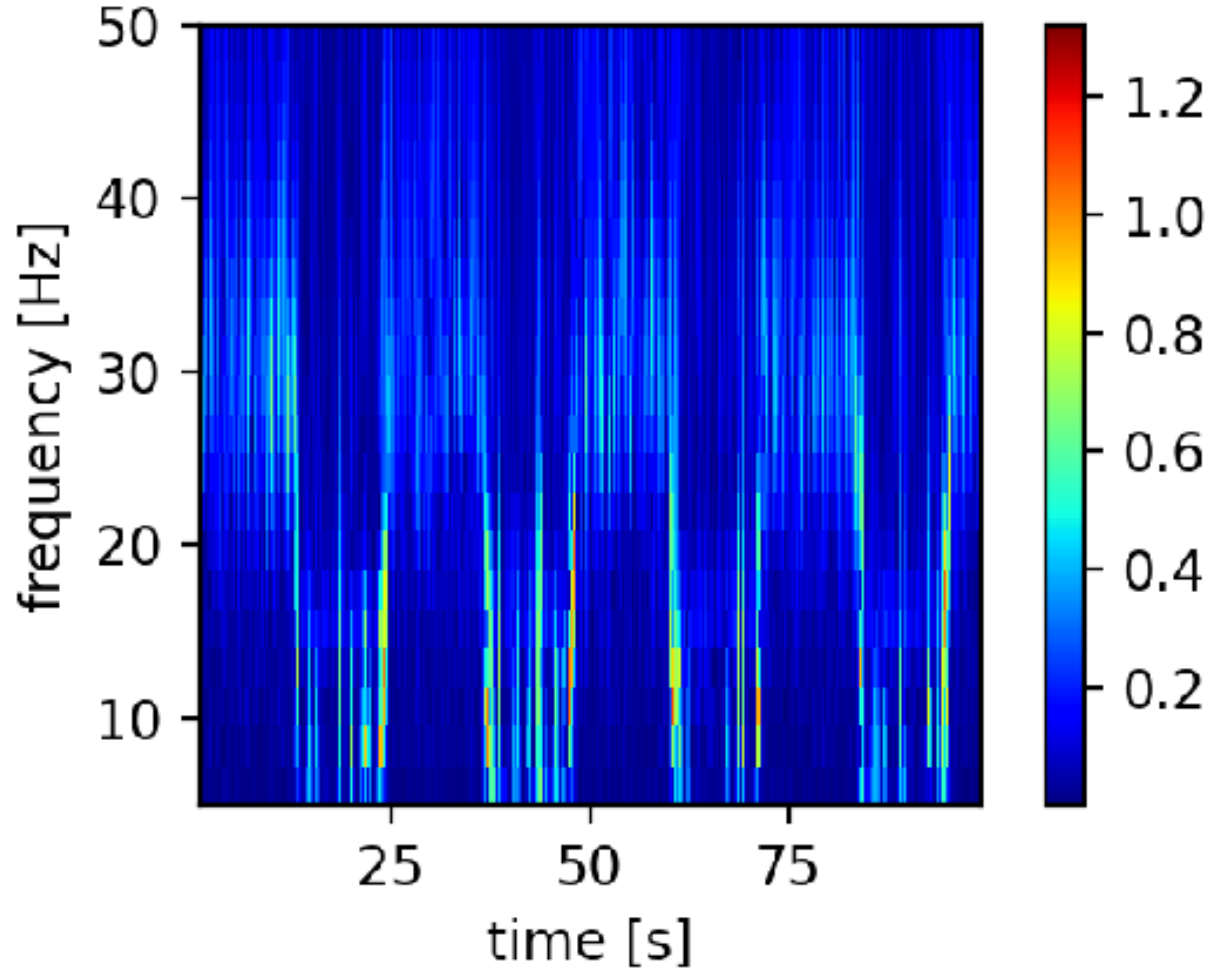
from: A. Hutt and J. Lefebvre,
Arousal fluctuations govern oscillatory transitions between dominant gamma- and alpha occipital activity during eyes open/closed conditions,
Brain Top. (2021)

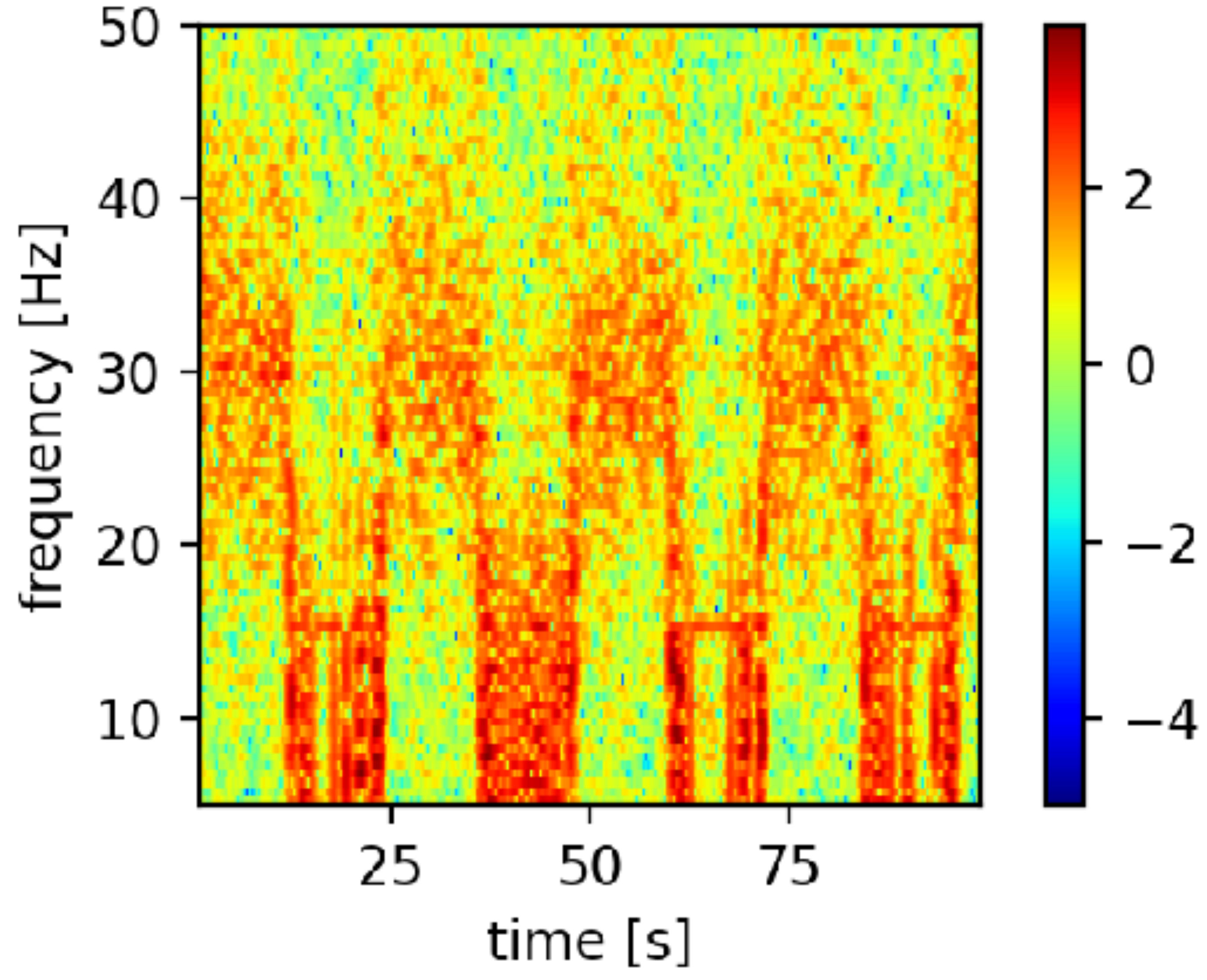
Example:

simulated cortical activity



STFT





TimeFrequency_3.py

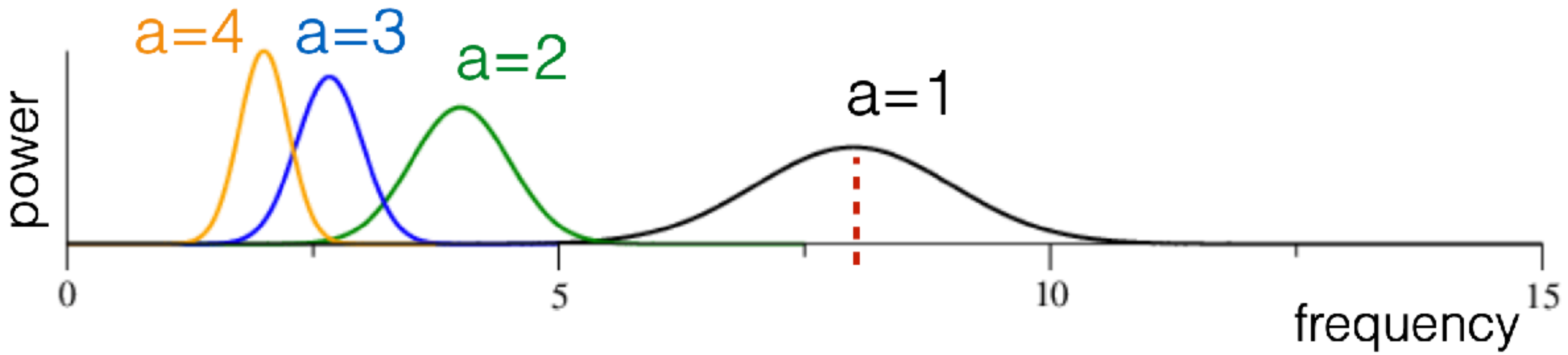
data taken from: A. Hutt and J. Lefebvre, Brain Top. (2021); doi:10.1007/s10548-021-00855-z

related Python libraries

PyWavelets

https://github.com/PyWavelets/pywt

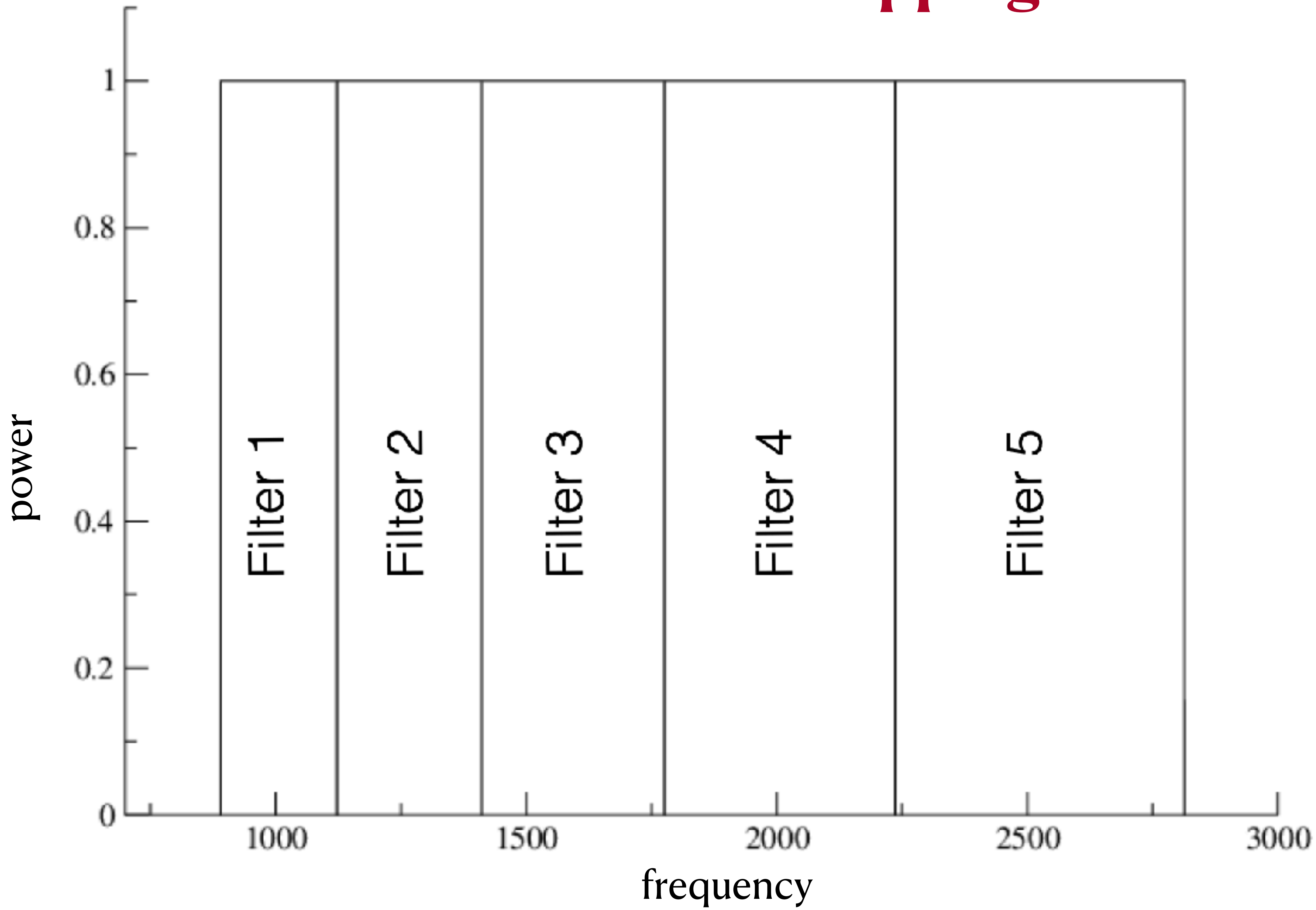
comment: what is the Discrete Wavelet Transform ?



overlapping frequency bands for different a !!

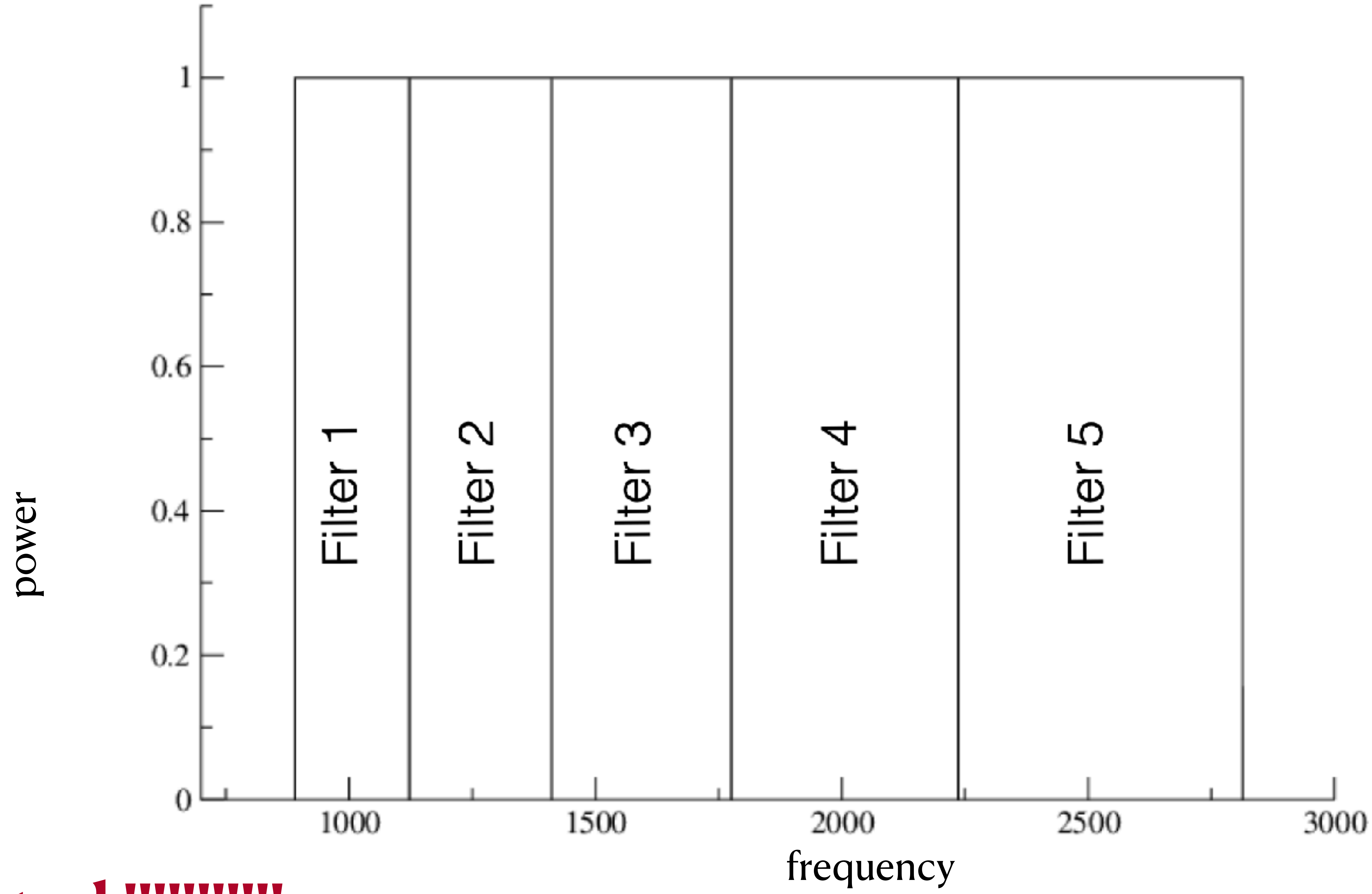
comment: what is the Discrete Wavelet Transform ?

filter bank with **non-overlapping** bands



comment: what is the Discrete Wavelet Transform ?

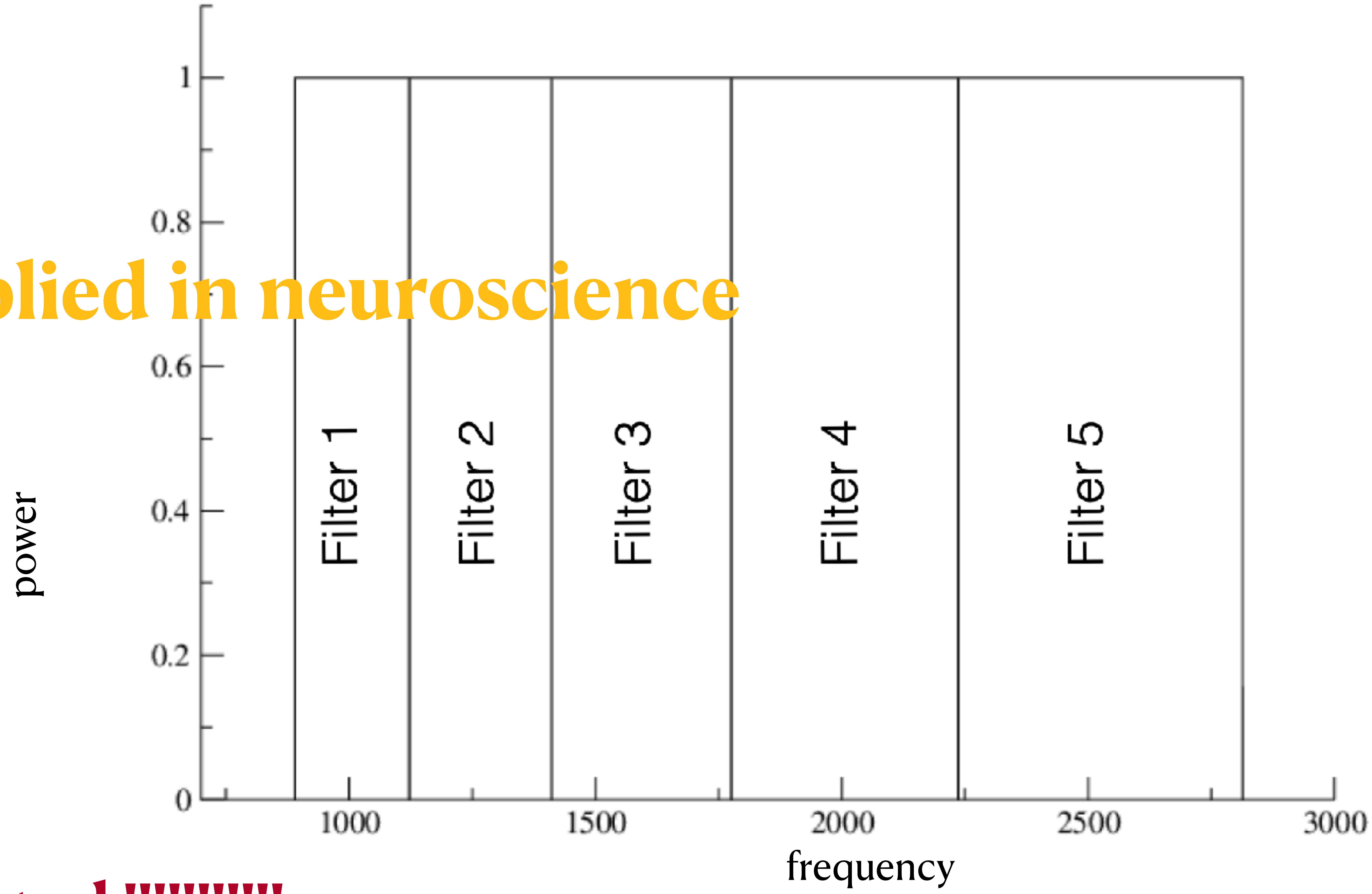
filter bank with **non-overlapping** bands



optimal decomposition of signal !!!!!!!!!!!

comment: what is the Discrete Wavelet Transform ?

filter bank with **non-overlapping** bands



but rarely applied in neuroscience

optimal decomposition of signal !!!!!!!!!!!

data sampling

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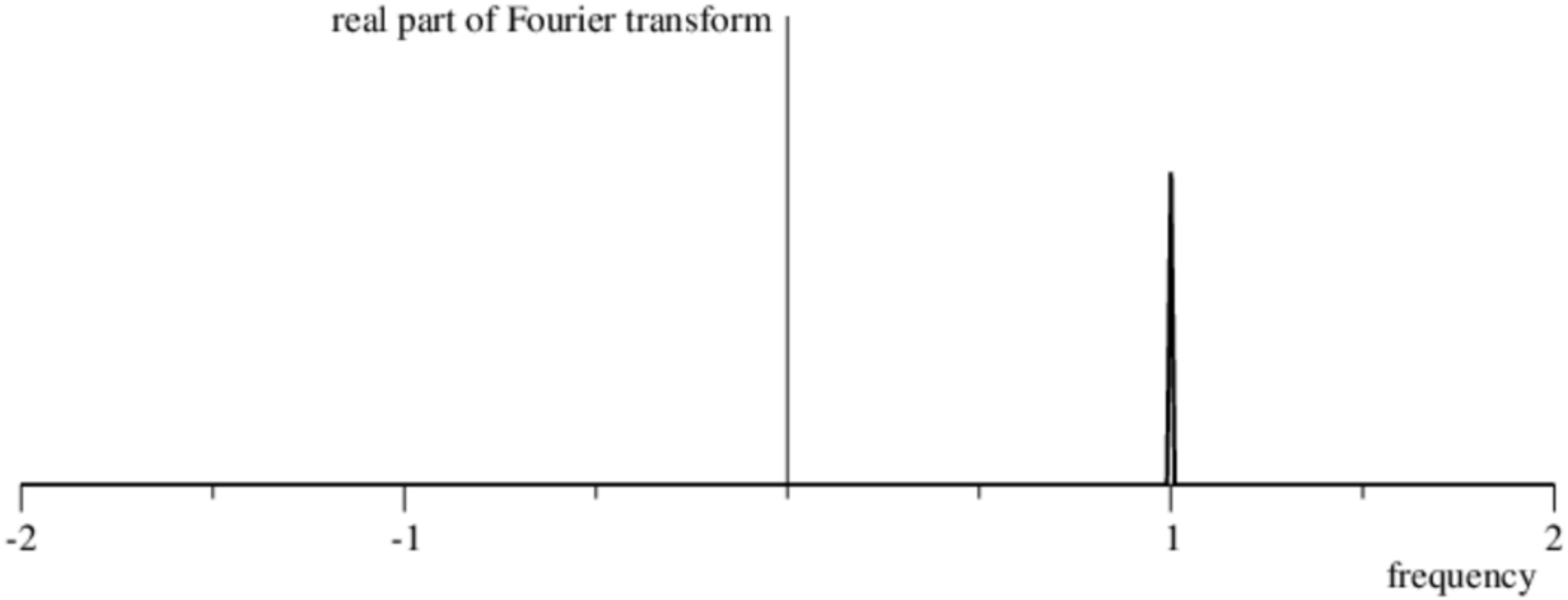
Hilbert Transform

$s(t) = \cos(2\pi f_0 t)$

an oscillation with a single frequency has a power spectrum with negative frequency

Hilbert Transform

$$s_a(t) = \cos(2\pi f t) + i \sin(2\pi f t) = e^{i2\pi f t}$$



analytical signal $s_a(t)$ contains a single positive frequency