Spectral analysis

Fundamental elements

Axel Hutt

preamble

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import numpy as np
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class signal_class:
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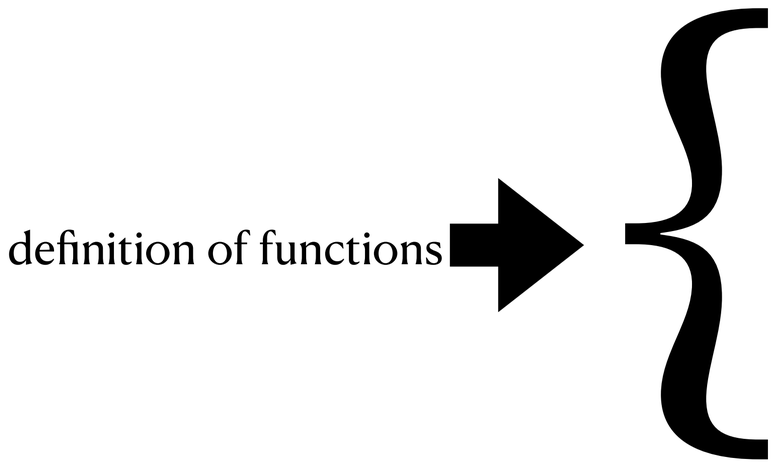
definition of functions



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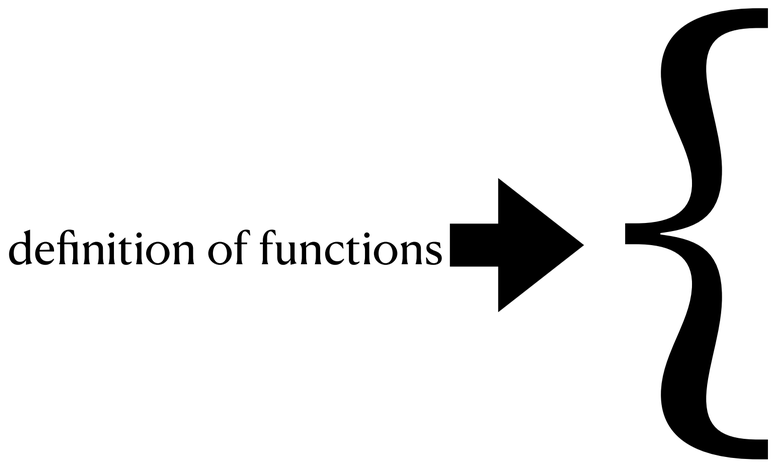


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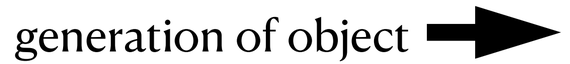


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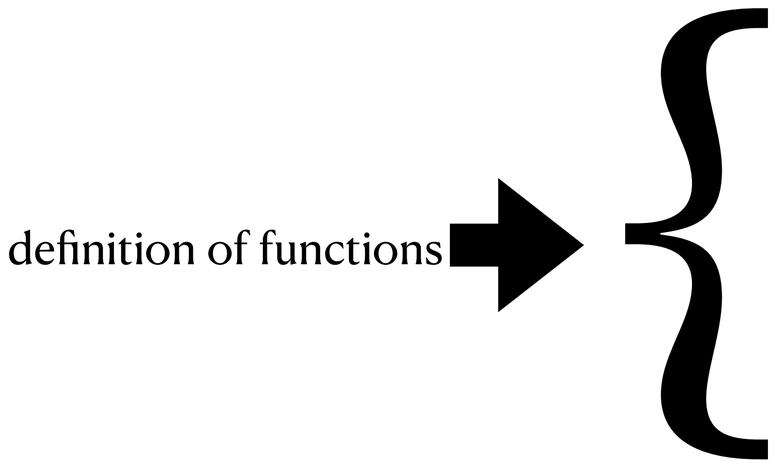


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generation of object →

call of functions →

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data sampling

Fourier analysis

errors in analysis

linear filters

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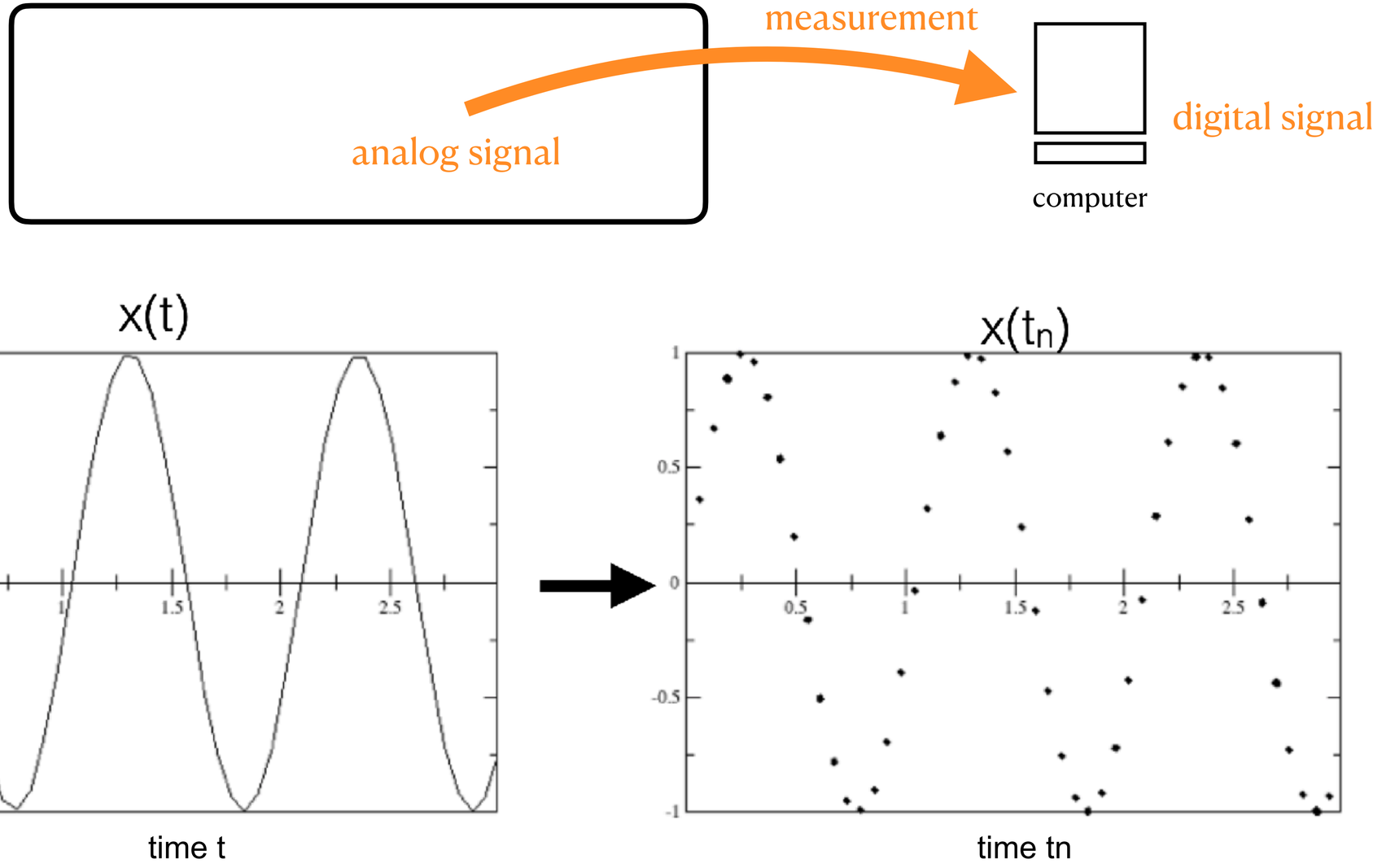
measurement
analog signal
digital signal
computer
x(t)
1
0.5
0
-0.5
-1
0.5
1
1.5
2
2.5
time t
x(tn)
1
0.5
0
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1
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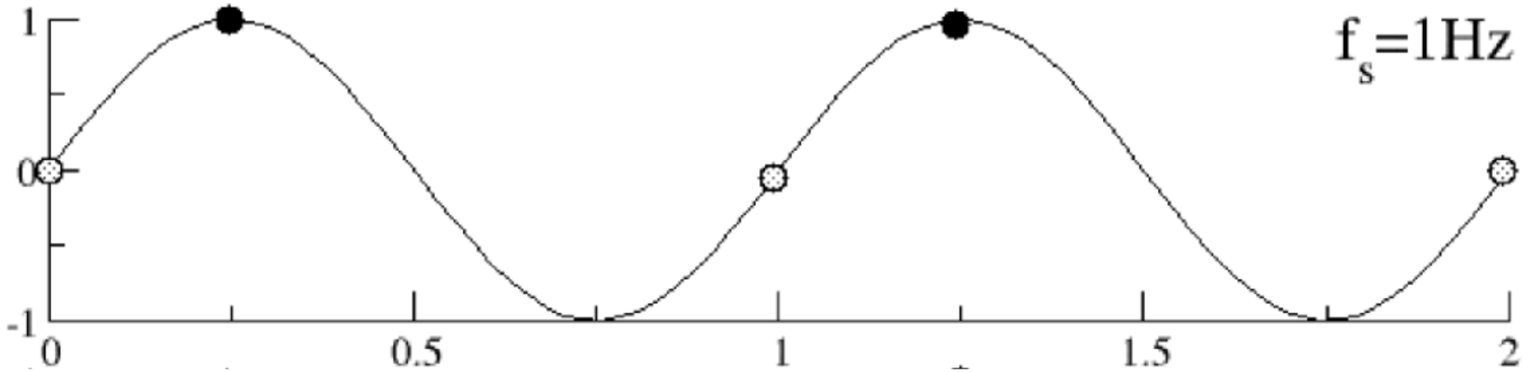
$T_m = 1s$



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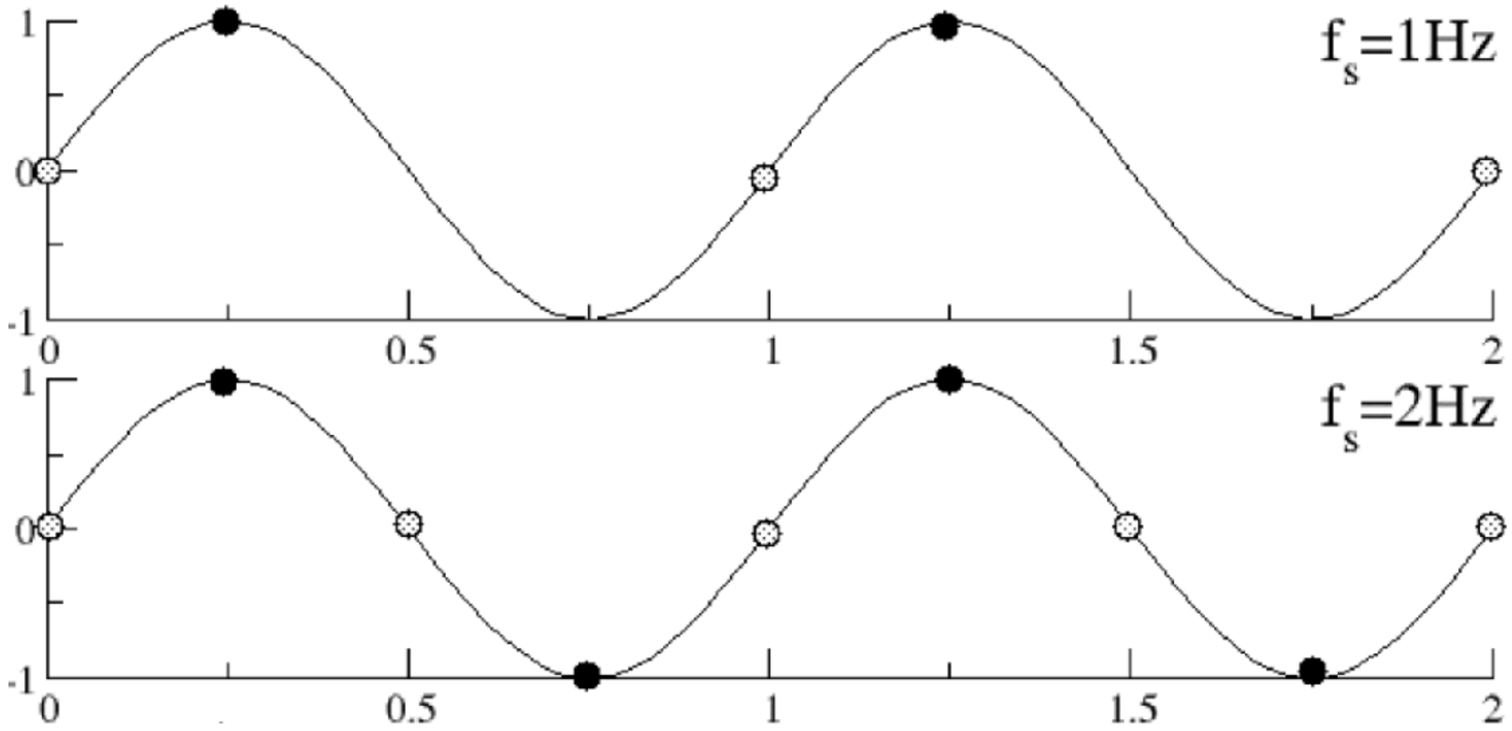

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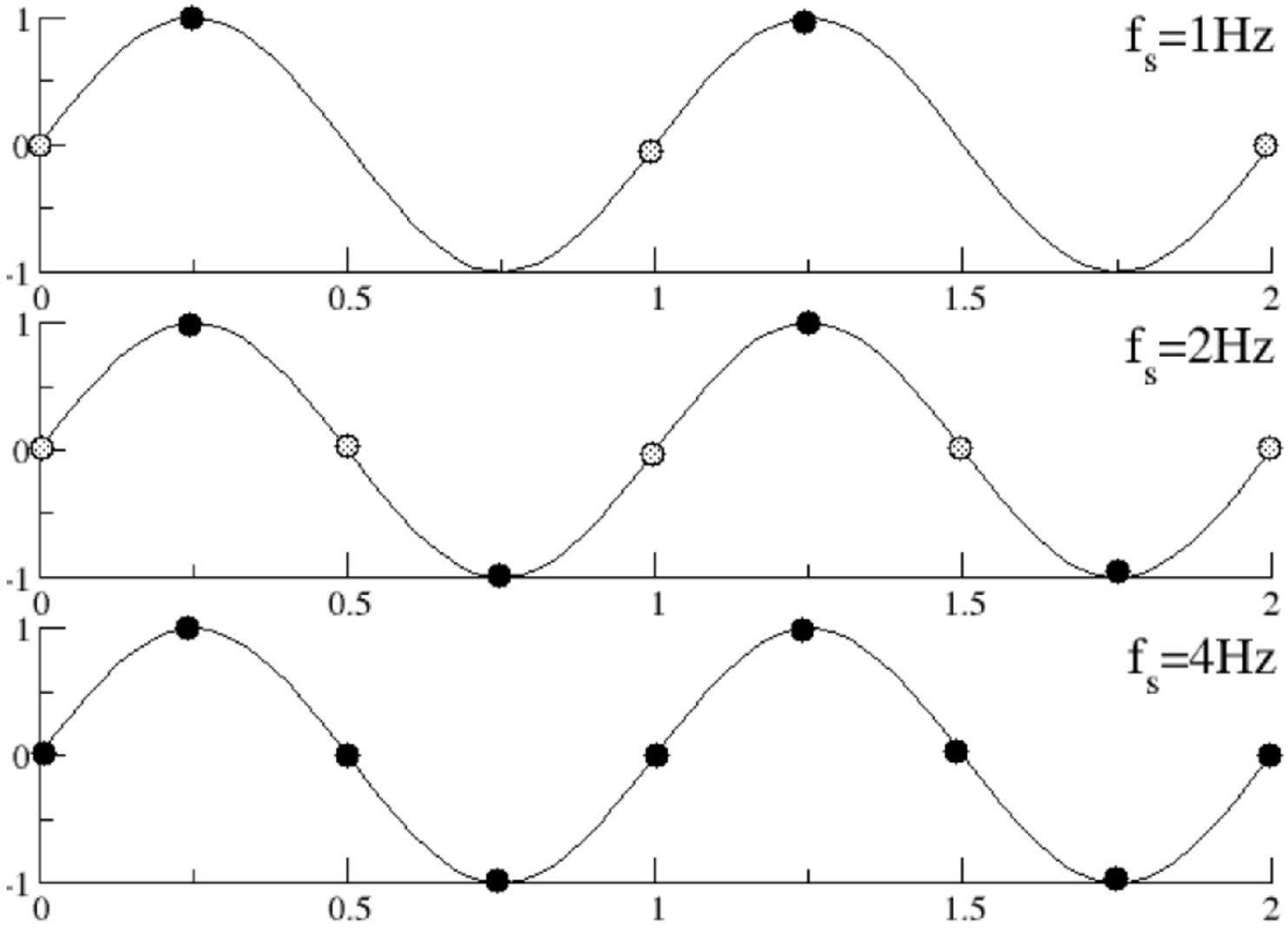
$T_s = 0.5s$

$T_m = 1s$



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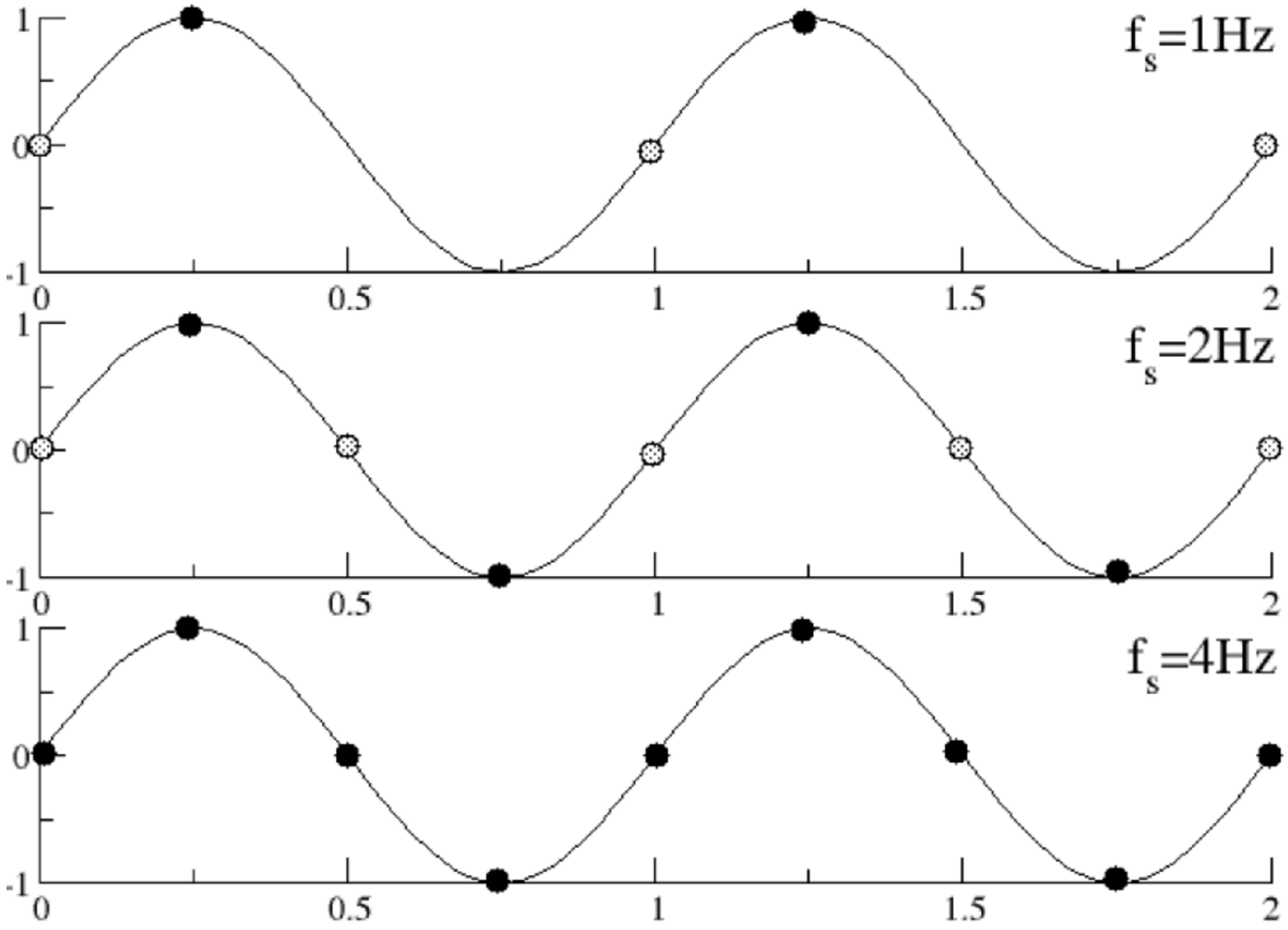

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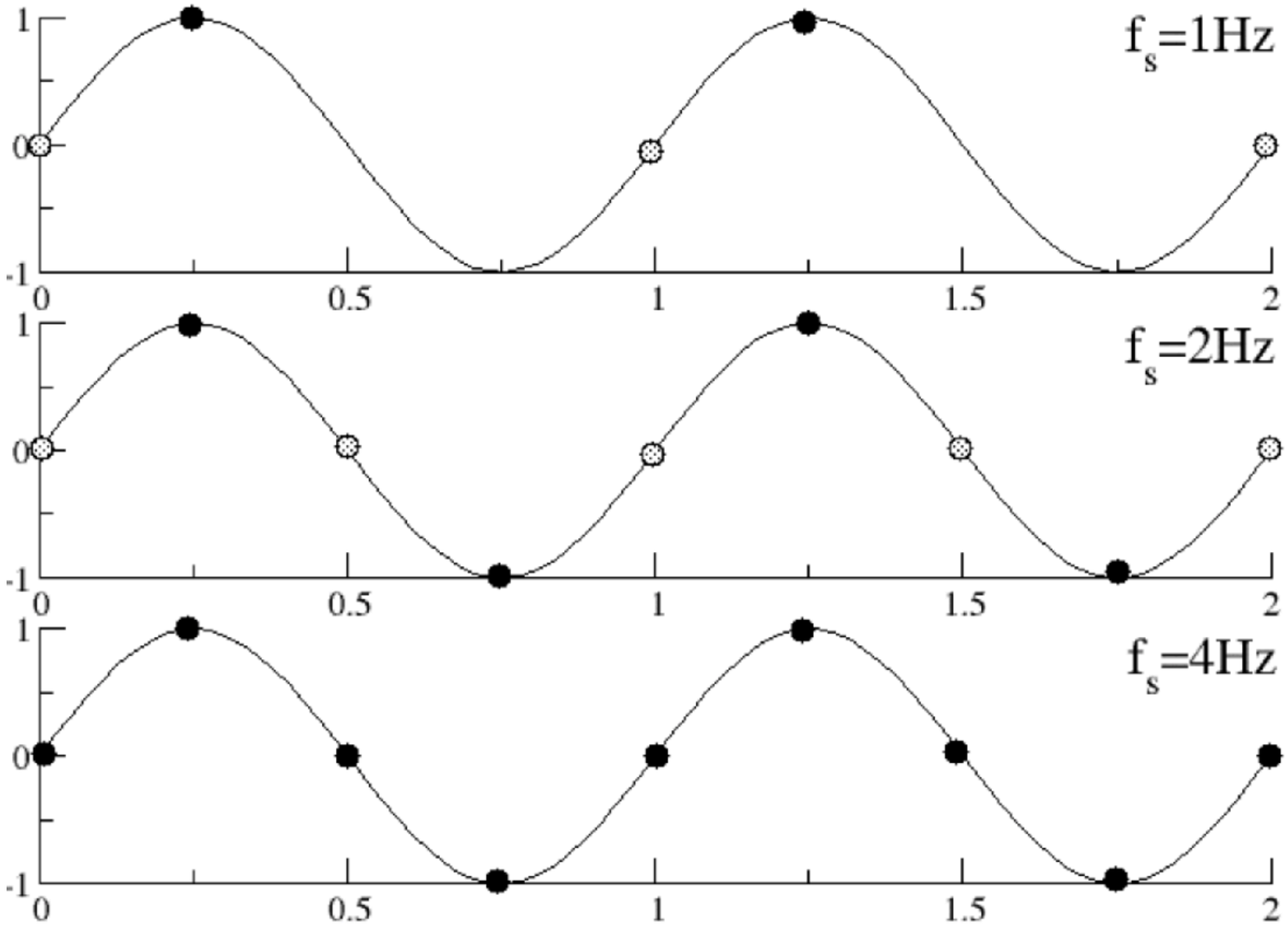
good sampling: $T_s \leq T_m/2$

$T_m = 1s$



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Nyquist law : $f_s \geq 2f_m$

Sampling theorem

(Shannon/Nyquist/Whitaker/Kotelnikov)

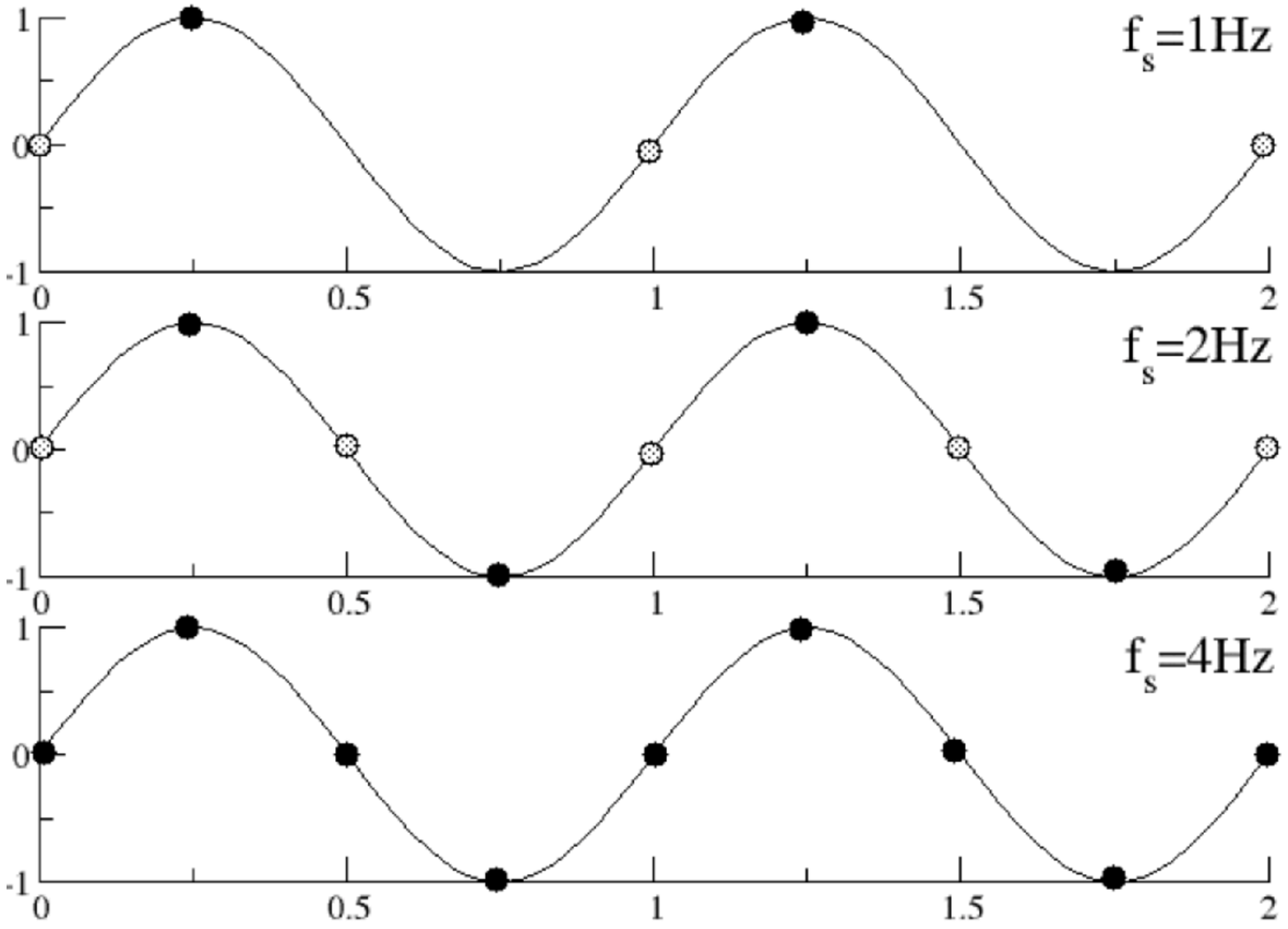
given:

- continuous function s(t)
- s(t) has maximum frequency f_m
- s(t) is sampled with frequency f_s , i.e. $s(t) \rightarrow s(t_n)\ ,\ t_n = n\Delta t = n/f_s$

hypothesis: information loss by sampling continuous function s(t)

objective: find sampling frequency f_s, for which no information loss occurs

solution: $f_s \geq 2f_m$



comment: in practice it is a good idea to choose $f_s \geq 4f_m$

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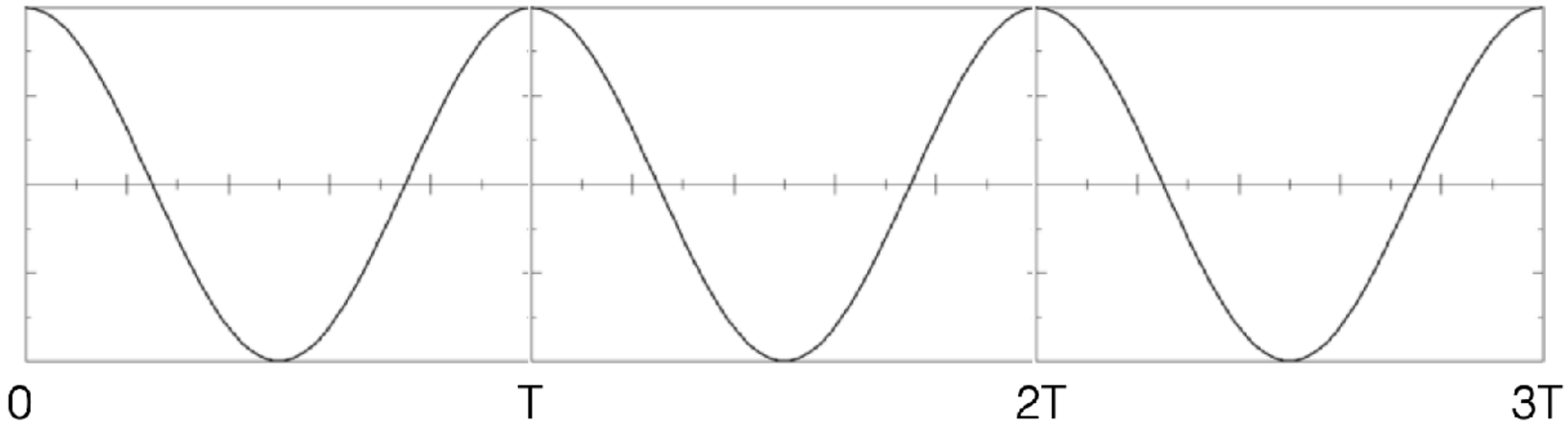
all-time assumption: the signal under study is periodic



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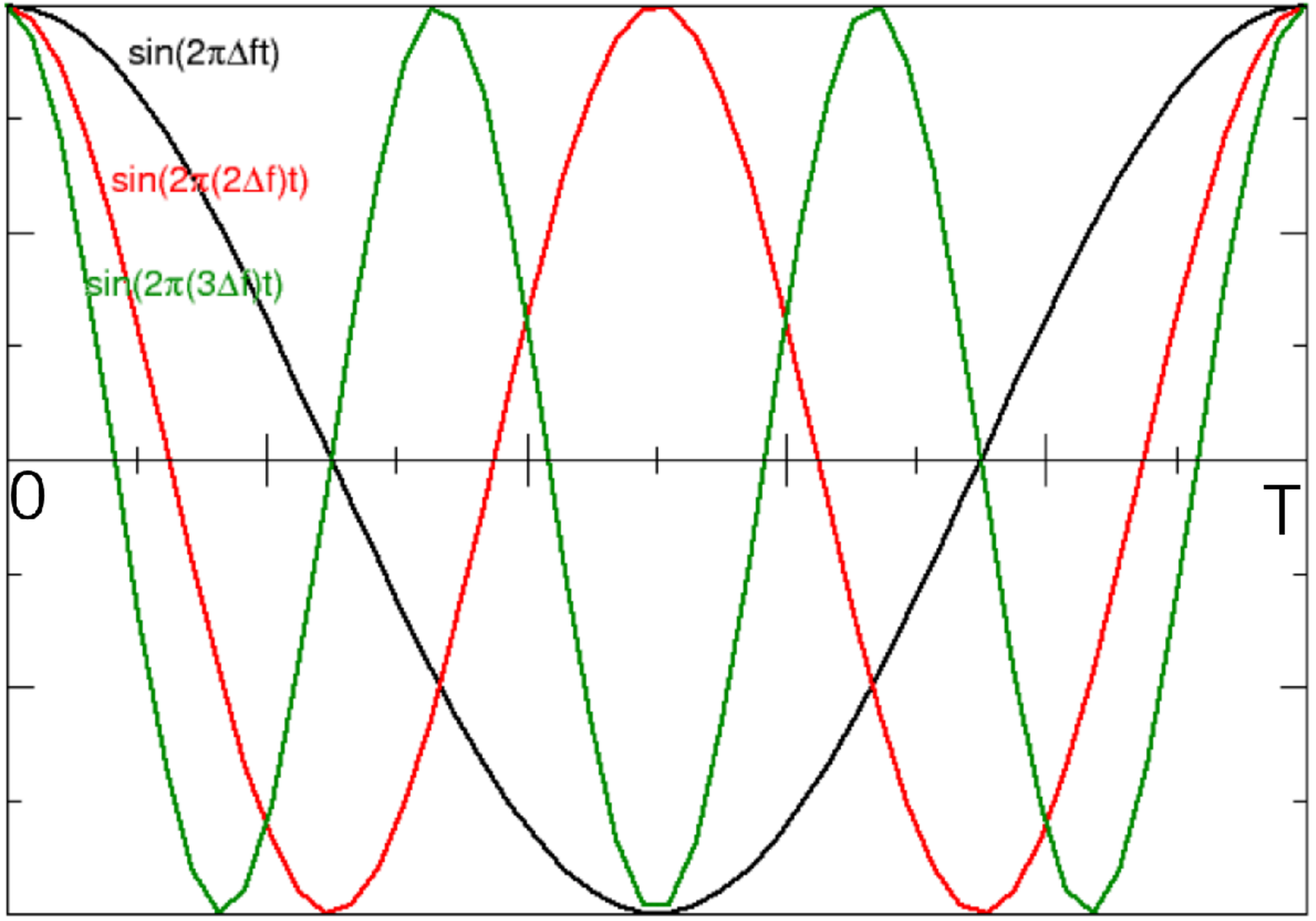

one period in interval

two periods in interval

three periods in interval

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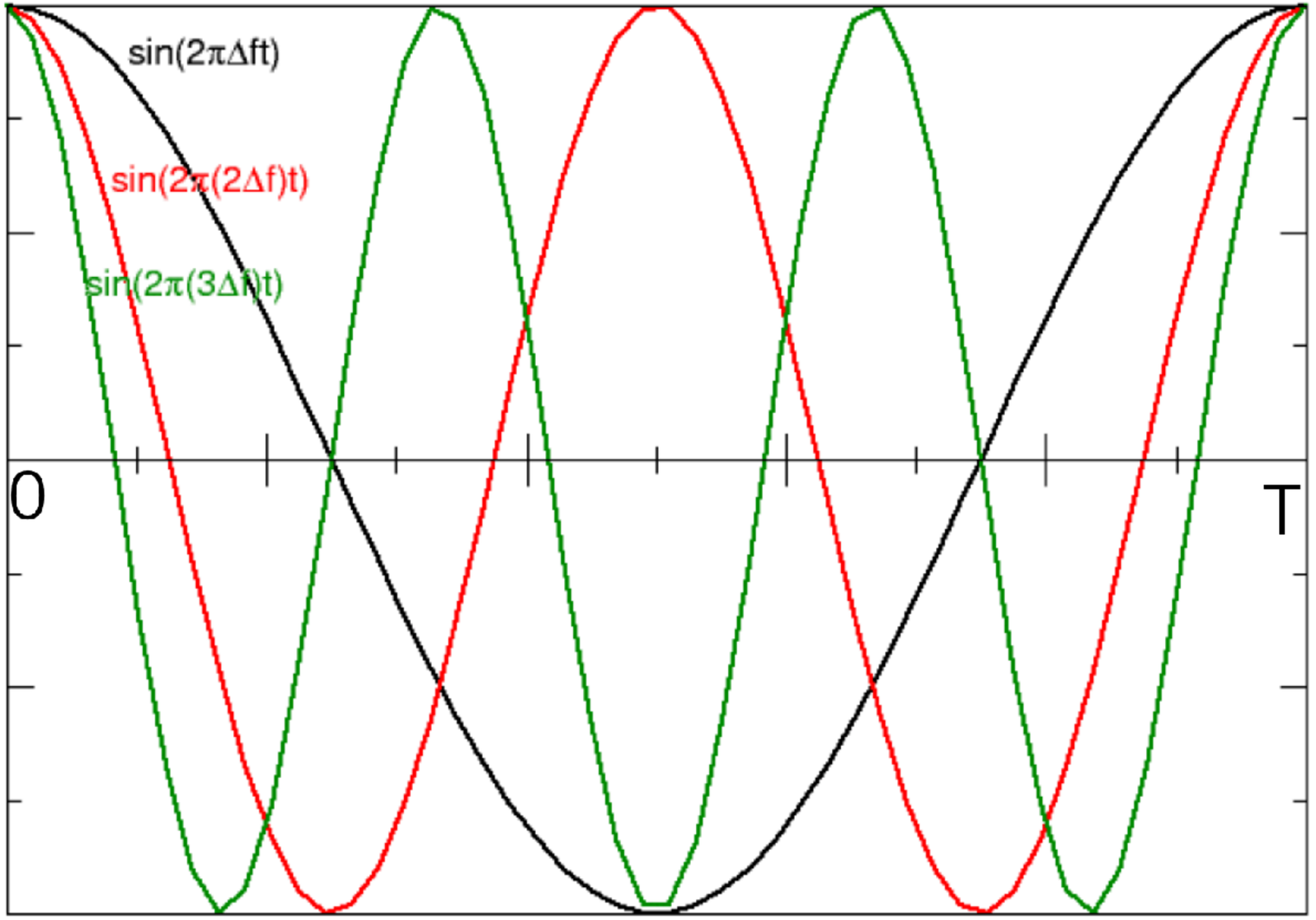

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smallest frequency: $\Delta f = \frac{1}{T}$

$f_n = n\Delta f$

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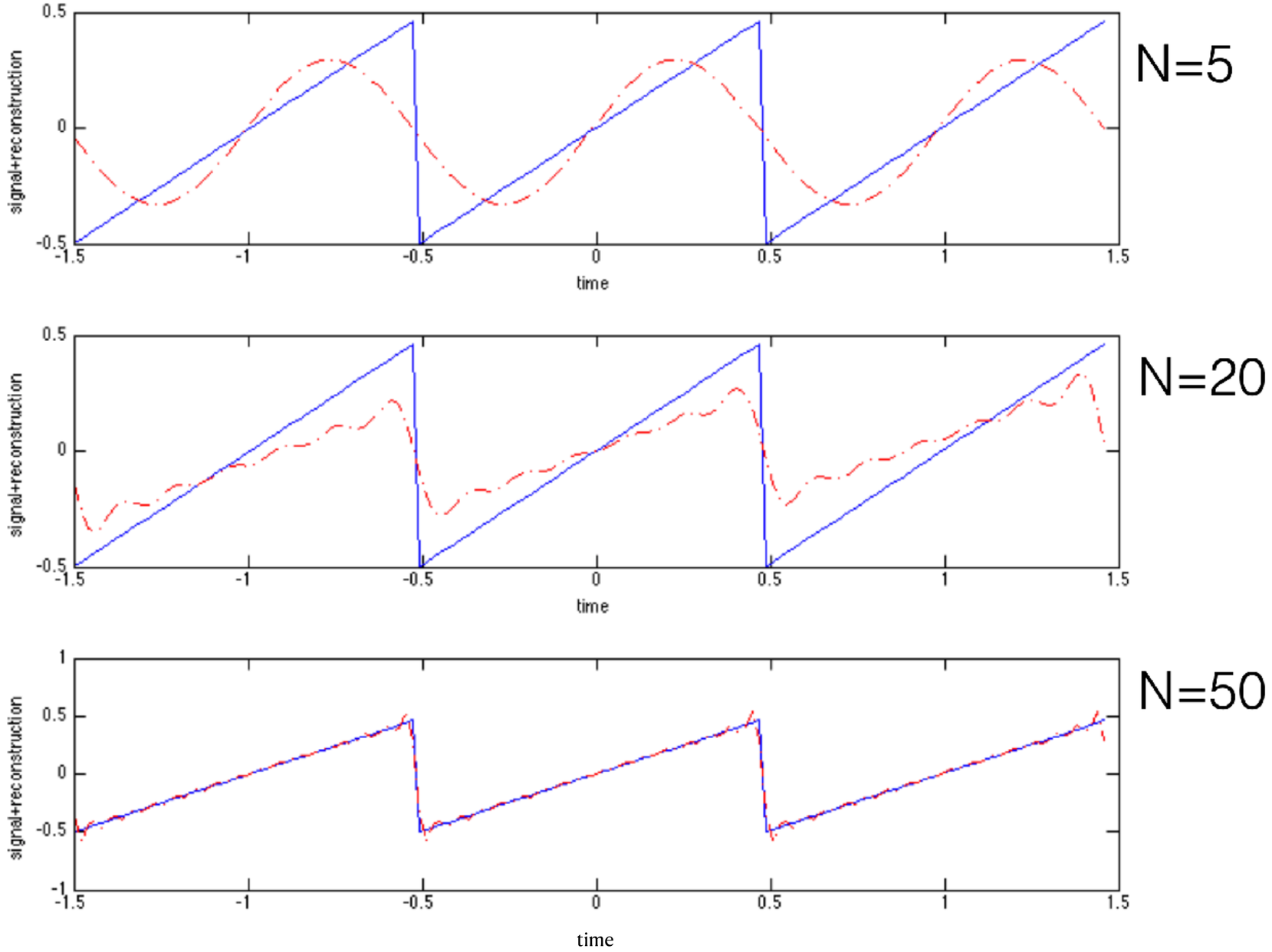
equivalent formulation: $$s(t) \approx \sum_{n=-N}^{N} c_n e^{i2\pi f_n t} \qquad c_n \in \mathbb{C}$$ (c_n are complex)

N to be chosen

approximation by different number of Fourier modes

approximation by different number of Fourier modes N



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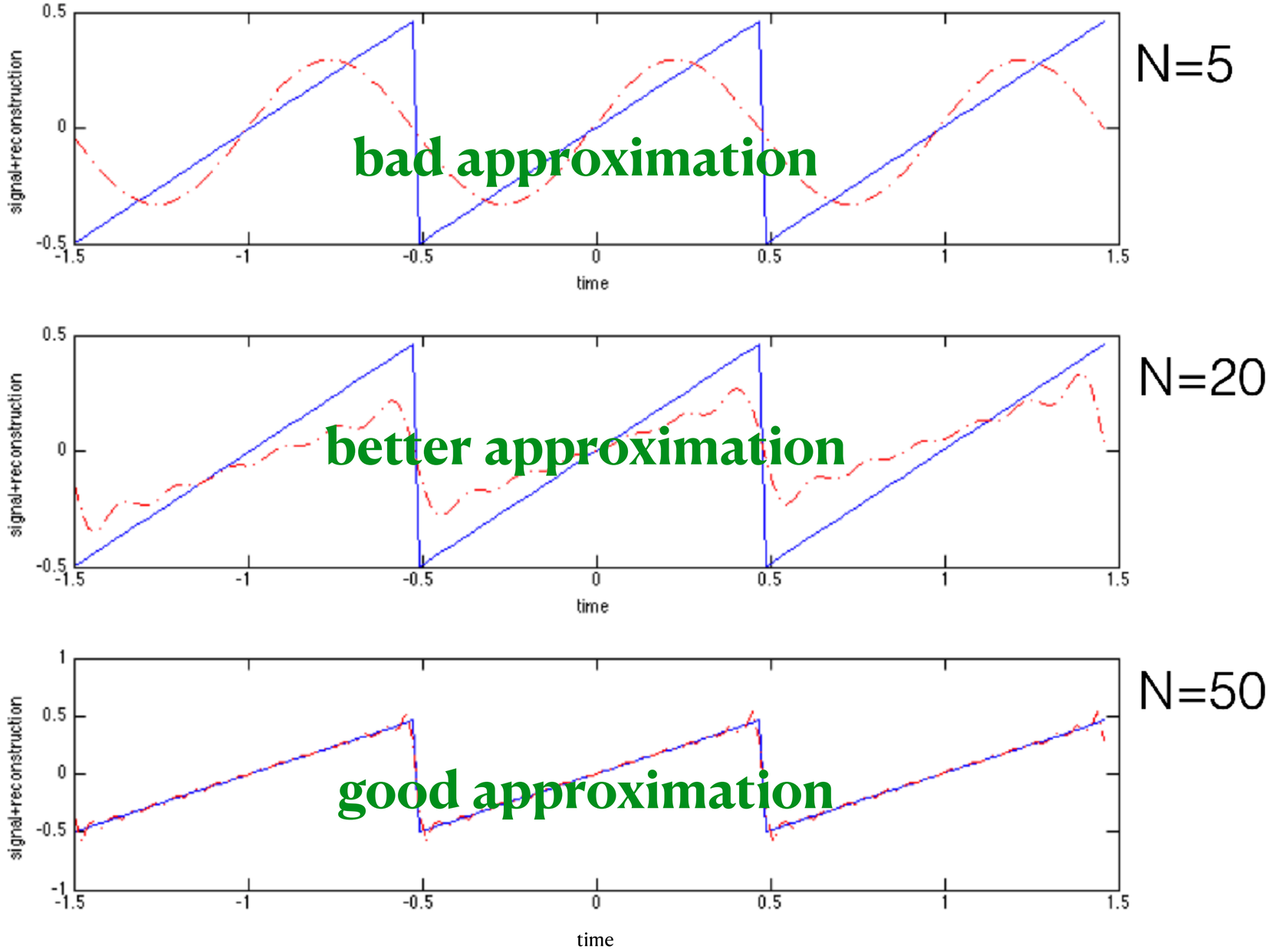

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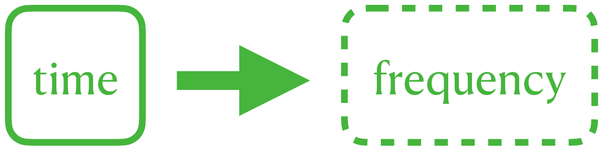

$$c_n = \sum_{k=1}^{N} s(t_k) e^{-i2\pi nk/N}$$

Discrete Fourier Transform (DFT)

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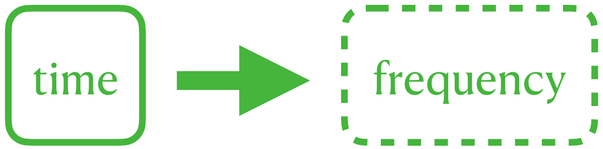

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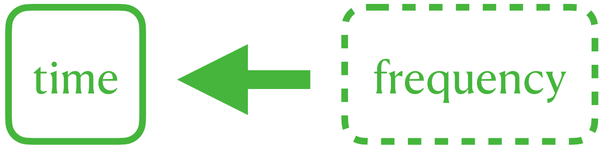

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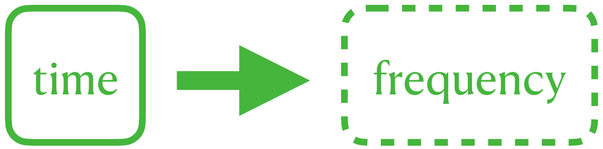

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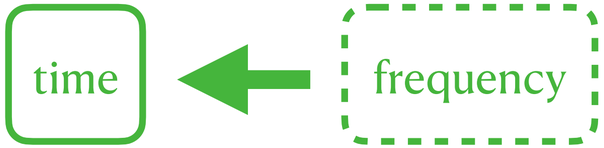

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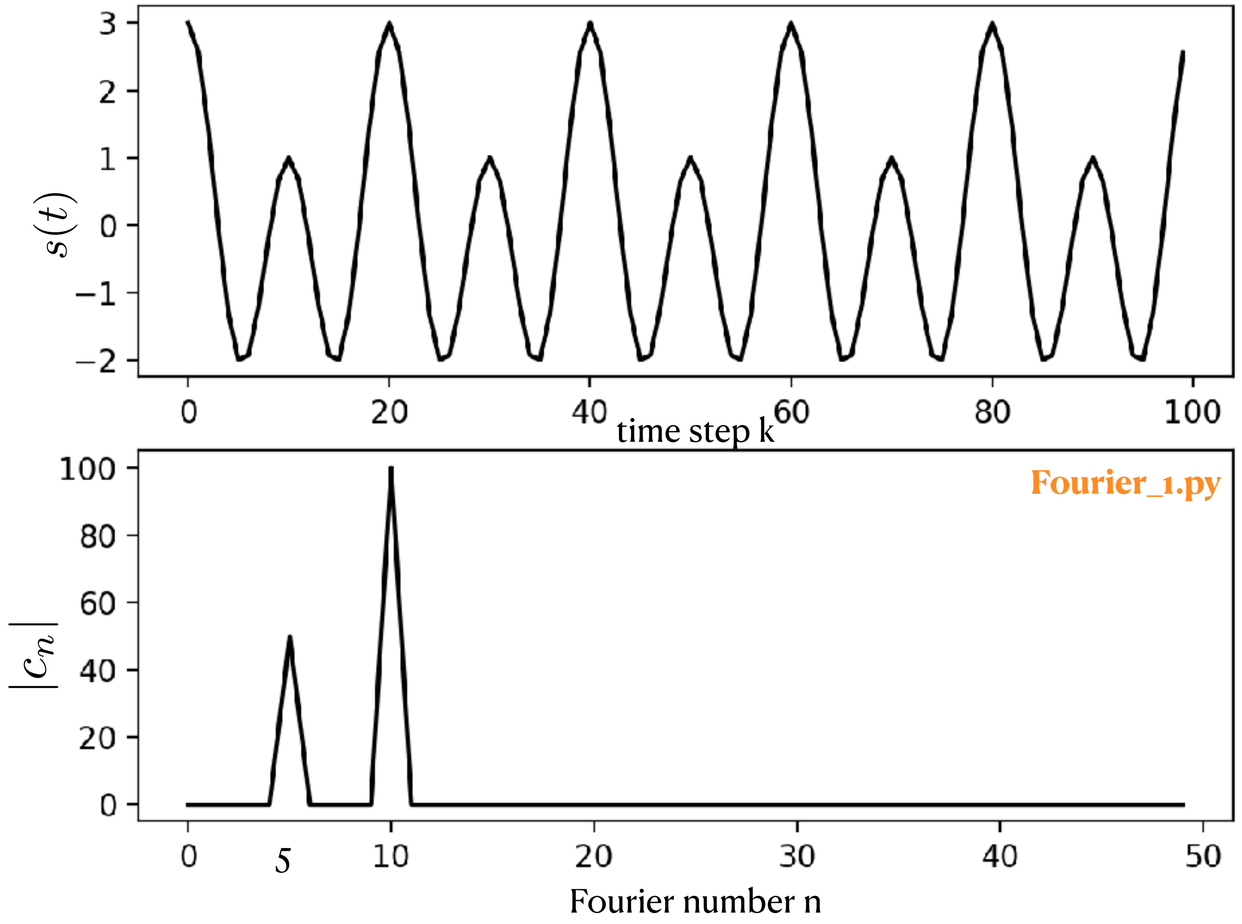

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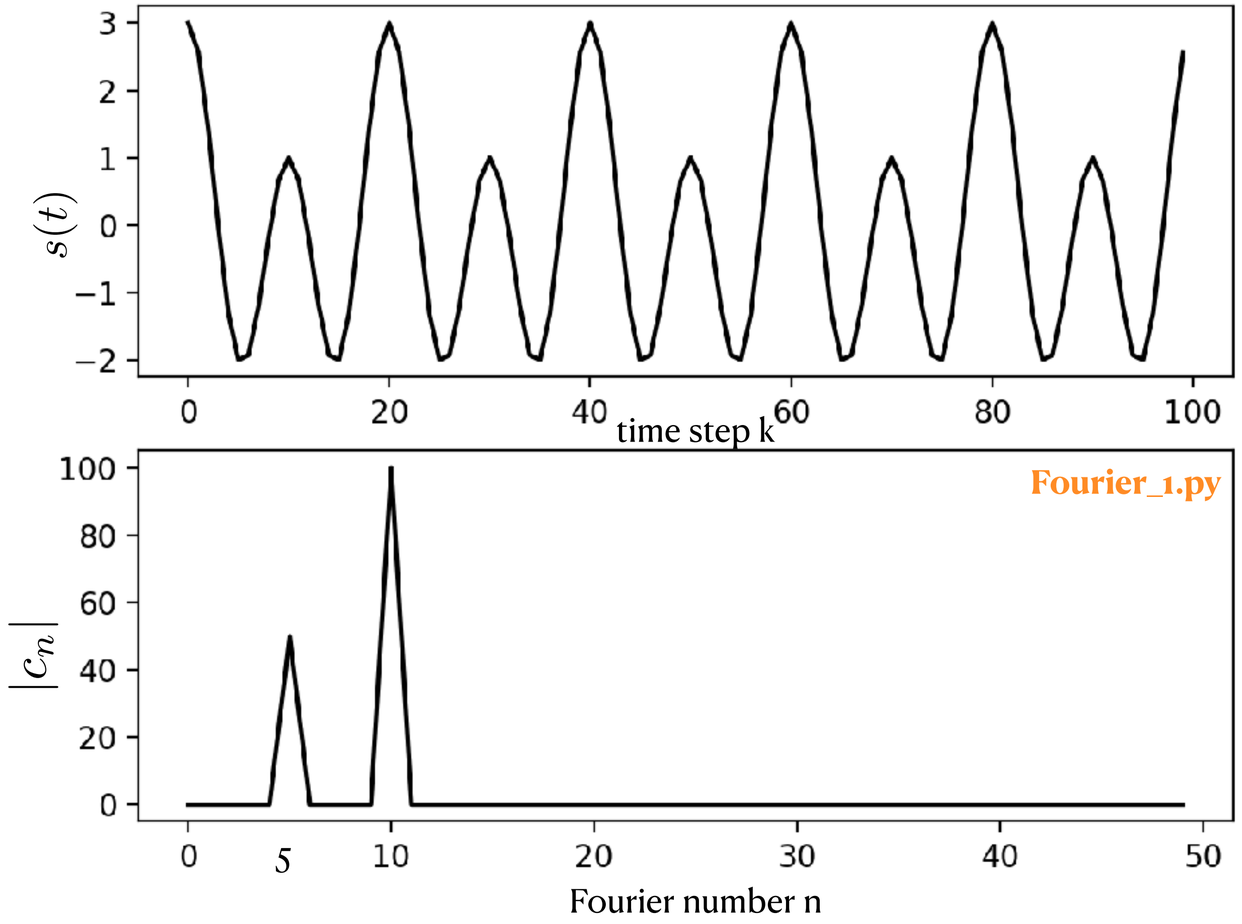

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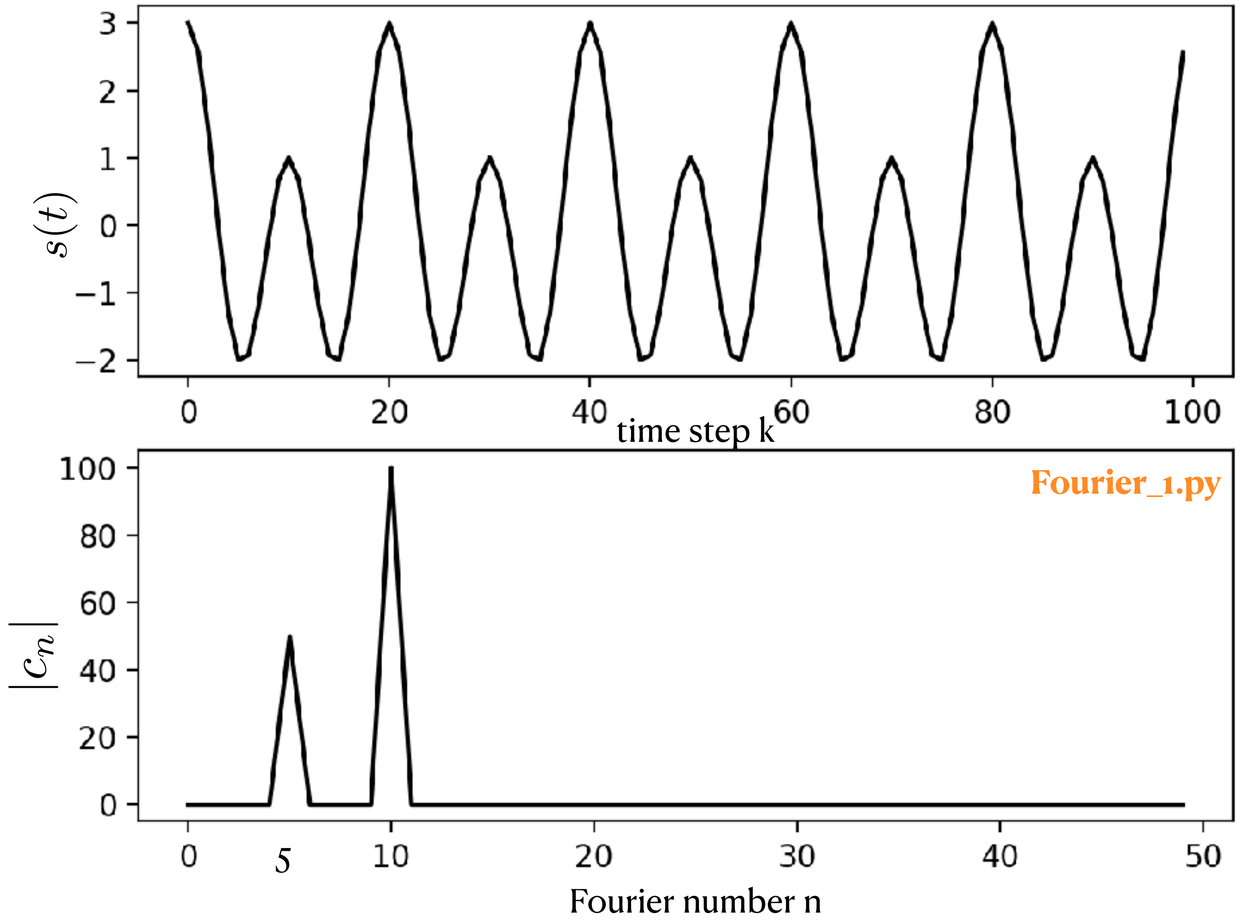

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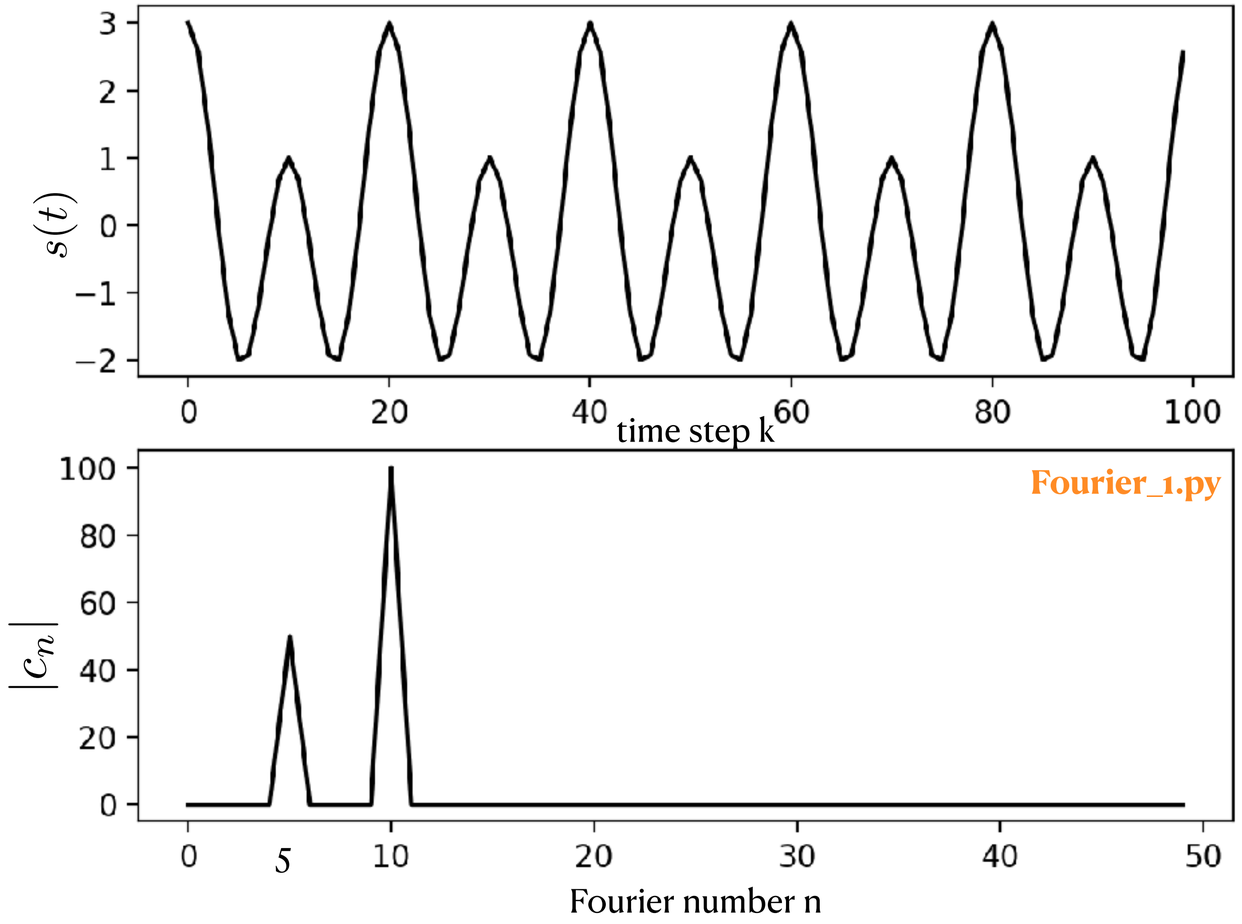

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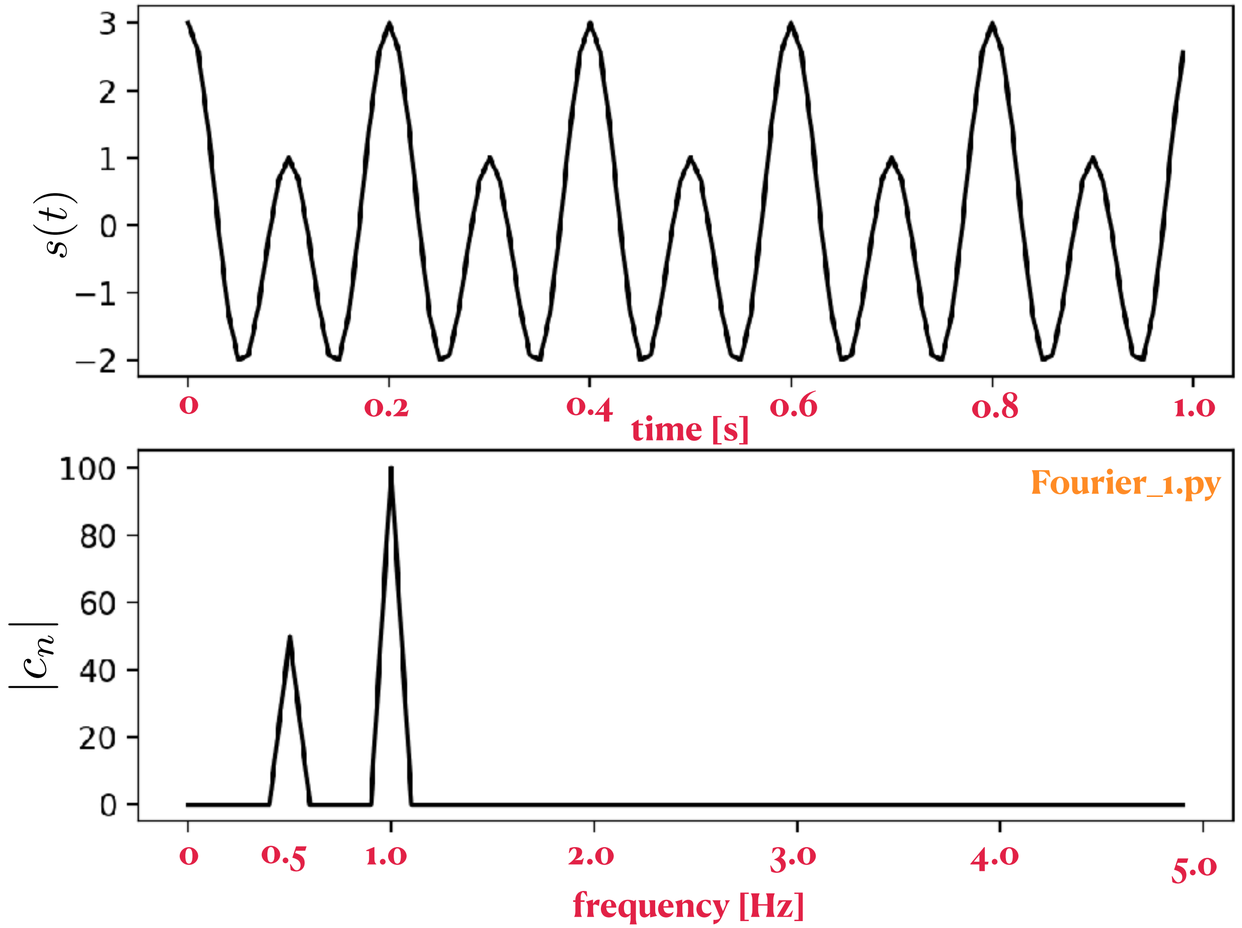
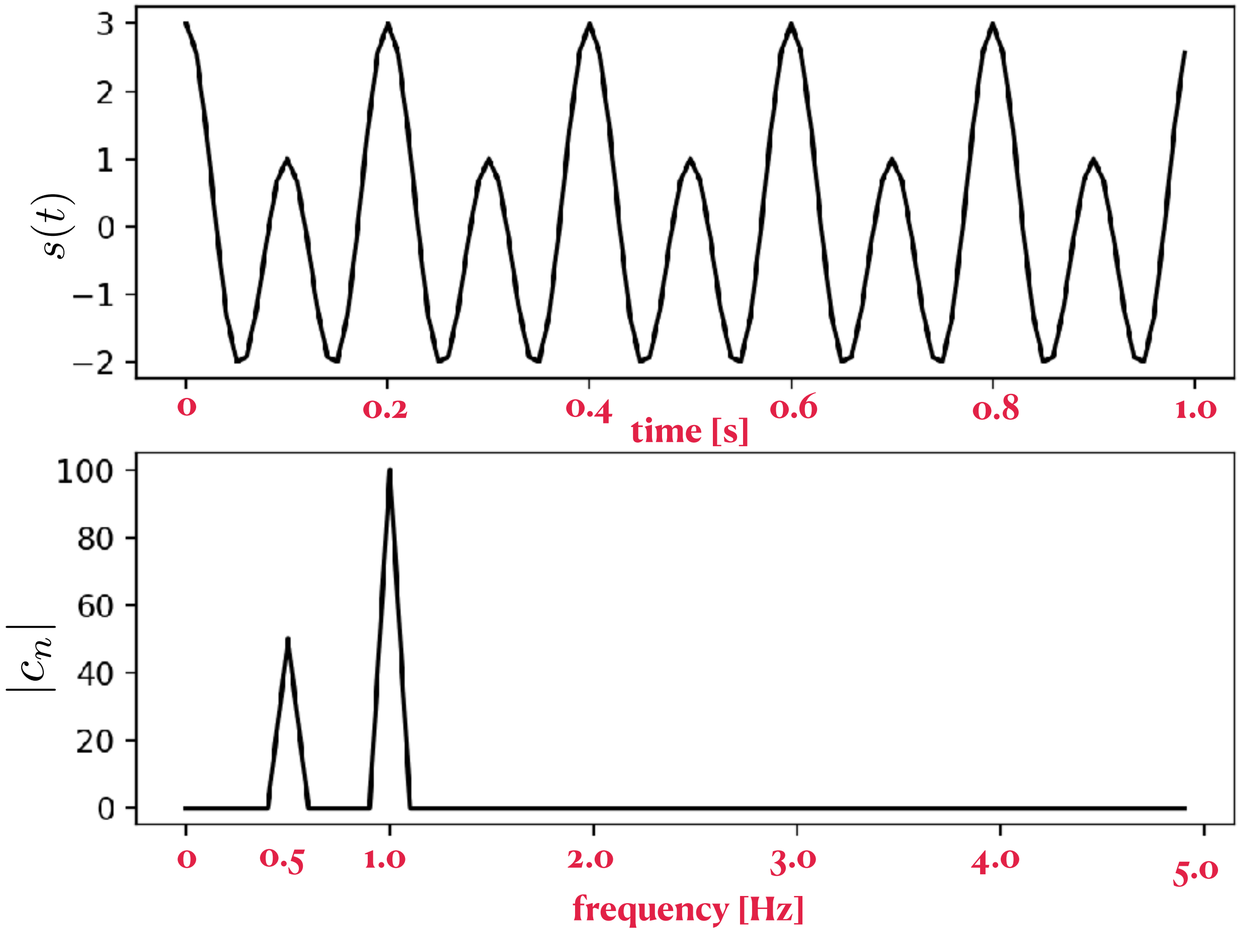


We note:

- for the spectral power it is sufficient to consider

$$0 \leq f \leq f_s/2$$



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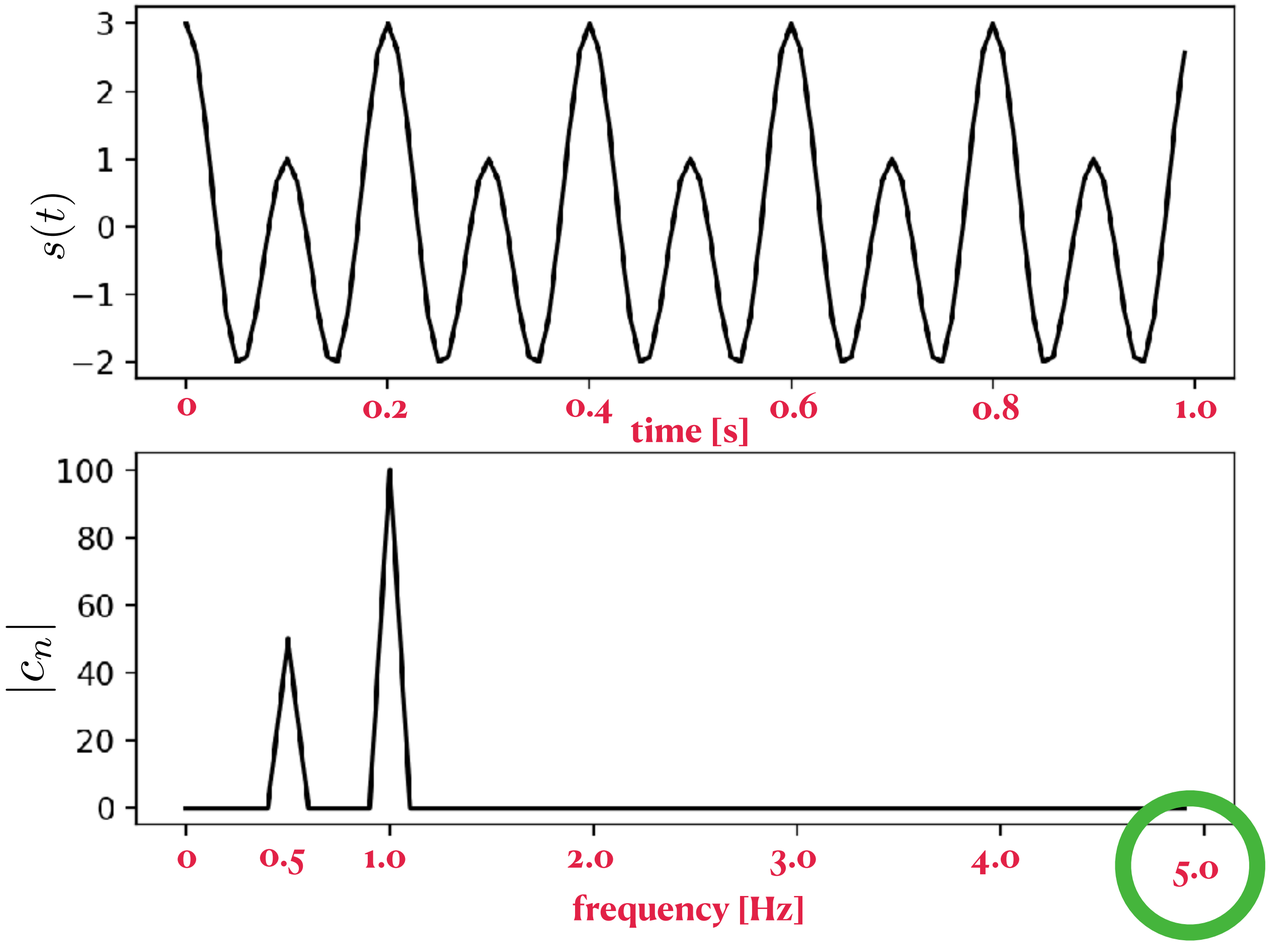
- maximum frequency of DFT is

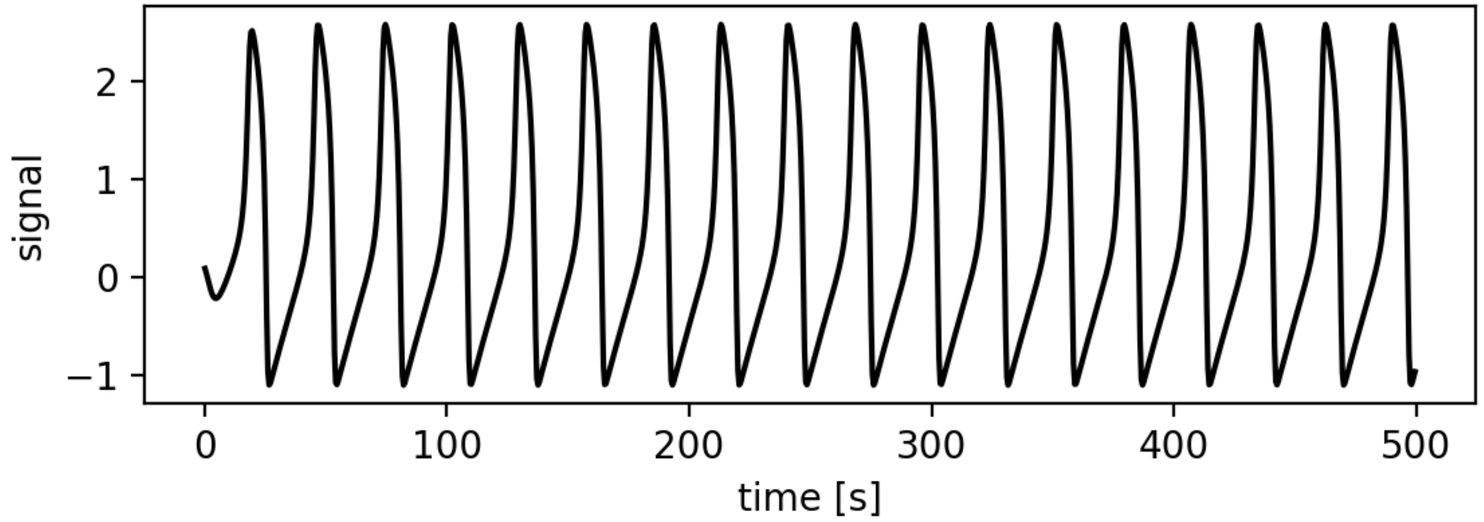
Nyquist frequency $f_s/2$

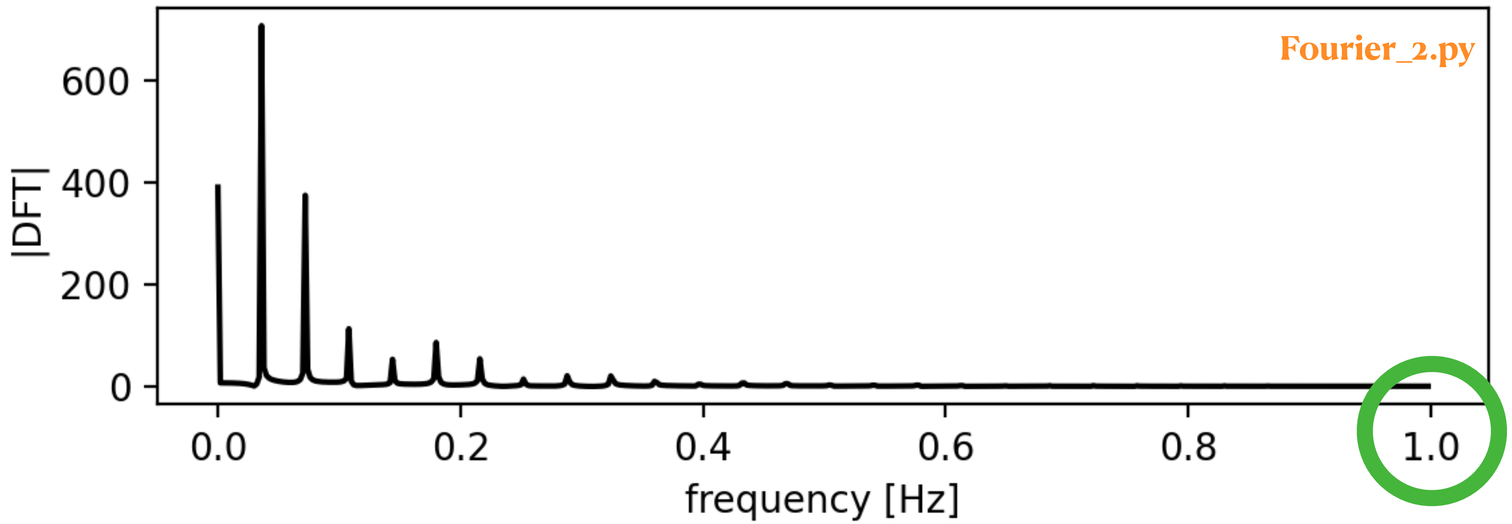


example signal: FitzHugh-Nagumo model activity

- T = 500s $\Delta f = 0.002Hz$
- f_s = 2Hz

Nyquist frequency is 1Hz
= maximum frequency





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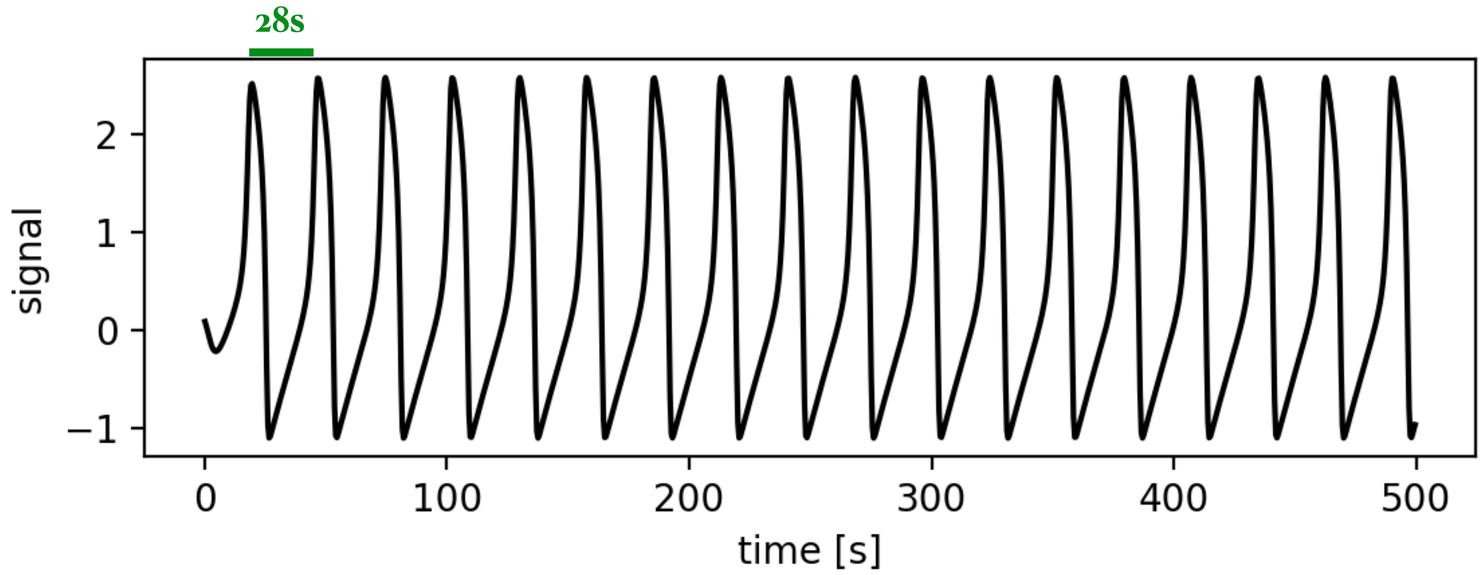
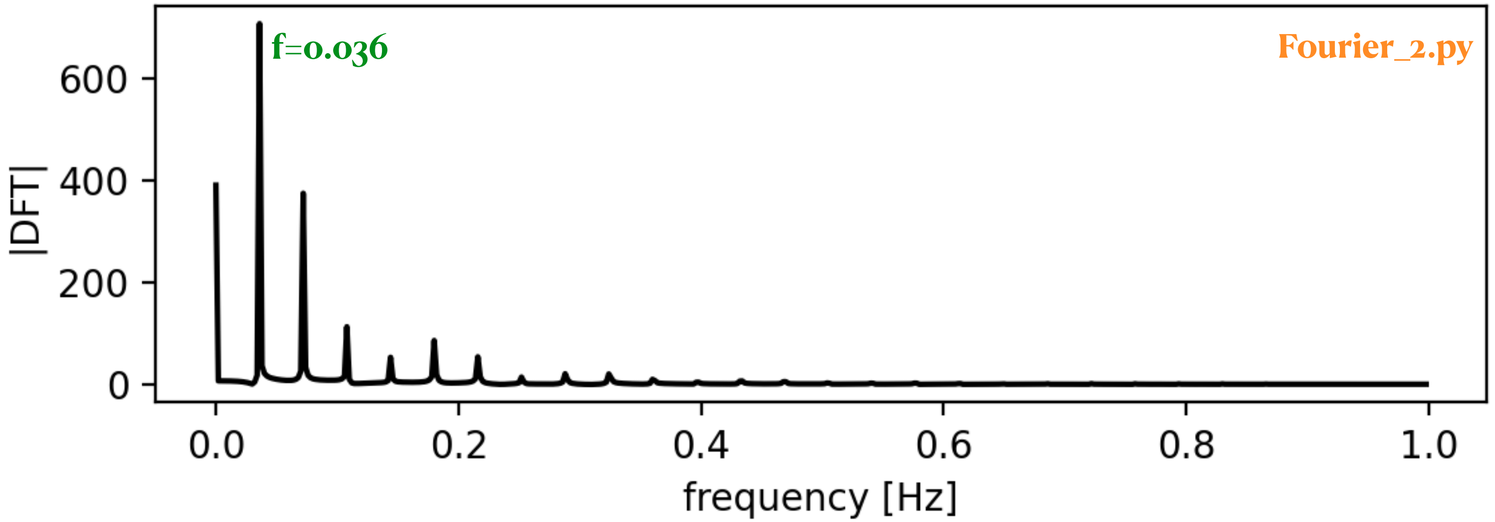


Nyquist frequency is 1Hz
= maximum frequency

principal data frequency
≈ highest power frequency



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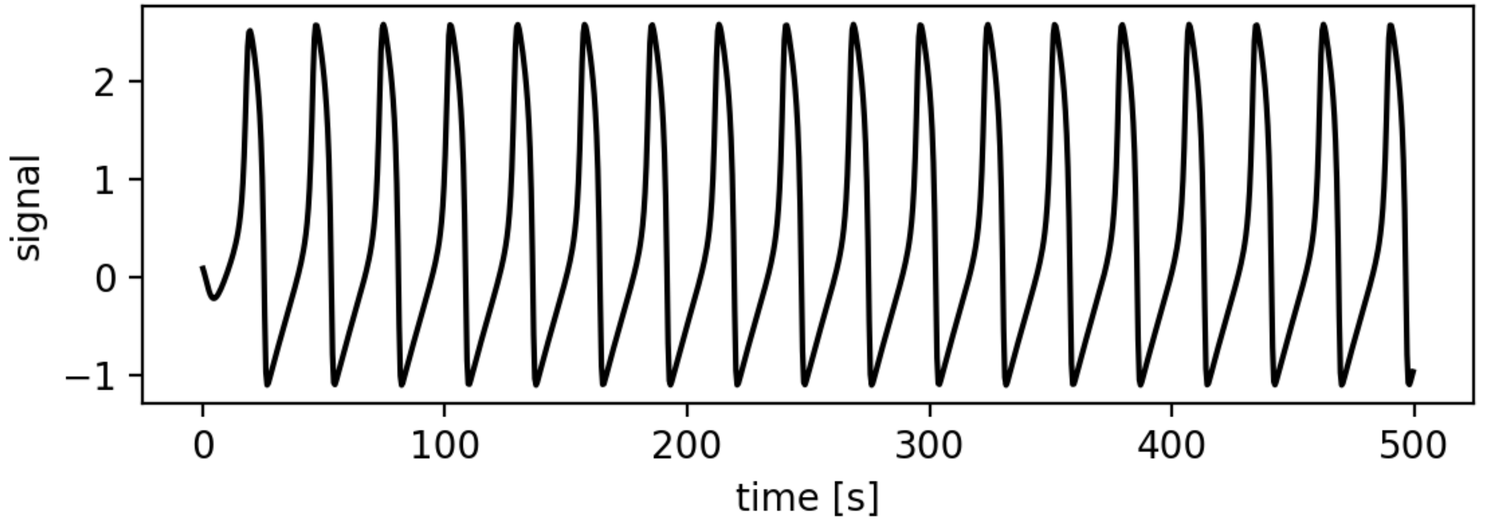

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DFT has multiple power peaks

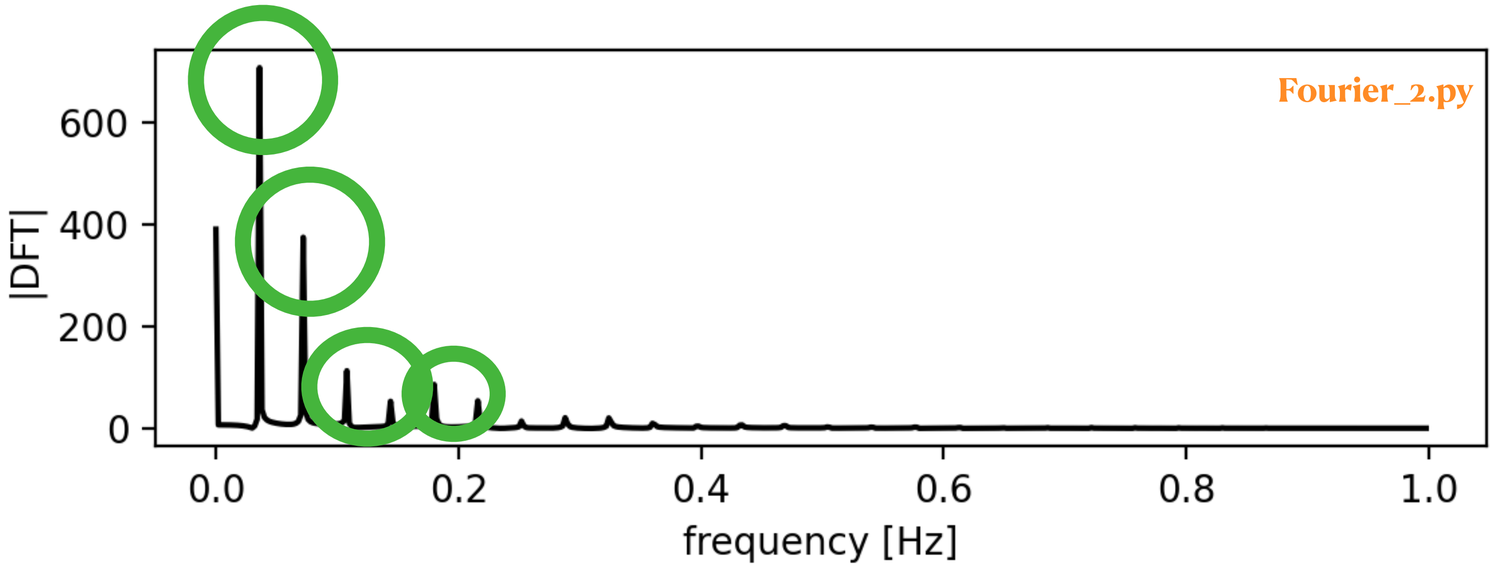
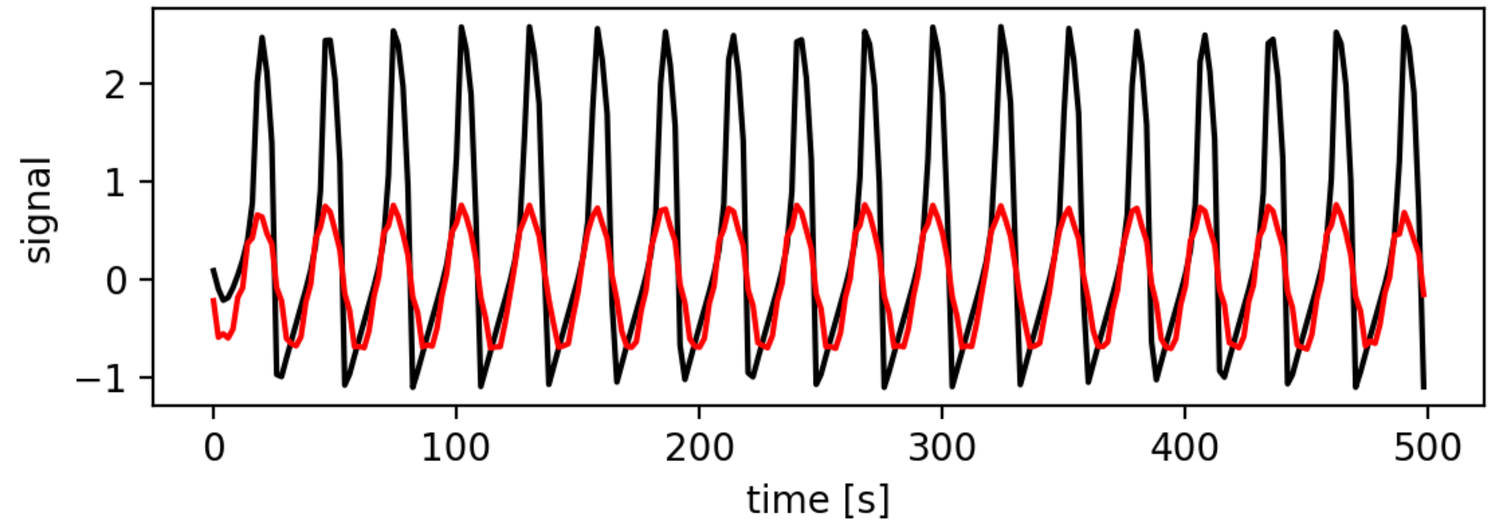
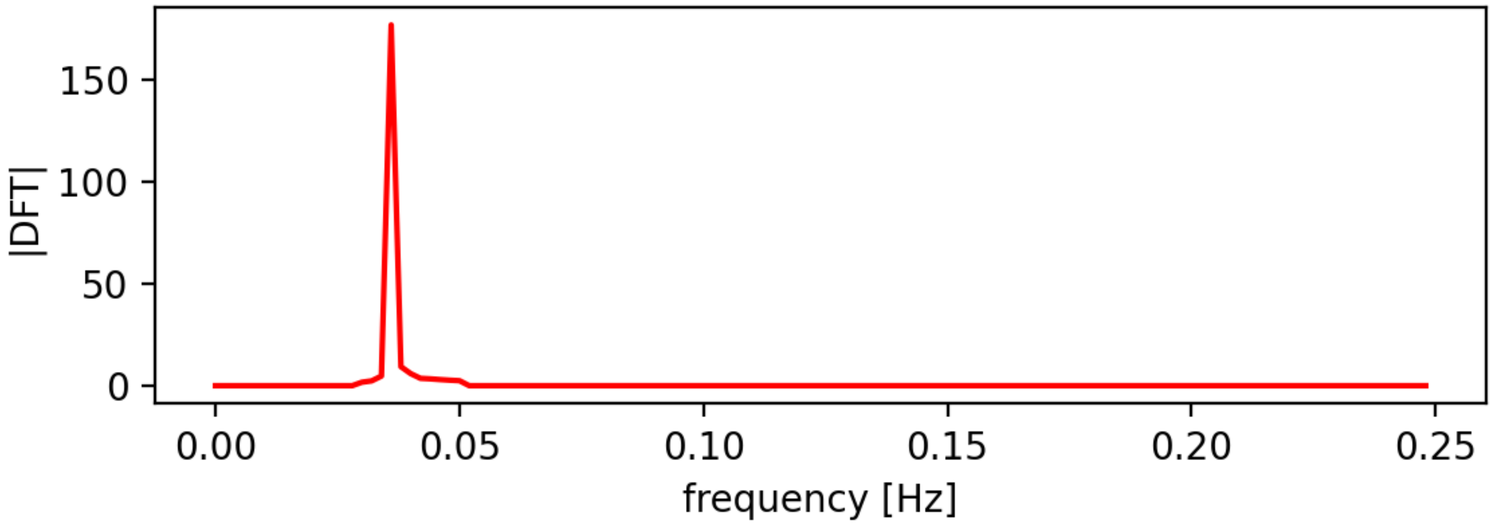
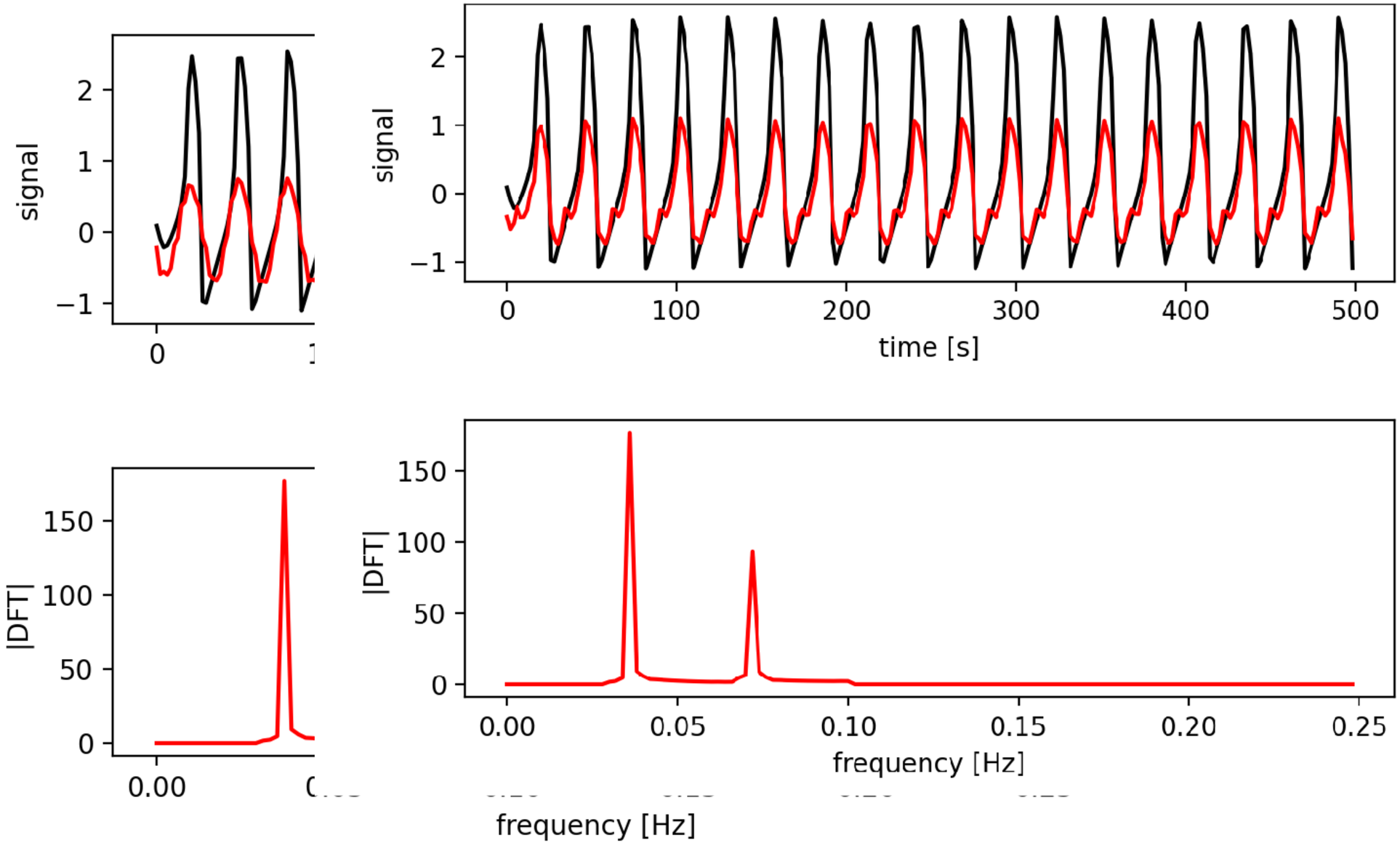
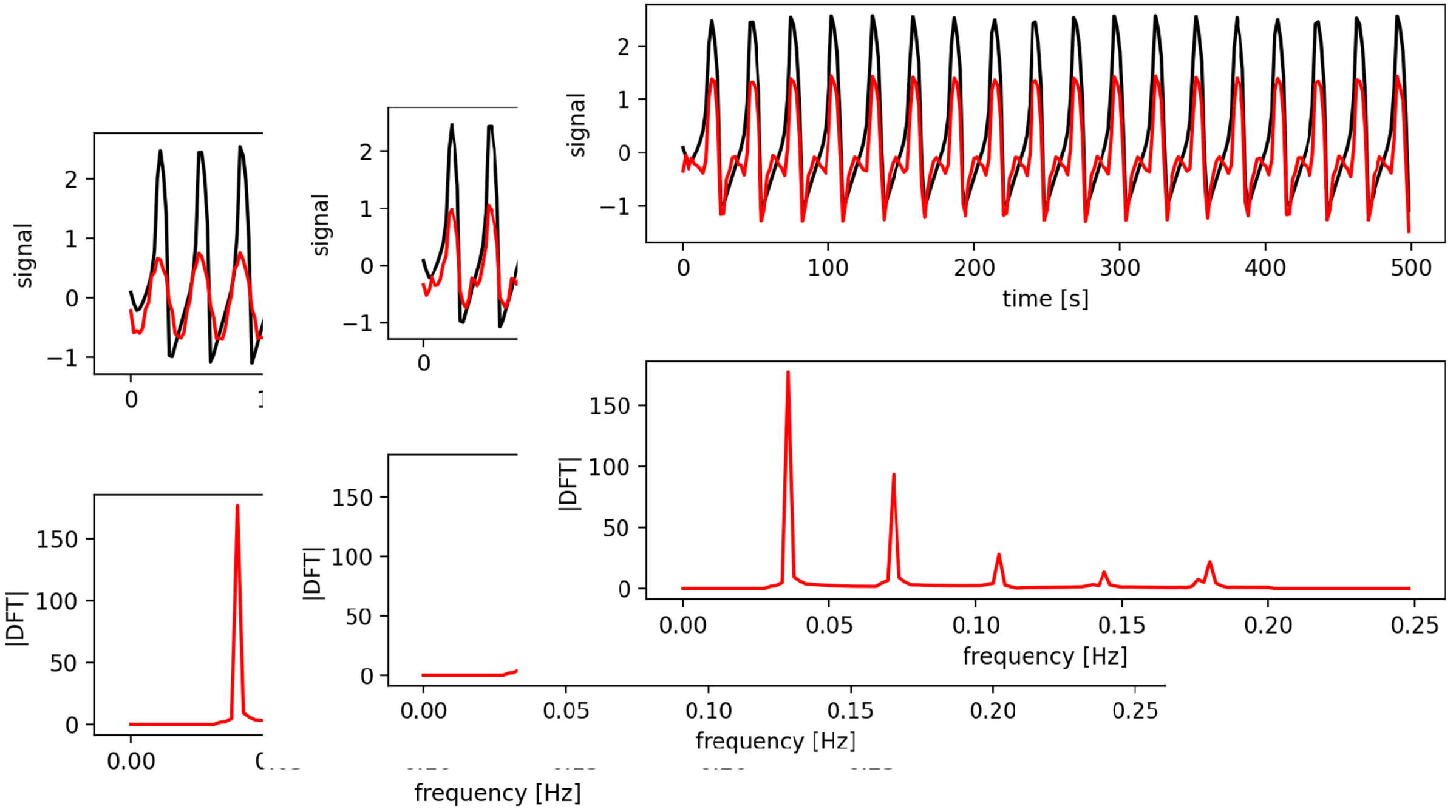


illustration for spectral decomposition (Fourier_2.py)






signal
2
1
0
−1
0
signal
2
1
0
−1
0
100
200
300
400
500
time [s]
|DFT|
150
100
50
0
0.00
|DFT|
150
100
50
0
0.00
0.05
0.10
0.15
0.20
0.25
frequency [Hz]
frequency [Hz]


signal
time [s]
|DFT|
frequency [Hz]

data sampling

Fourier analysis

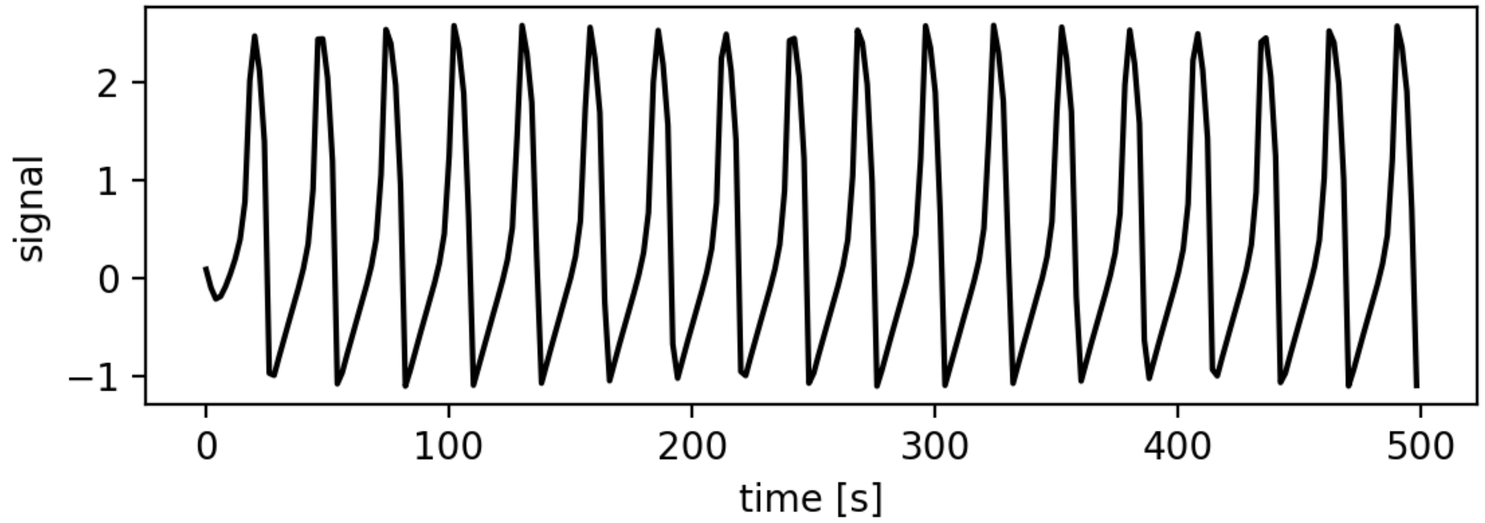
errors in analysis

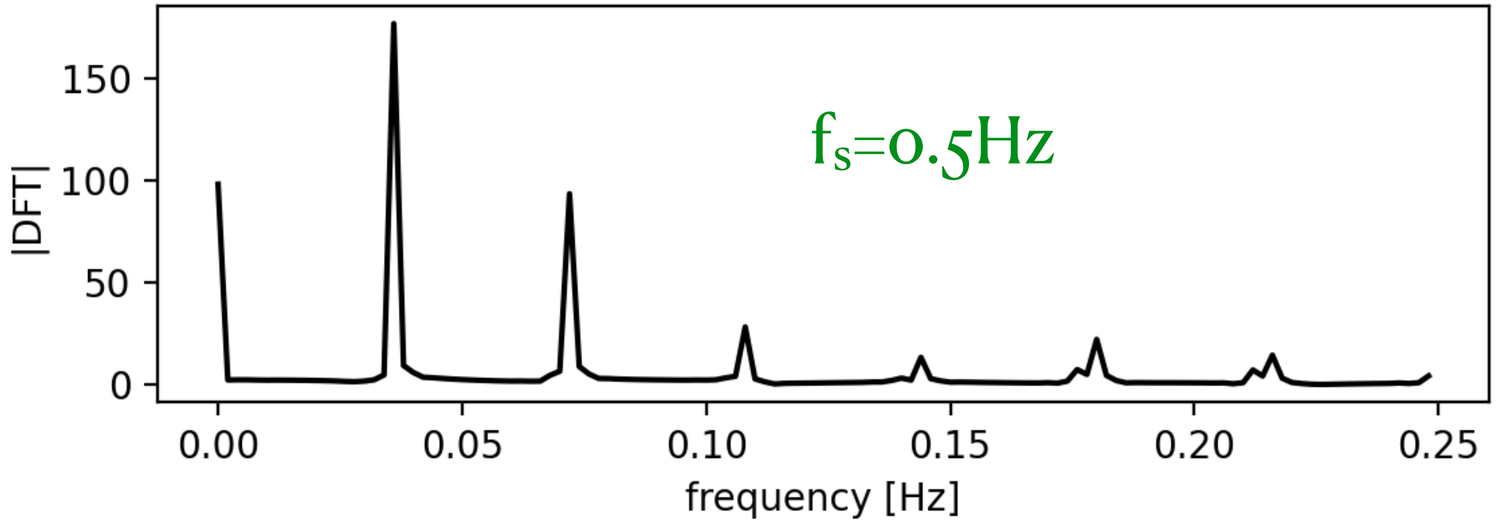
linear filters

time-frequency analysis

Error: subsampling

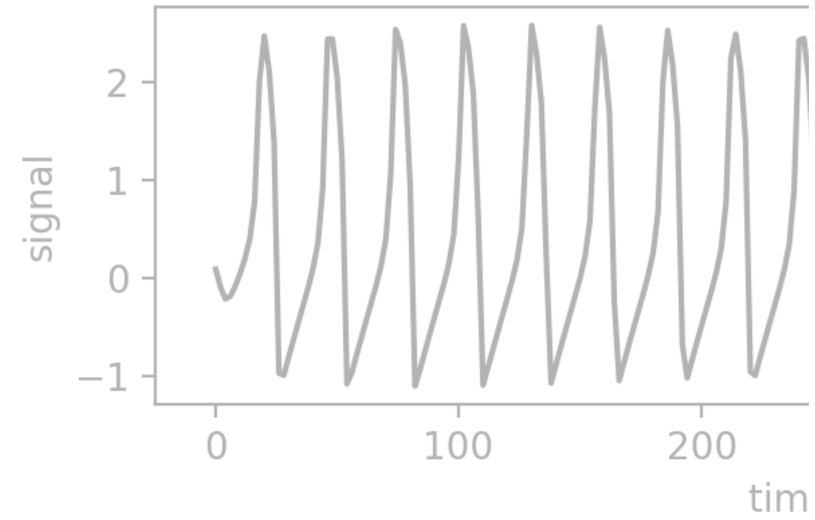
SamplingError_1.py

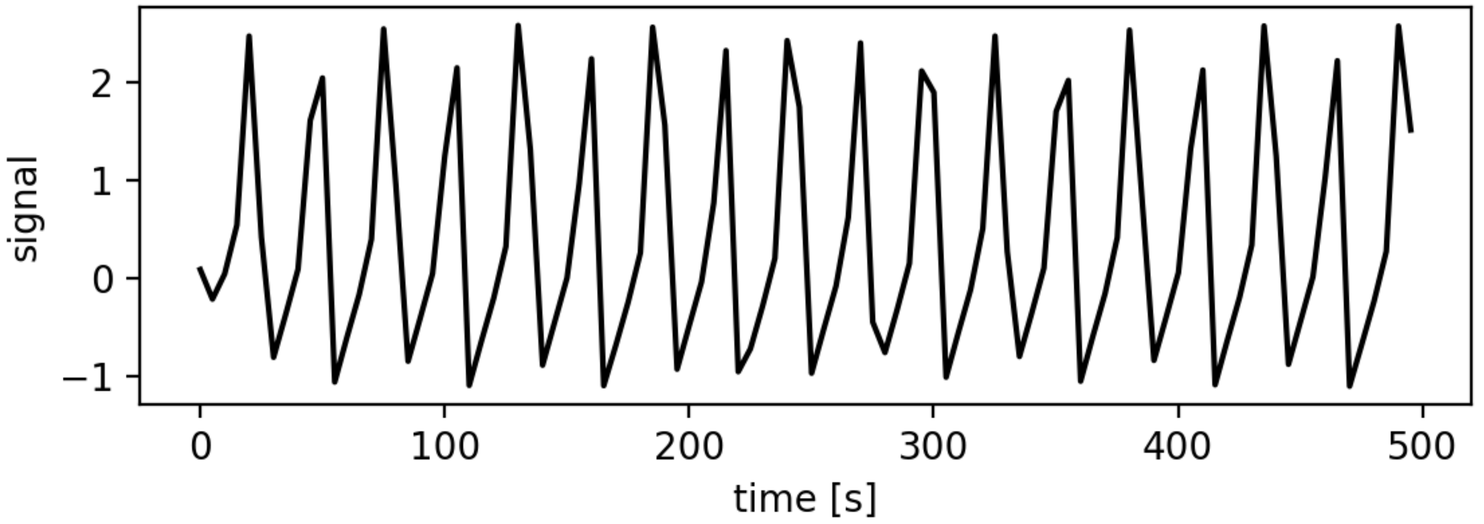


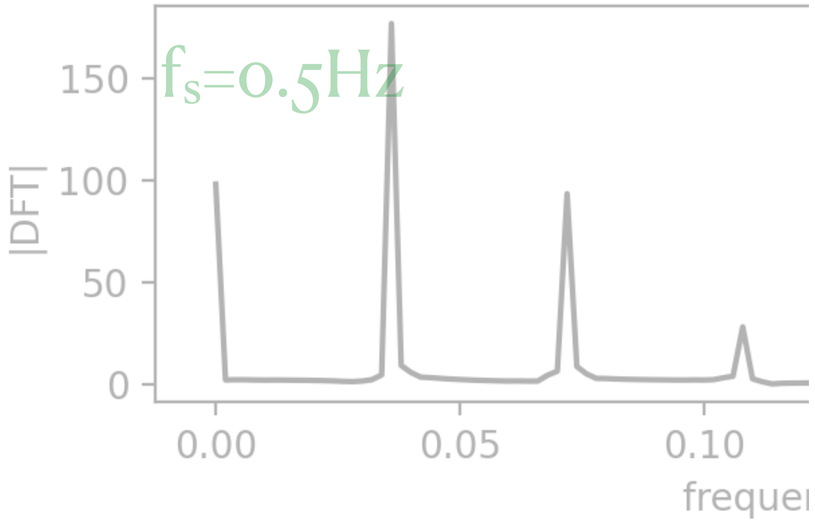


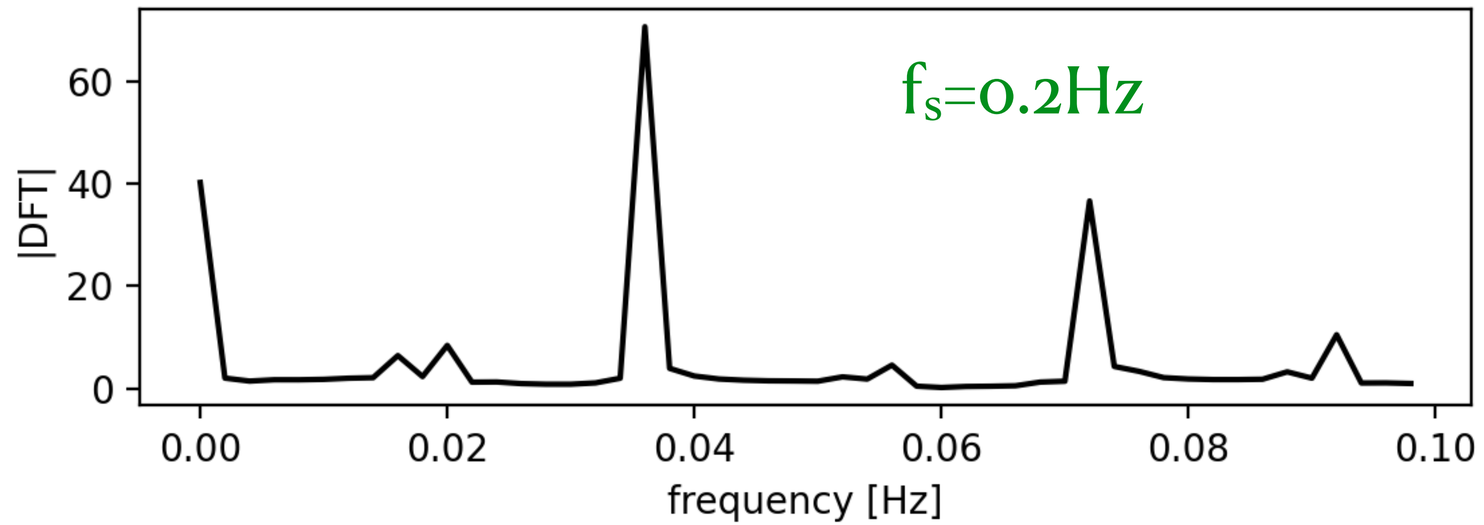
f_s=0.5Hz

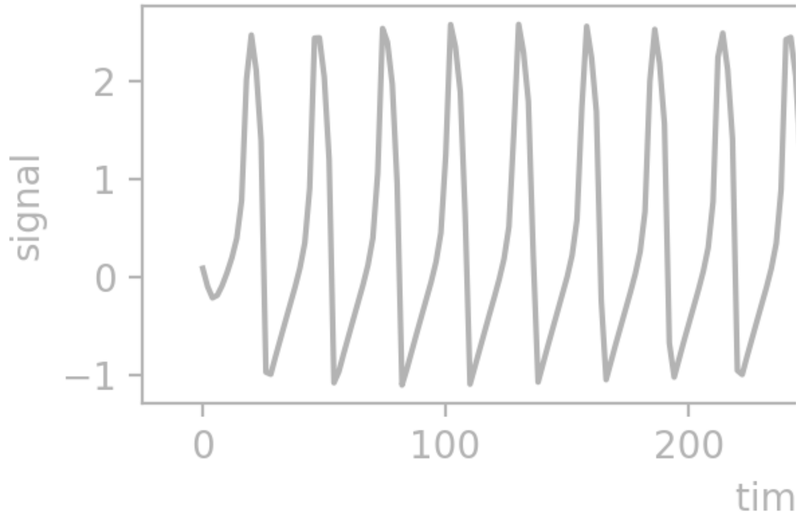
SamplingError_1.py

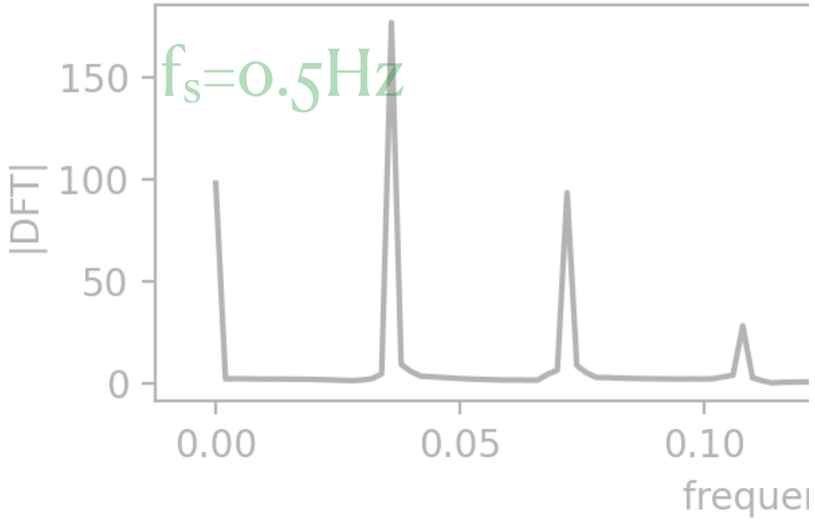




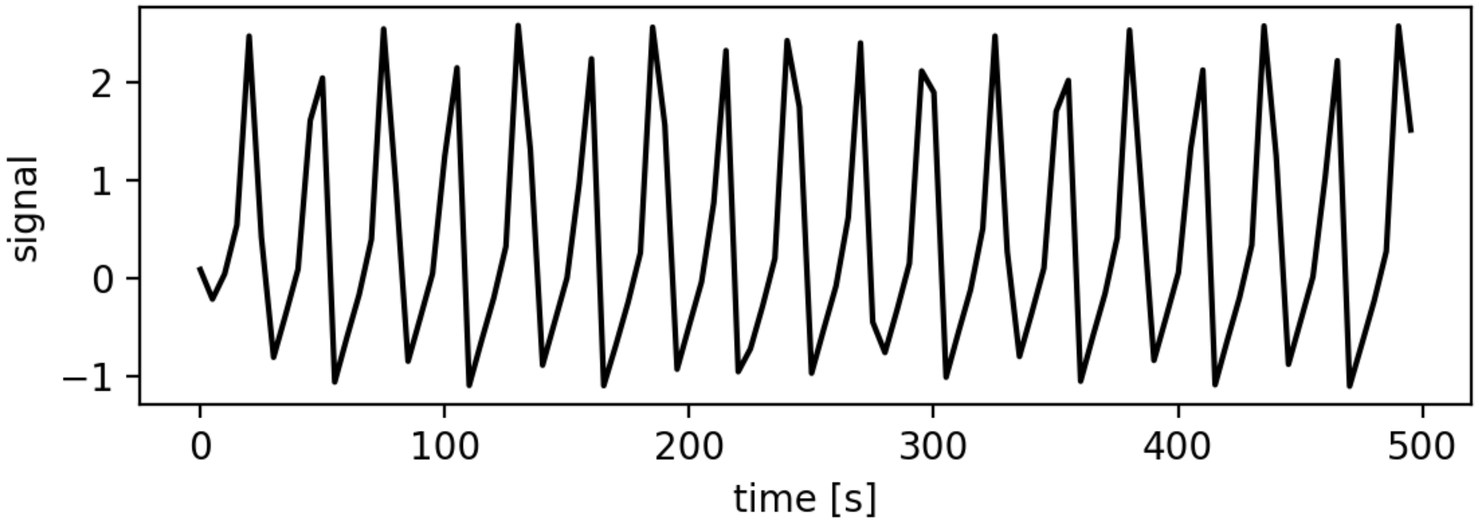


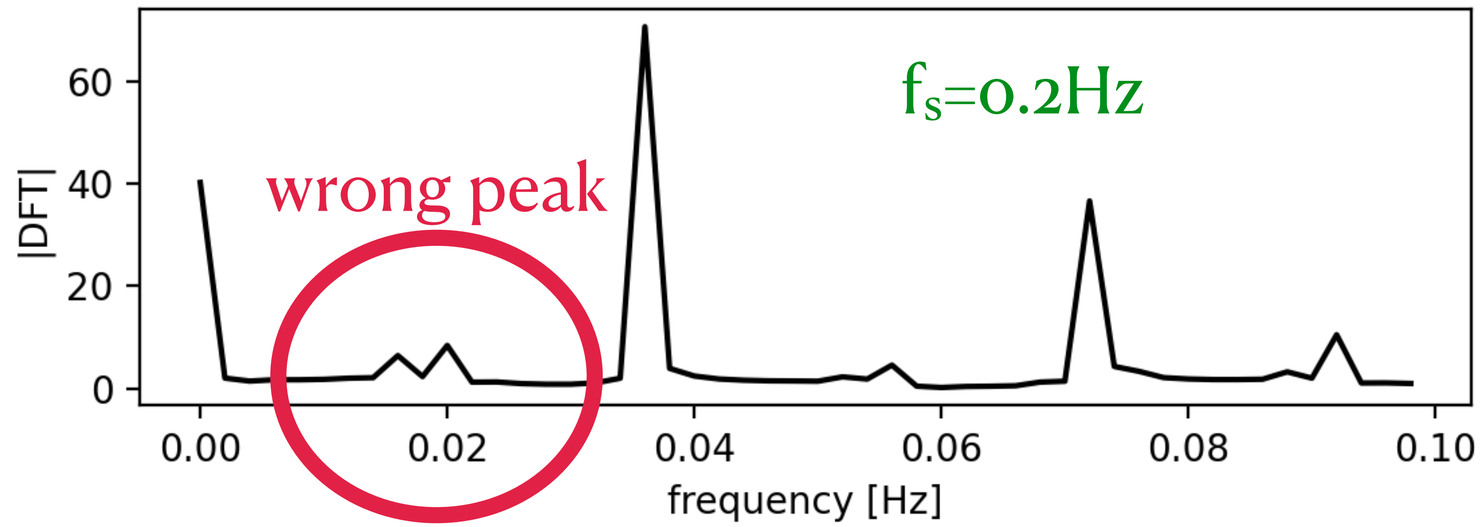






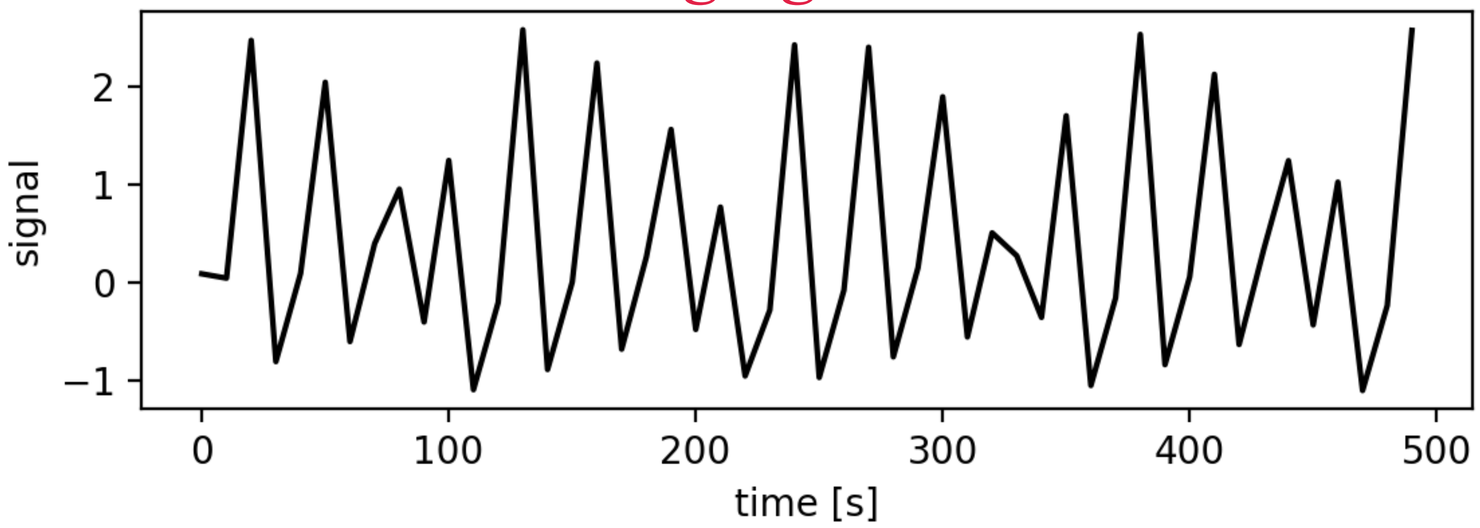
SamplingError_1.py

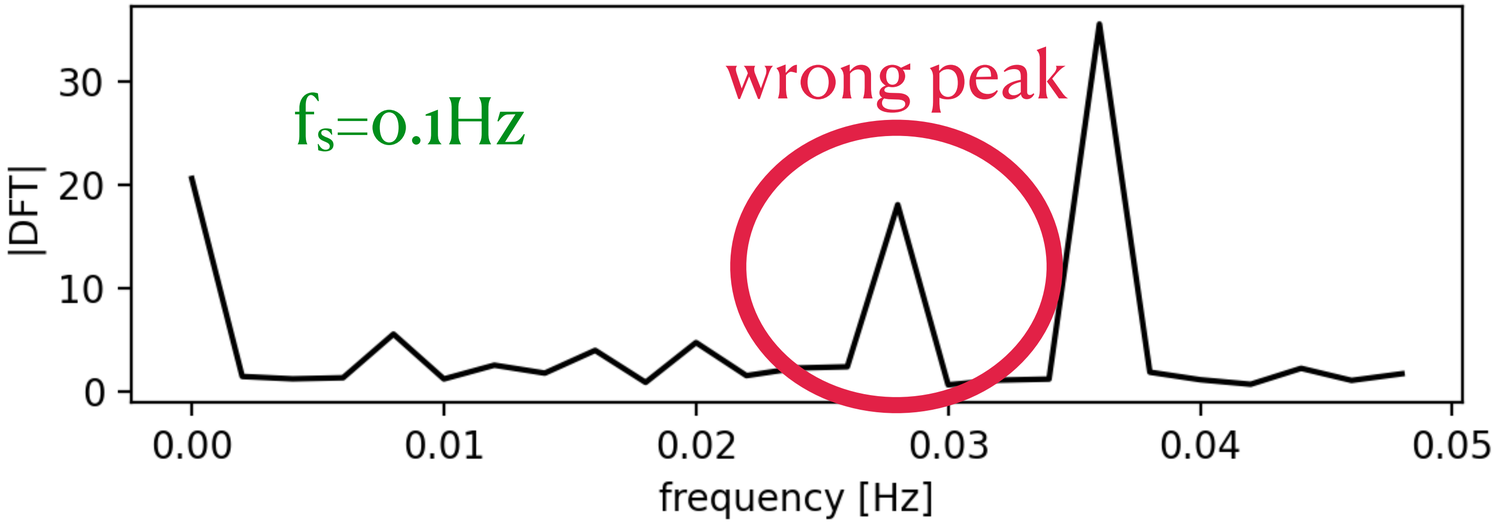


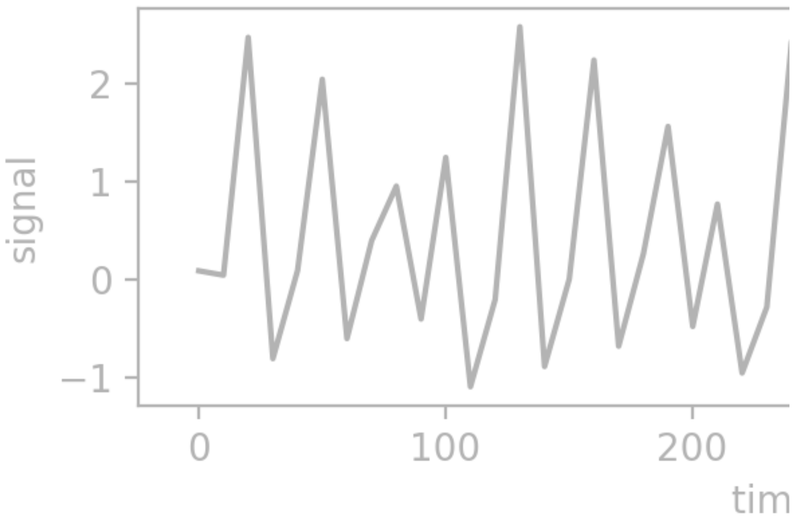


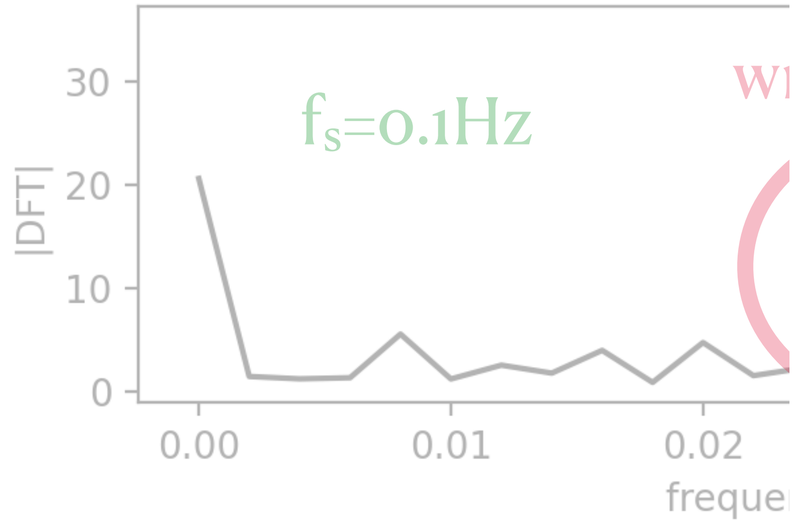
wrong signal

SamplingError_1.py





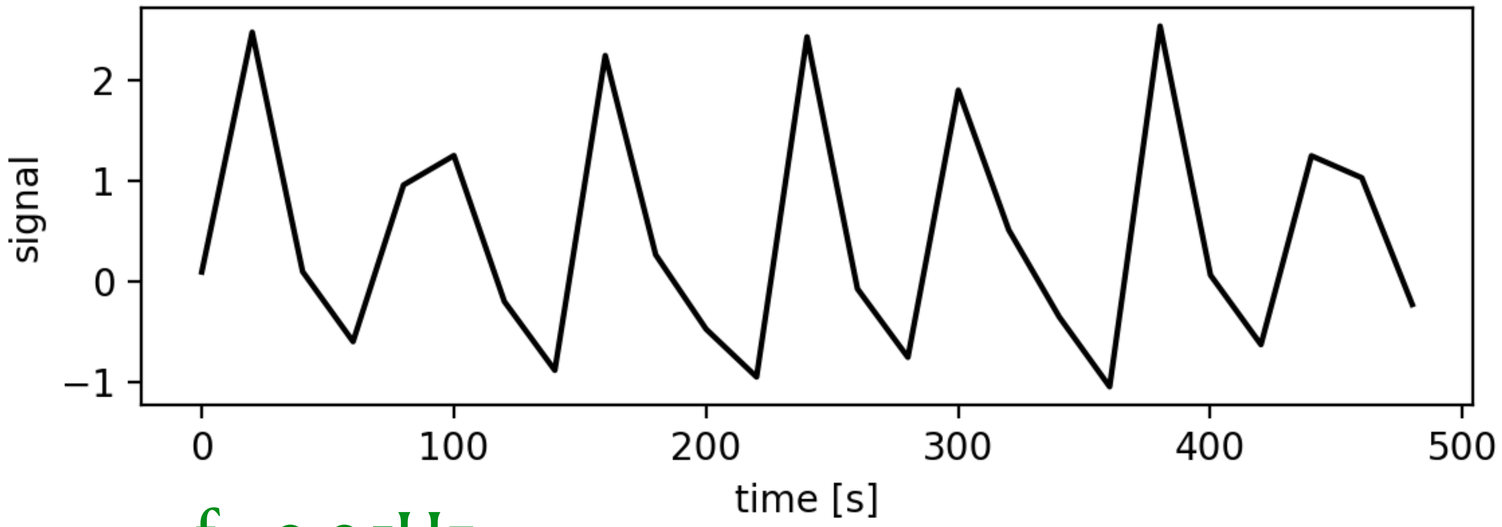




f_s=0.1Hz

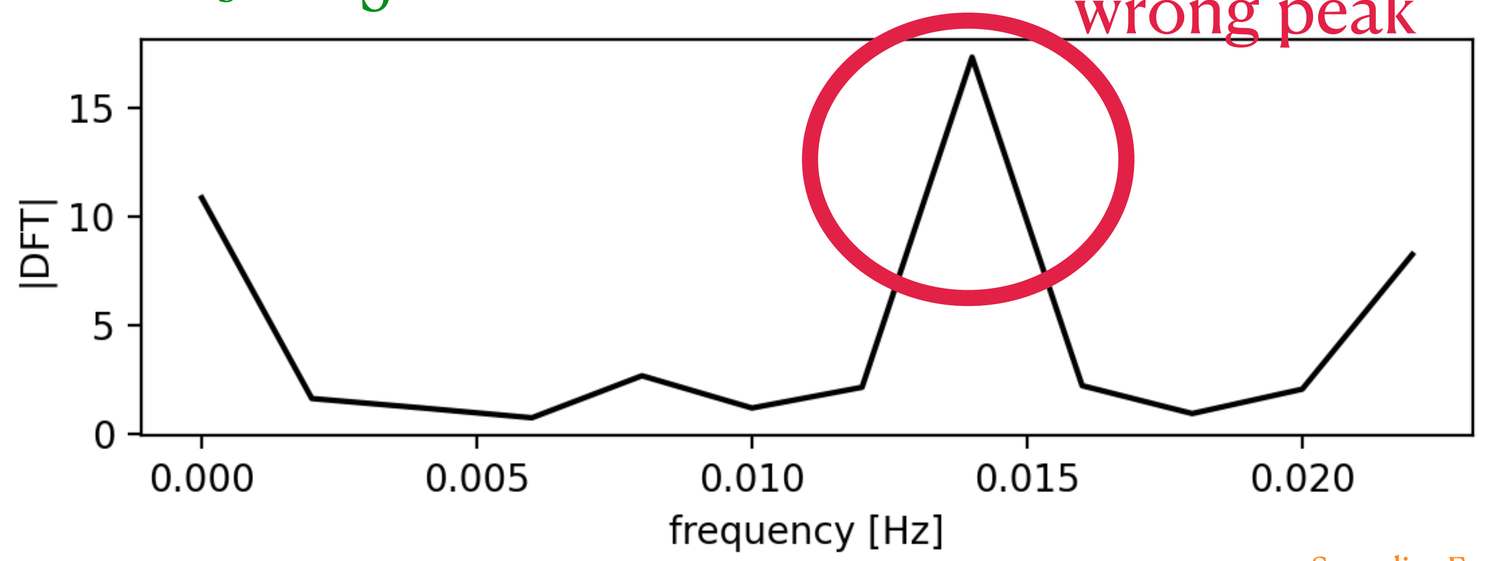
wr

wrong signal



f_s=0.05Hz

wrong peak



SamplingError_1.py

data sampling

Fourier analysis

errors in analysis **aliasing** spectral leakage spectral power

linear filters

time-frequency analysis

- assume: we have a signal s(t) and its Fourier Transform X(f)

- assume: we have a signal s(t) and its Fourier Transform X(f)

- then the Poisson summation formula yields:

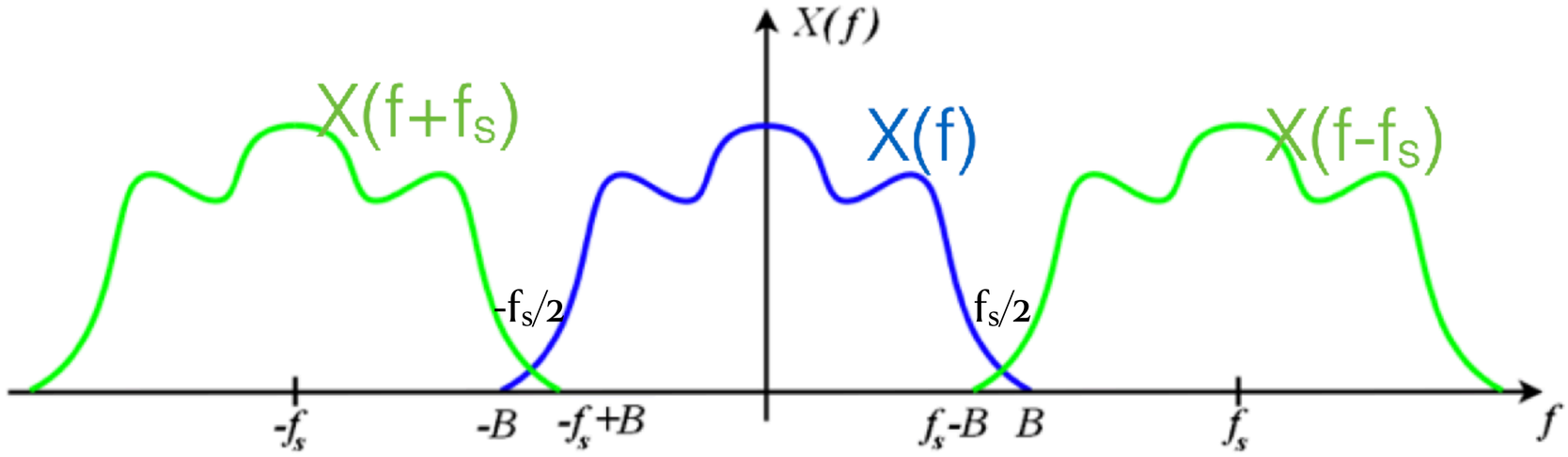
$$\sum_{k=1}^{N} s(t_k) e^{-i2\pi nk/N} = \frac{1}{\Delta t} \sum_{p=-\infty}^{\infty} X(f_n - p f_s)$$

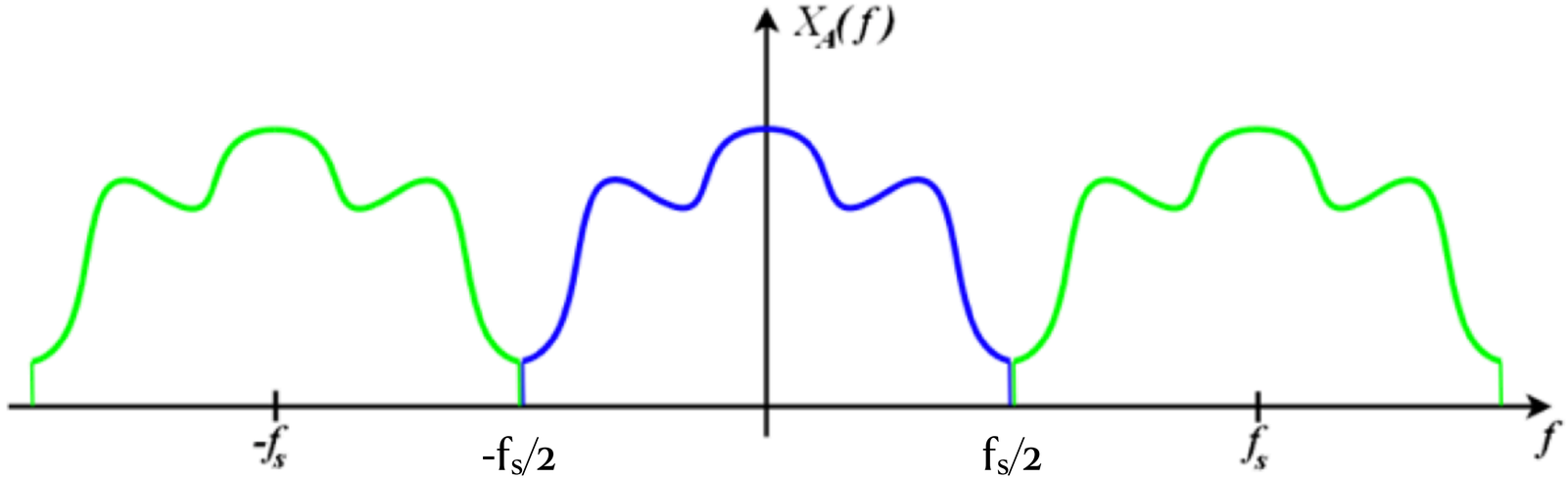
DFT

periodic continuation
of Fourier transform with
period f_s

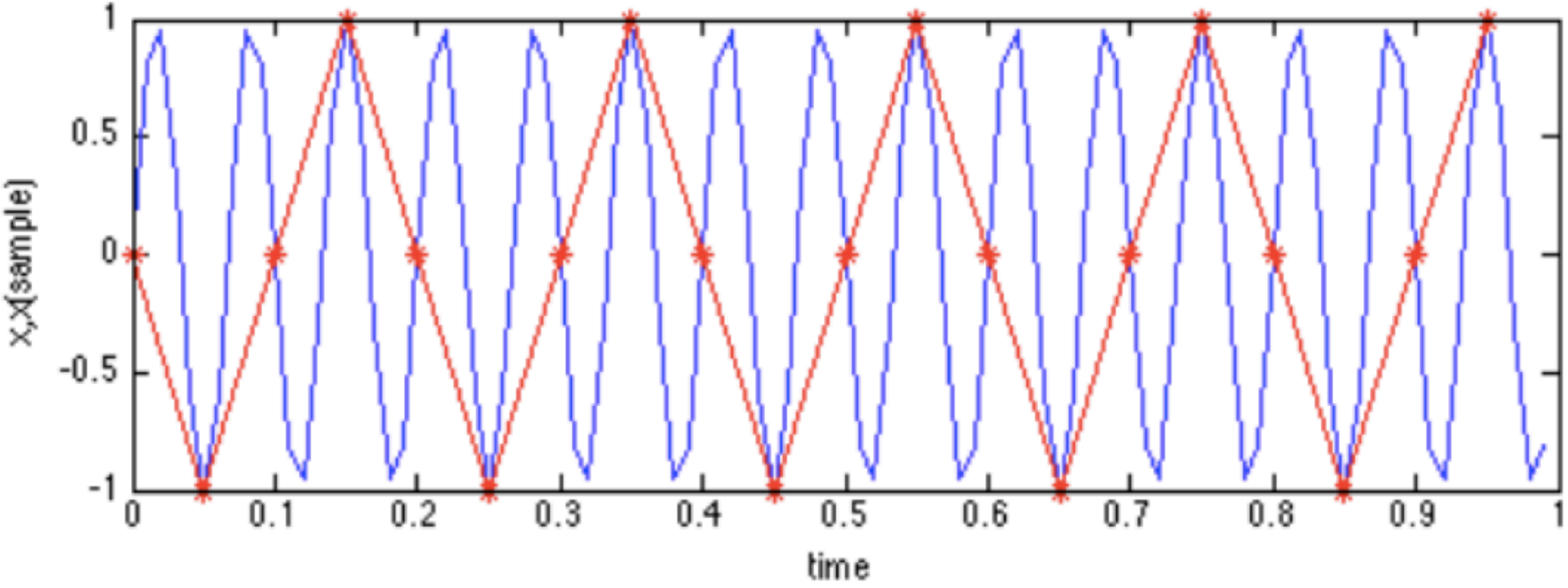
$$\mathrm{DFT} = X_A \sim X(f_n) + X(f_n - f_s) + X(f_n + f_s) + \cdots$$

$$\text{DFT} = X_A \sim X(f_n) + X(f_n - f_s) + X(f_n + f_s) + \cdots$$



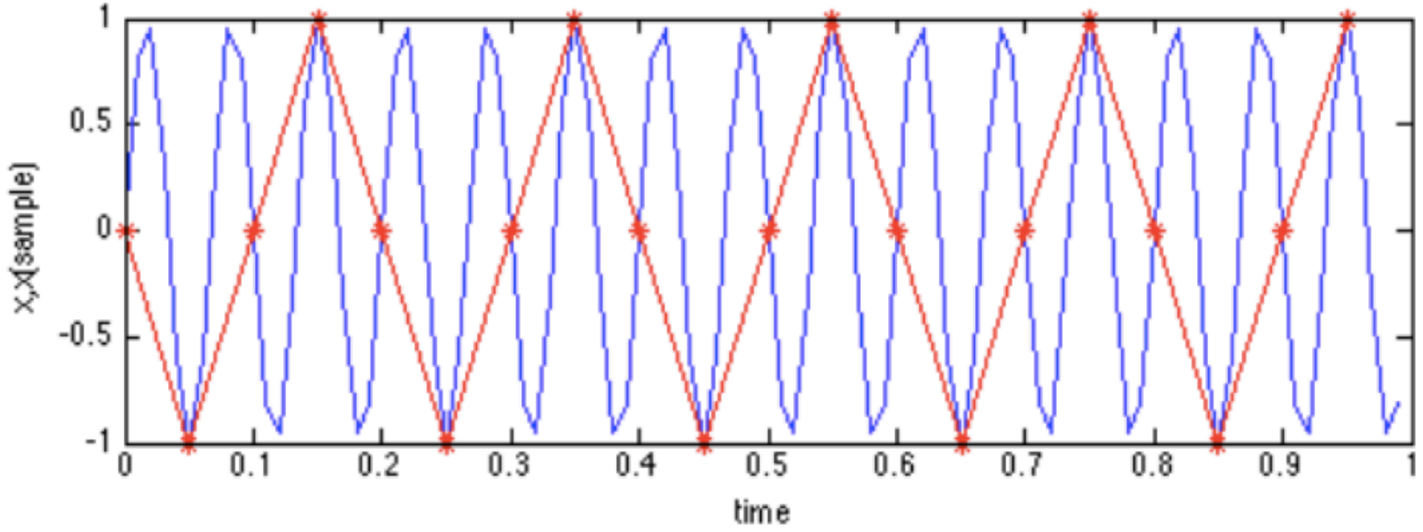


Example: sub-sampled periodic signal



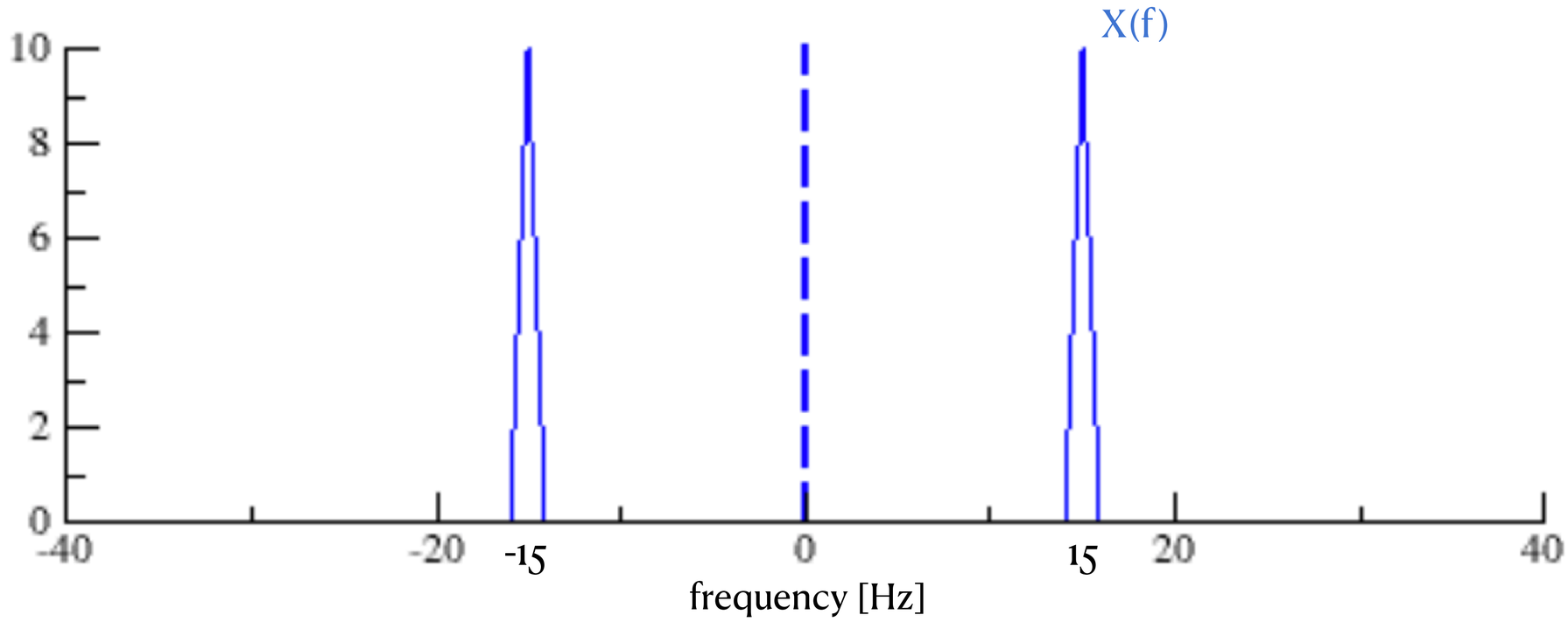
original time series with f_m=15Hz

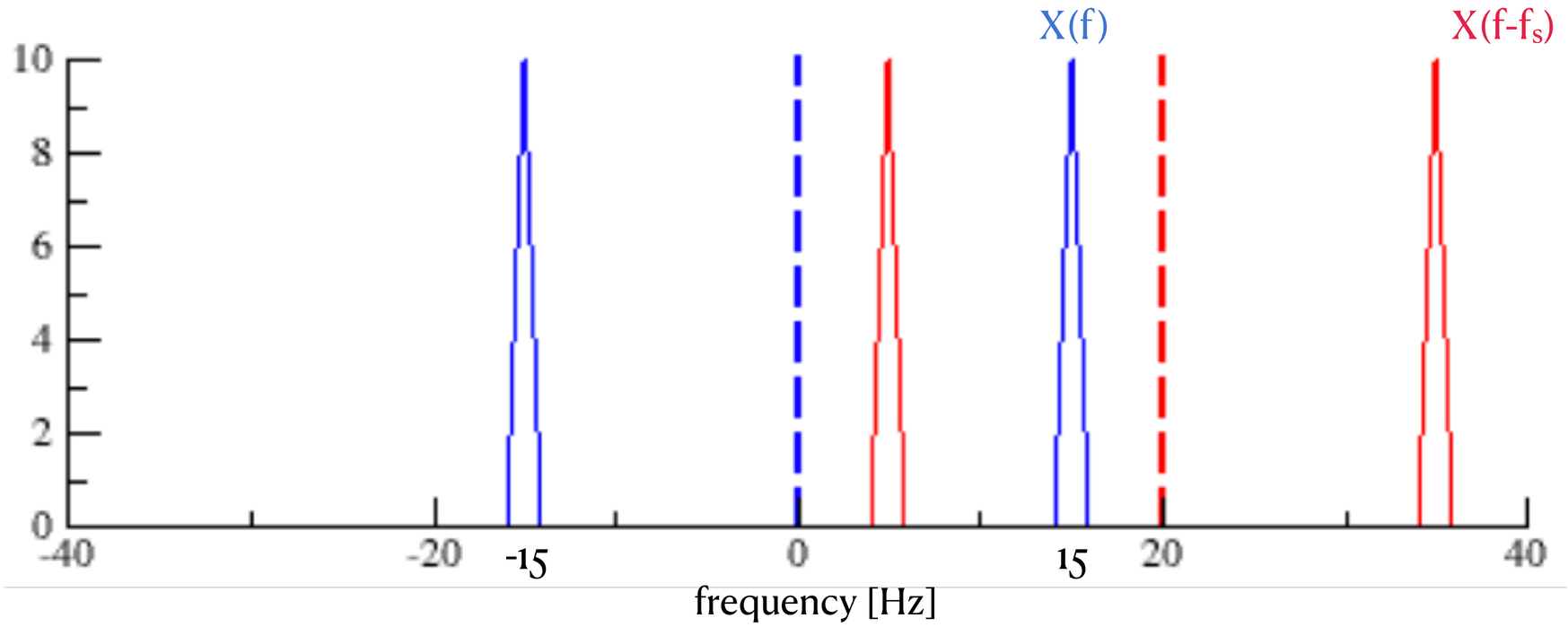
subsampled time series with f_s=20Hz

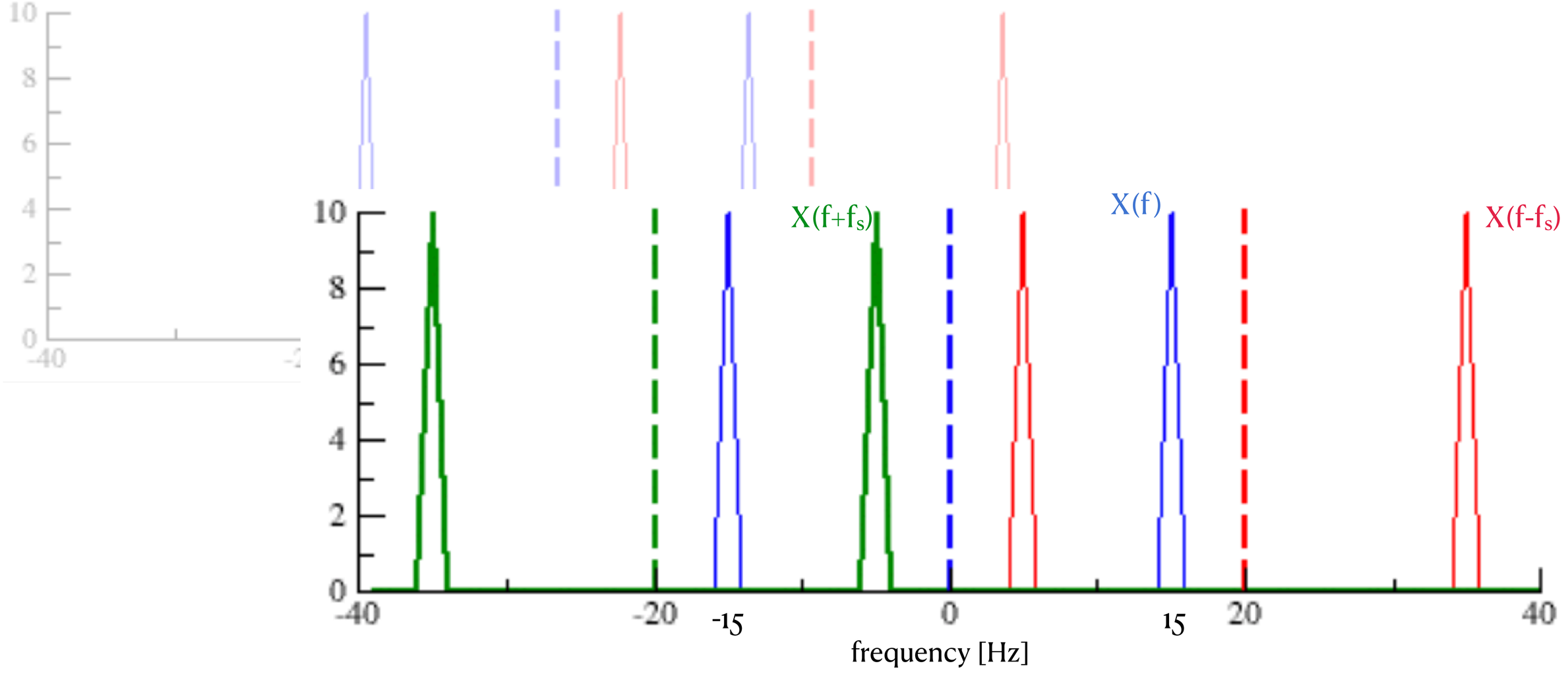


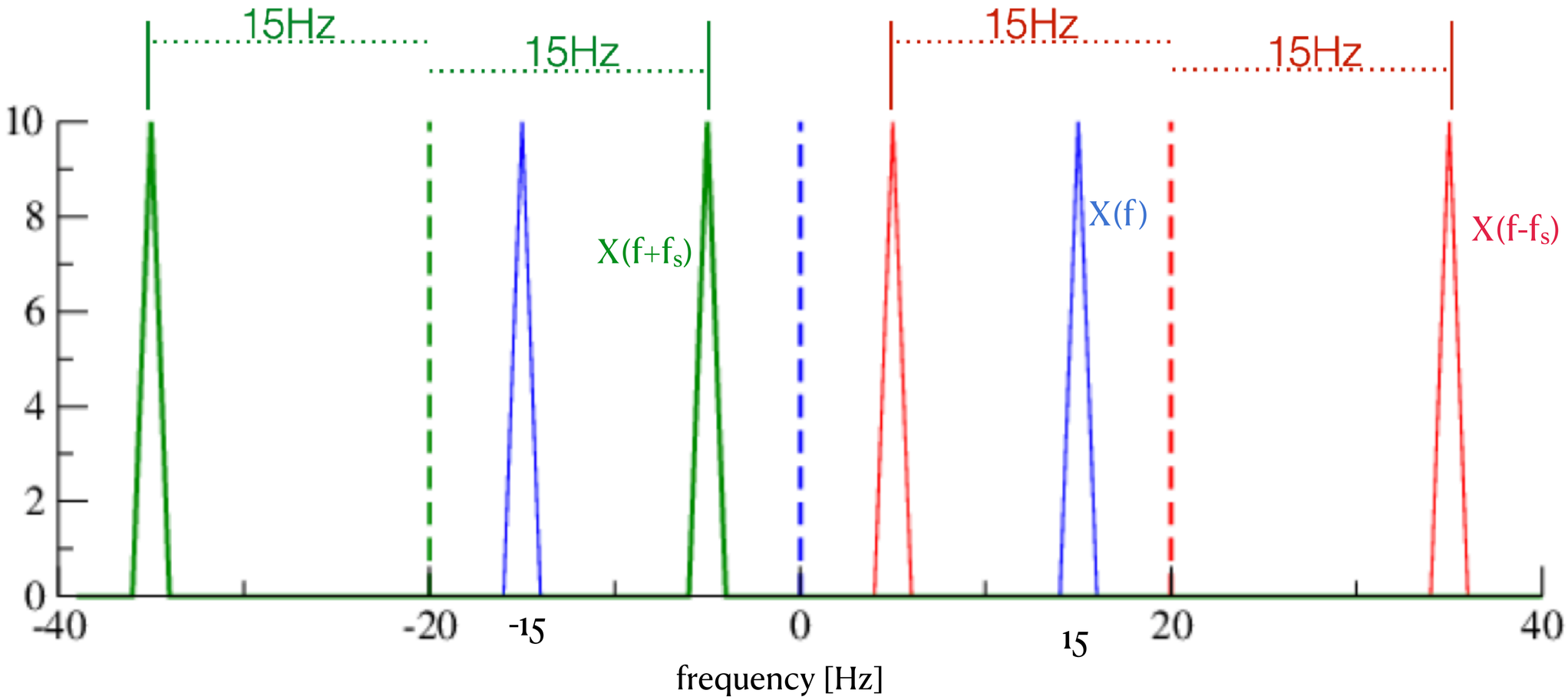
original time series with f_m=15Hz

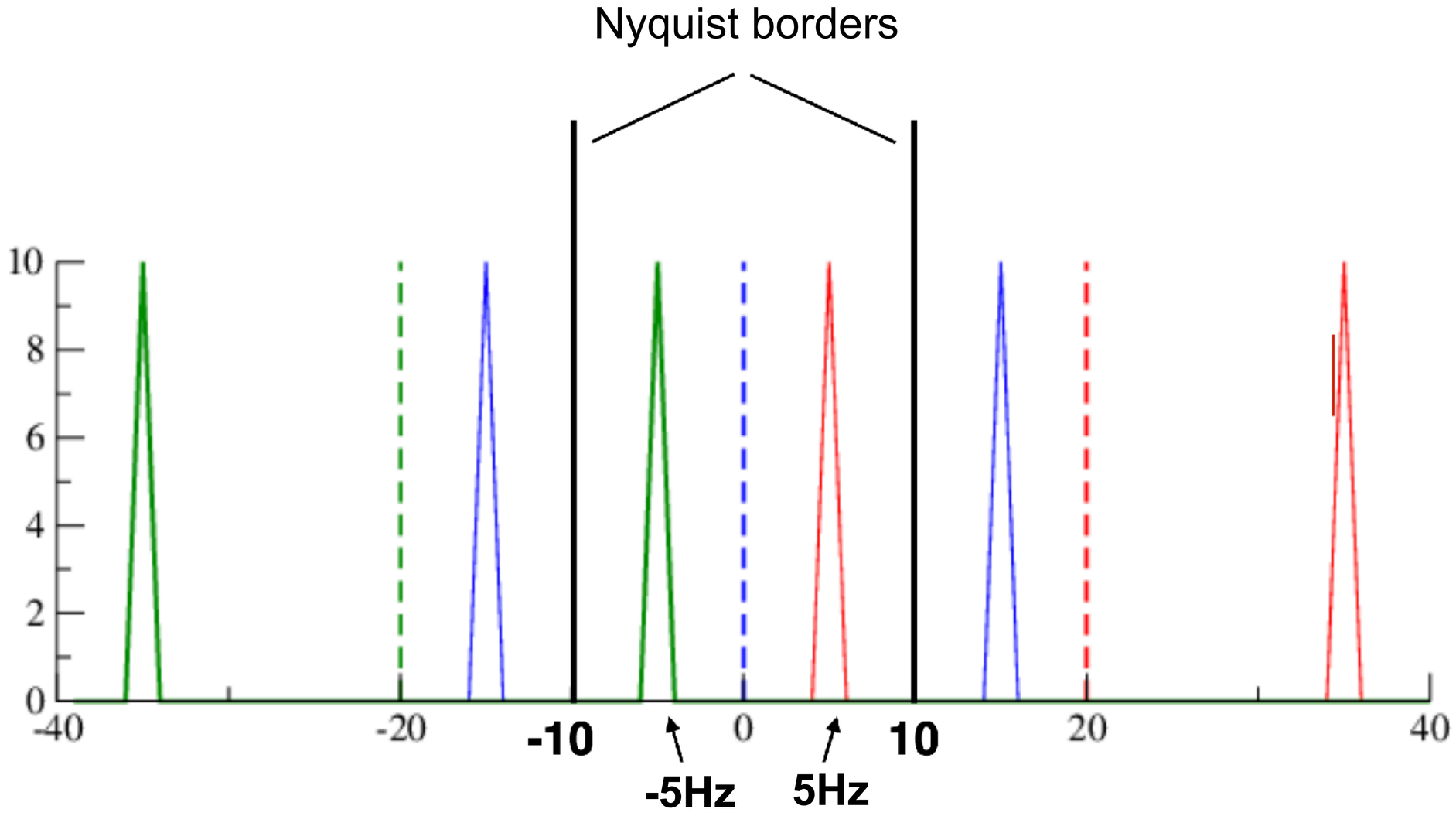
subsampled time series with f_s=20Hz

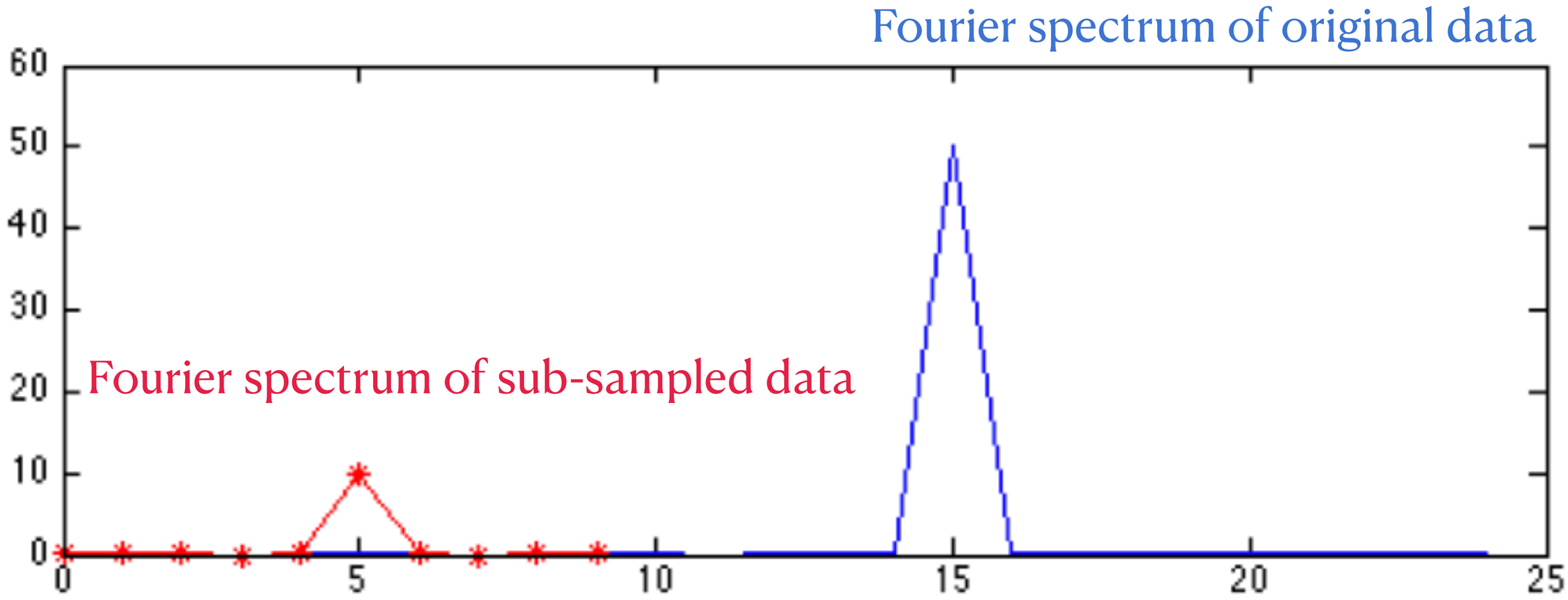



X(f)
X(f-fs)
10
8
6
4
2
0
-40
-20
-15
0
15
20
40
frequency [Hz]

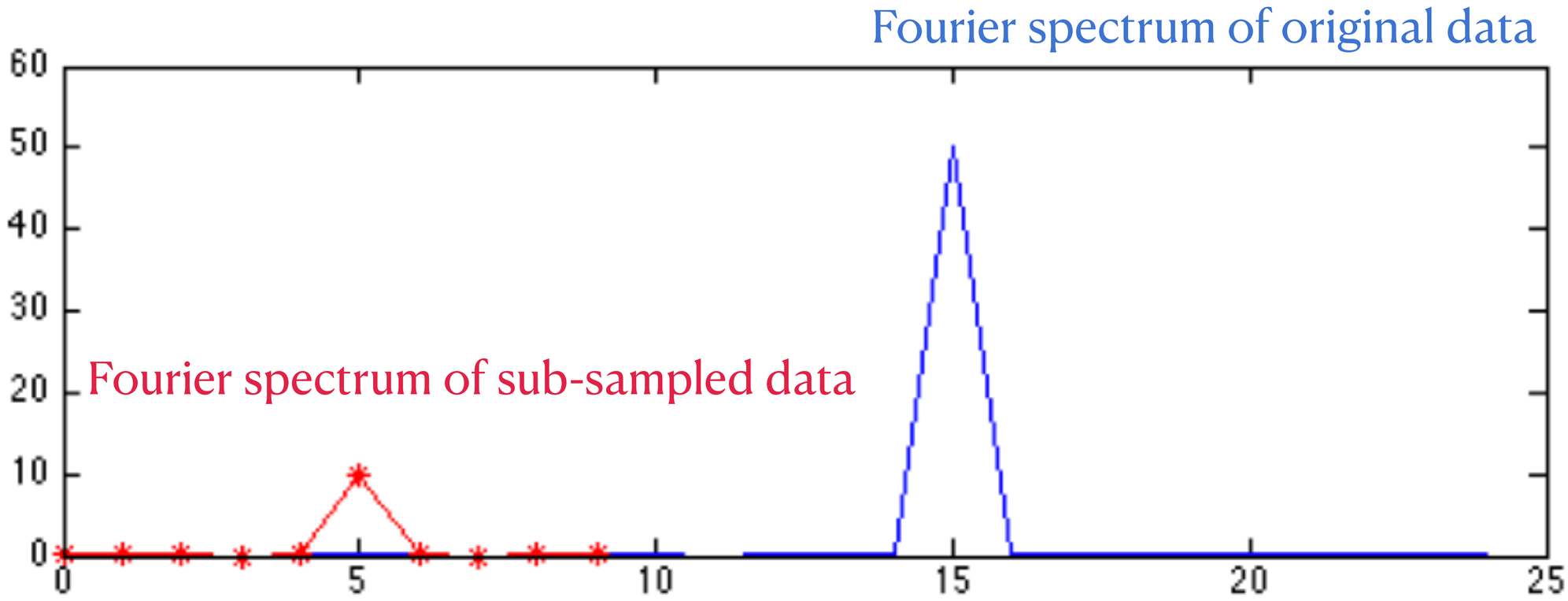

10
8
6
4
2
0
-40
X(f+fs)
X(f)
X(f-fs)
10
8
6
4
2
0
-40
-20
-15
0
15
20
40
frequency [Hz]


15Hz
15Hz
15Hz
15Hz
10
8
6
4
2
0
X(f+fs)
X(f)
X(f-fs)
-40
-20
-15
0
15
20
40
frequency [Hz]


Nyquist borders
10
8
6
4
2
0
-40
-20
-10
0
10
20
40
-5Hz
5Hz

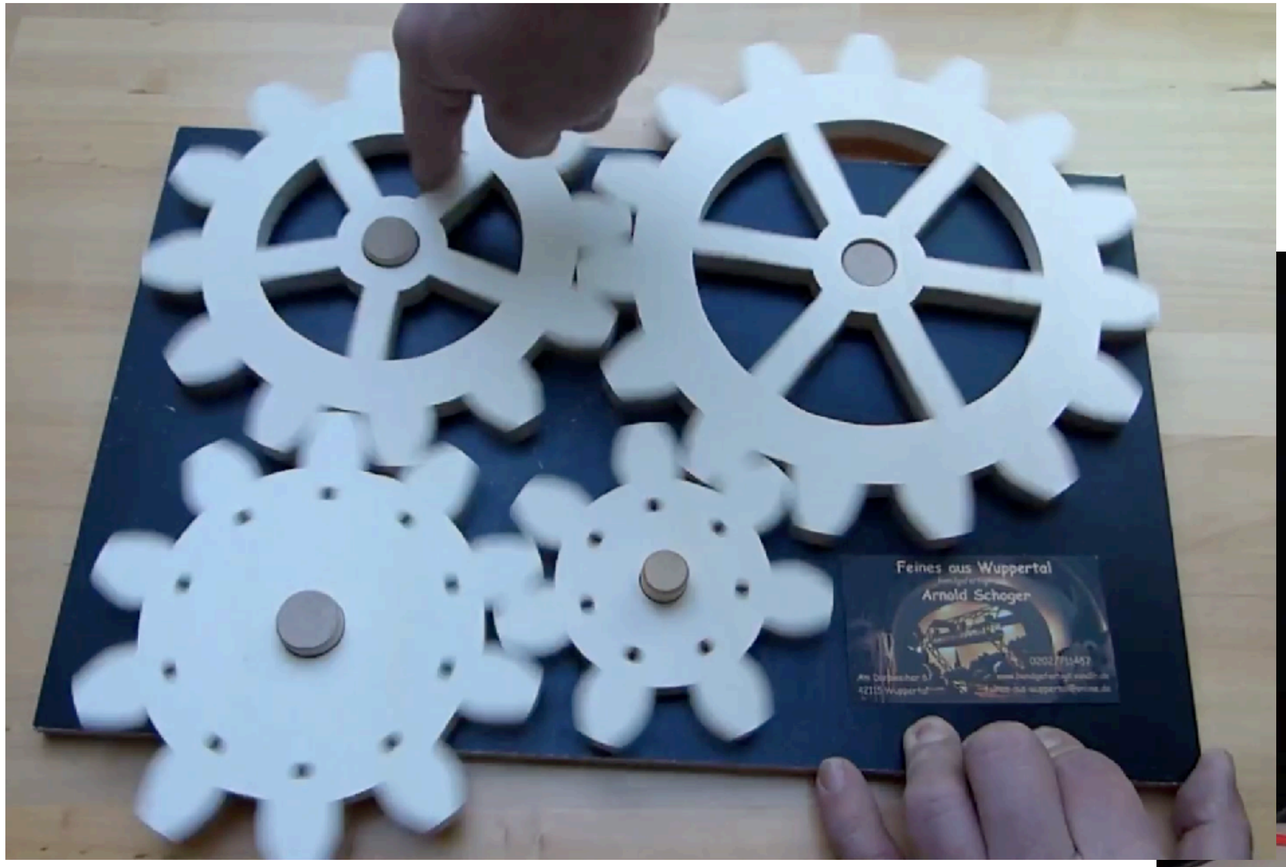

Fourier spectrum of original data
Fourier spectrum of sub-sampled data
60
50
40
30
20
10
0
0
5
10
15
20
25

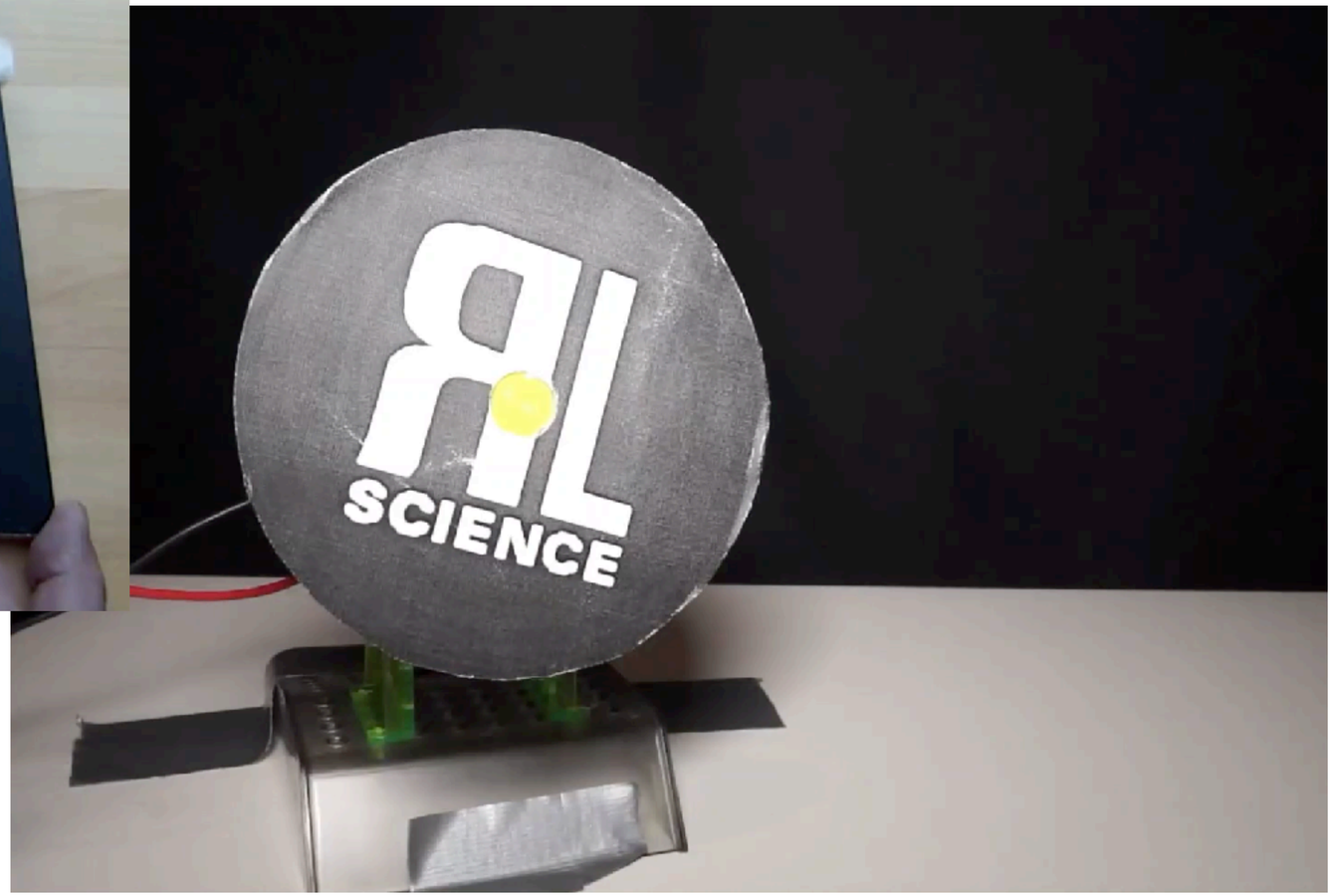
aliasing



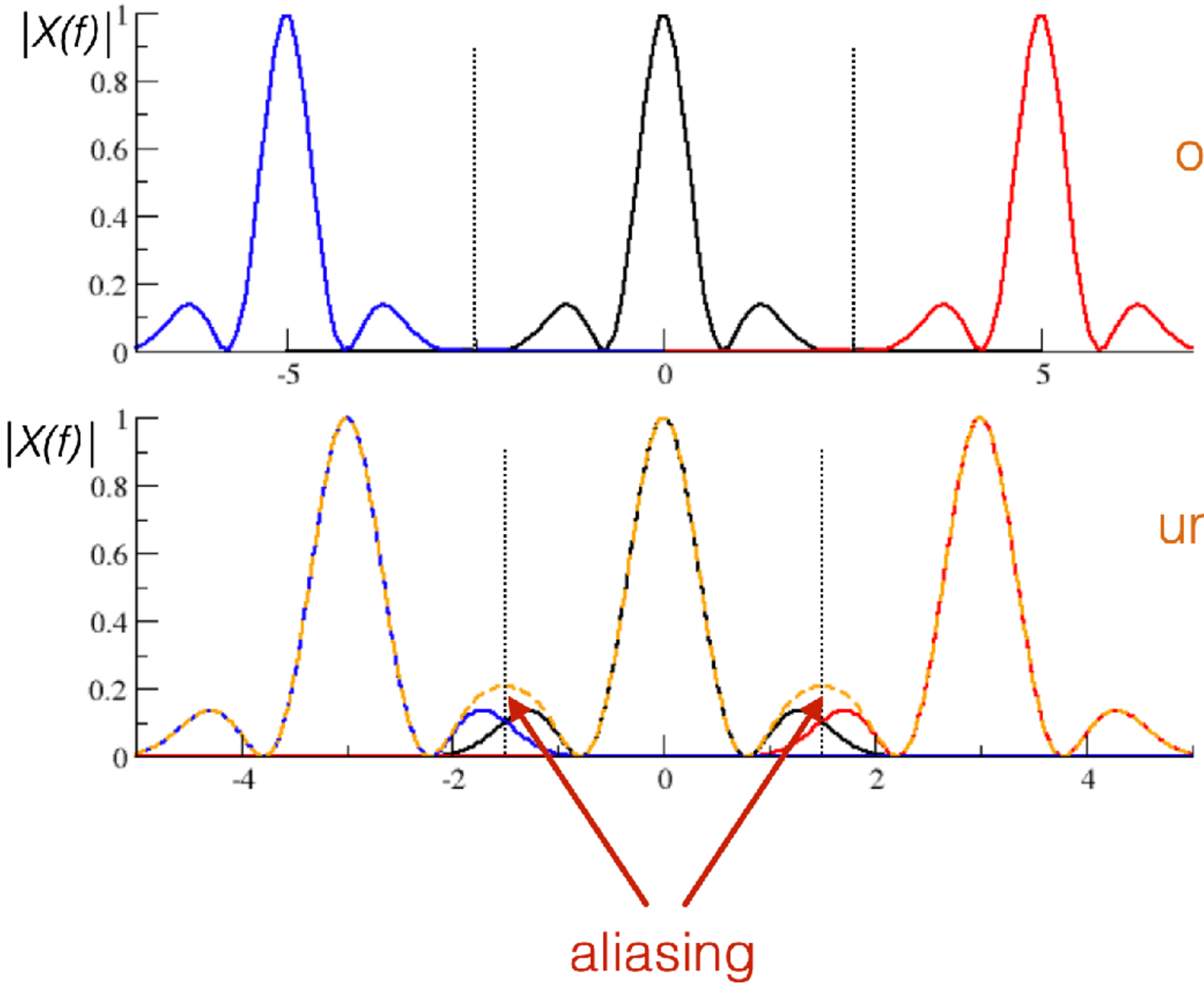
erroneous peak

aliasing = stroboscope effect





summary for aliasing



choose sampling rate that is more than double the maximum frequency

data sampling

Fourier analysis

errors in analysis

aliasing **spectral leakage** spectral power

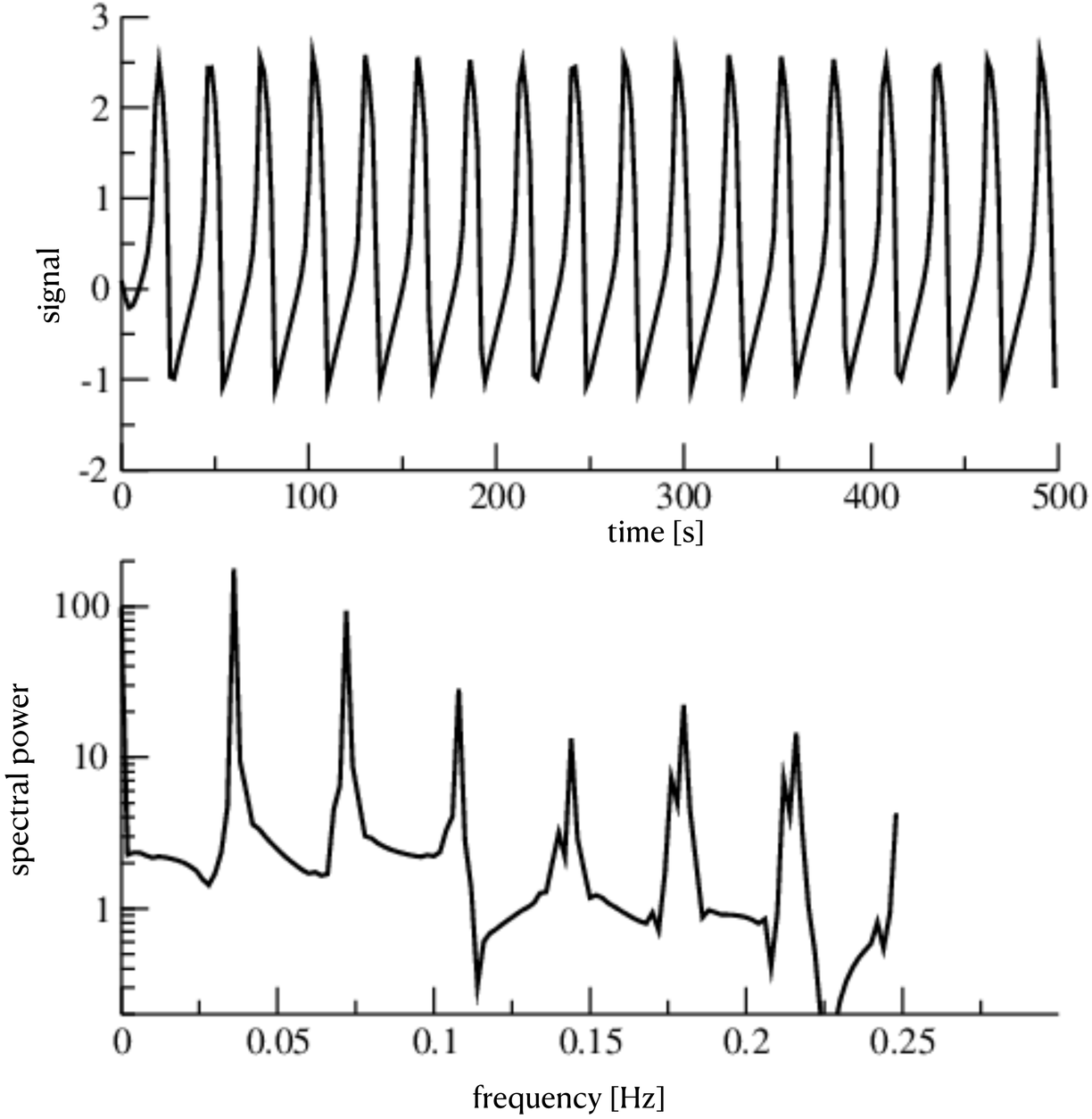
linear filters

time-frequency analysis

Example: FHN oscillator

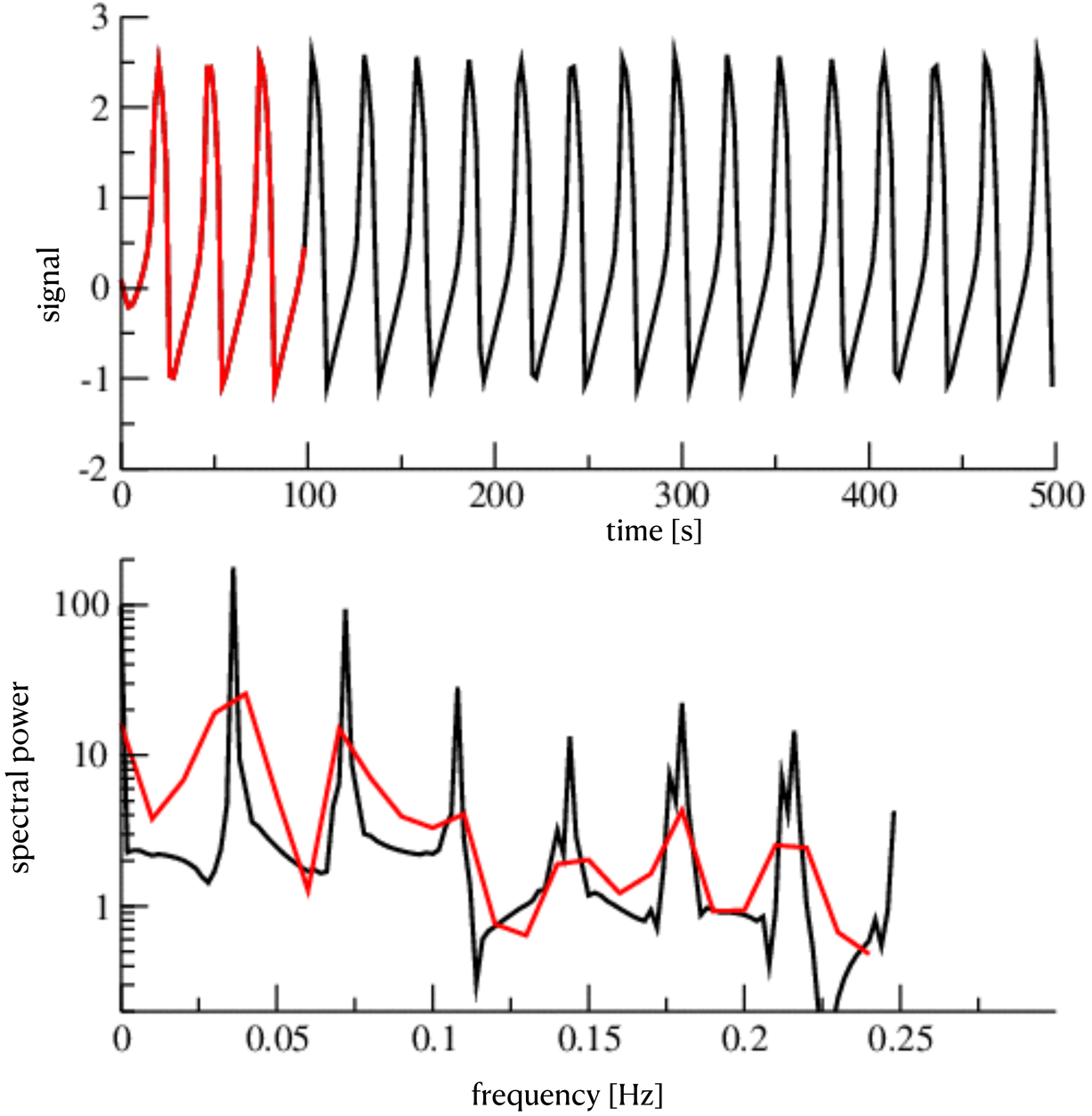
$f_s = 0.5\text{Hz}$

$T = 500\text{s}$

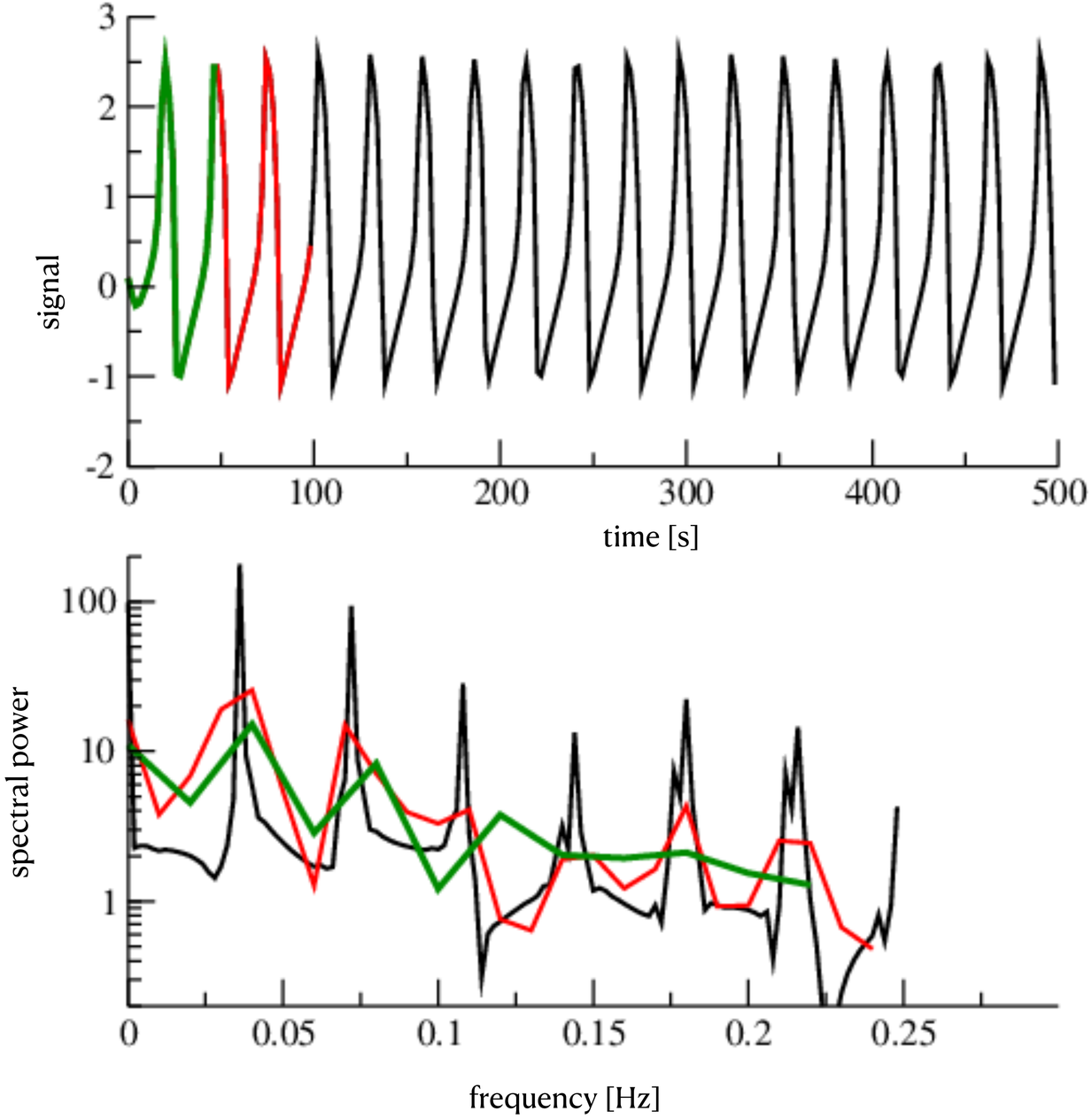


SamplingError_2.py

T= 100s


3
2
1
0
-1
-2
signal
0
100
200
300
400
500
time [s]
100
10
1
spectral power
0
0.05
0.1
0.15
0.2
0.25
frequency [Hz]

T= 50s

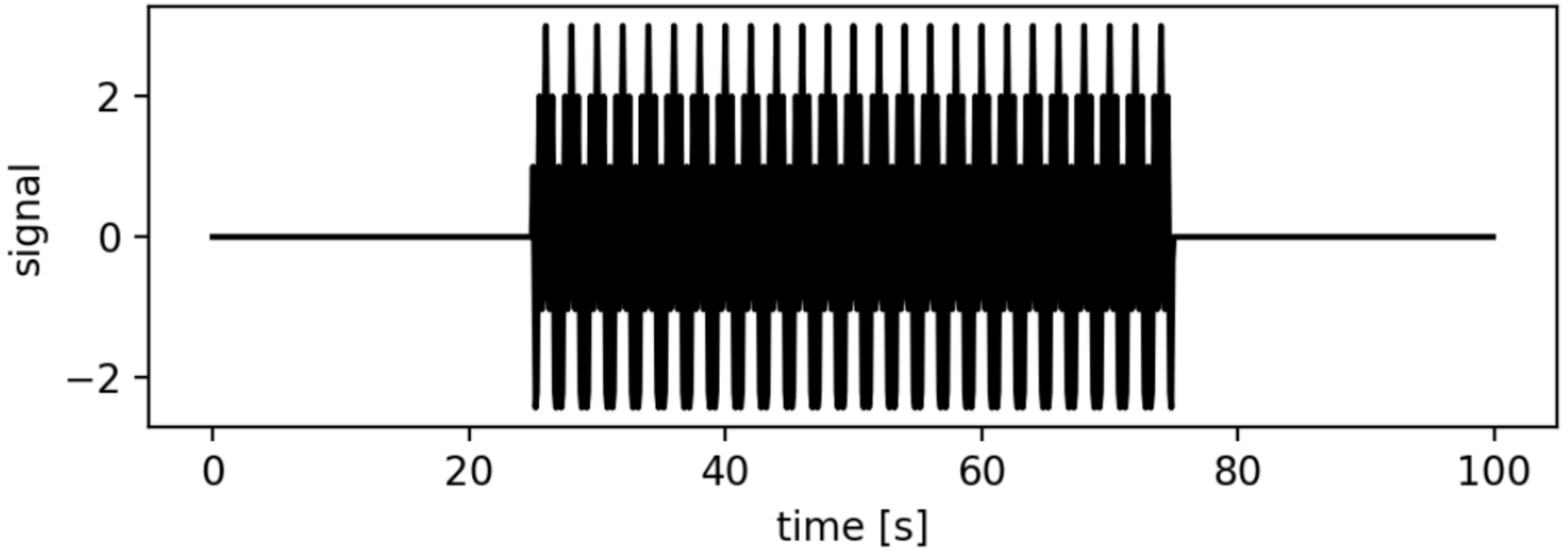


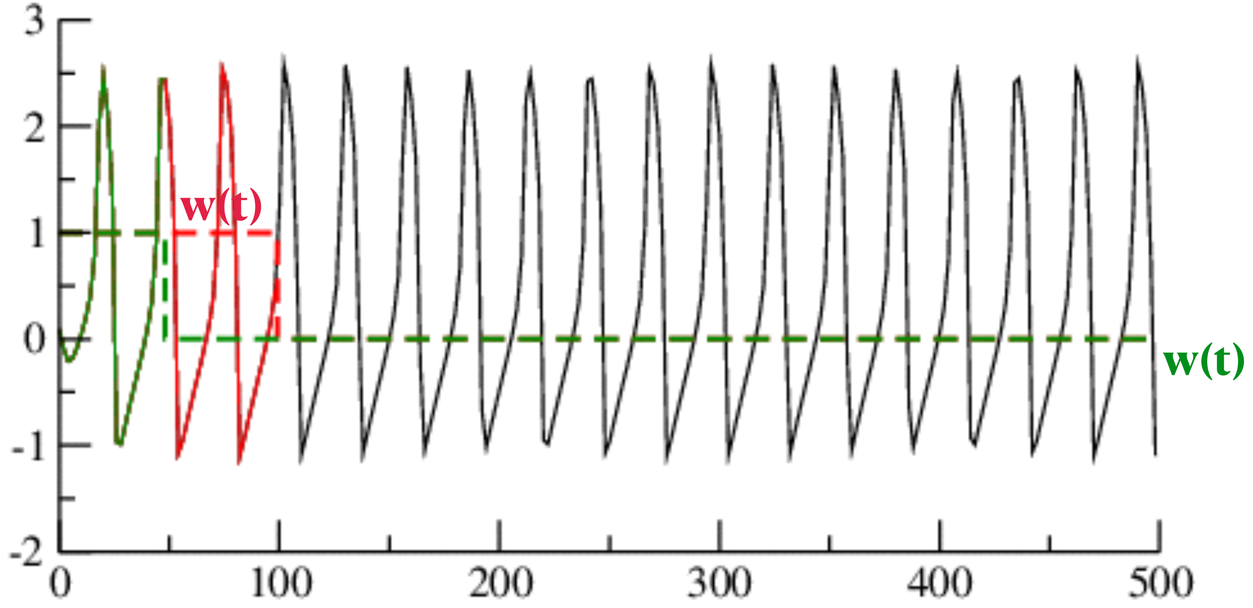
the shorter the time series, the worse the spectral estimate

how to improve the spectral estimate ?

how to improve the spectral estimate ?

idea: prolongation of signal by zero-padding

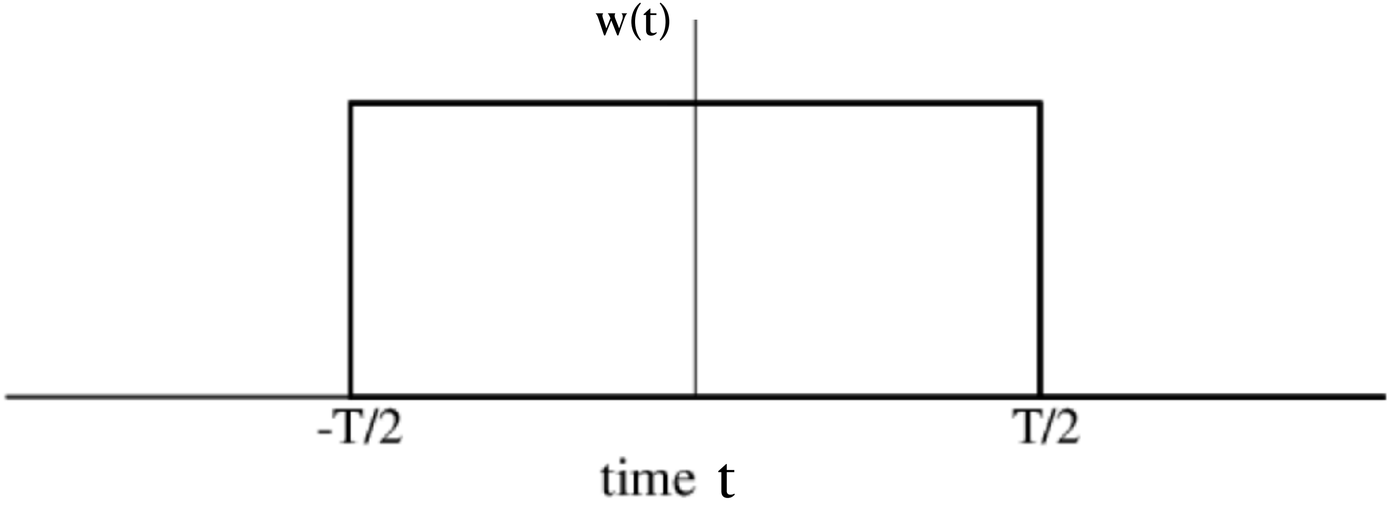




description of short signal:

multiplication by window function

$$s(t_k) \;\rightarrow\; s(t_k)w(t_k)$$



$$\text{DFT} = \sum_{k=1}^{N} s(t_k) w(t_k) e^{-i2\pi nk/N} \sim \sum_{m=-N/2}^{N/2} c_{n-m} d_m$$

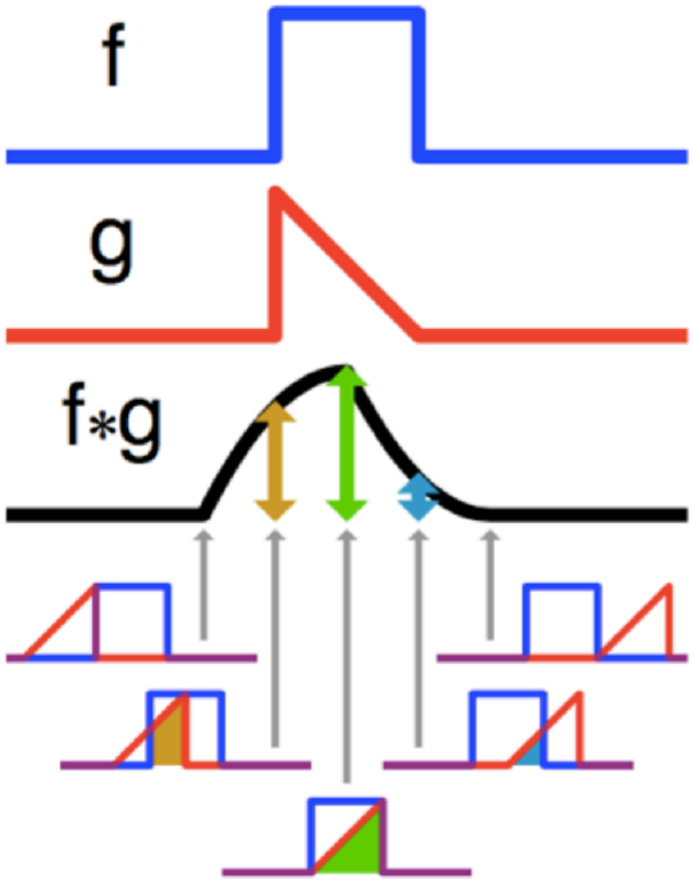
$$d_m = \sum_{k=1}^{N} w(t_k) e^{-i2\pi nk/N}$$

$$\text{DFT} = \sum_{k=1}^{N} s(t_k) w(t_k) e^{-i2\pi nk/N} \sim \sum_{m=-N/2}^{N/2} c_{n-m} d_m$$

$$d_m = \sum_{k=1}^{N} w(t_k) e^{-i2\pi nk/N}$$

$$\text{DFT} = \sum_{k=1}^{N} s(t_k) w(t_k) e^{-i2\pi nk/N} \sim \sum_{m=-N/2}^{N/2} c_{n-m} d_m$$

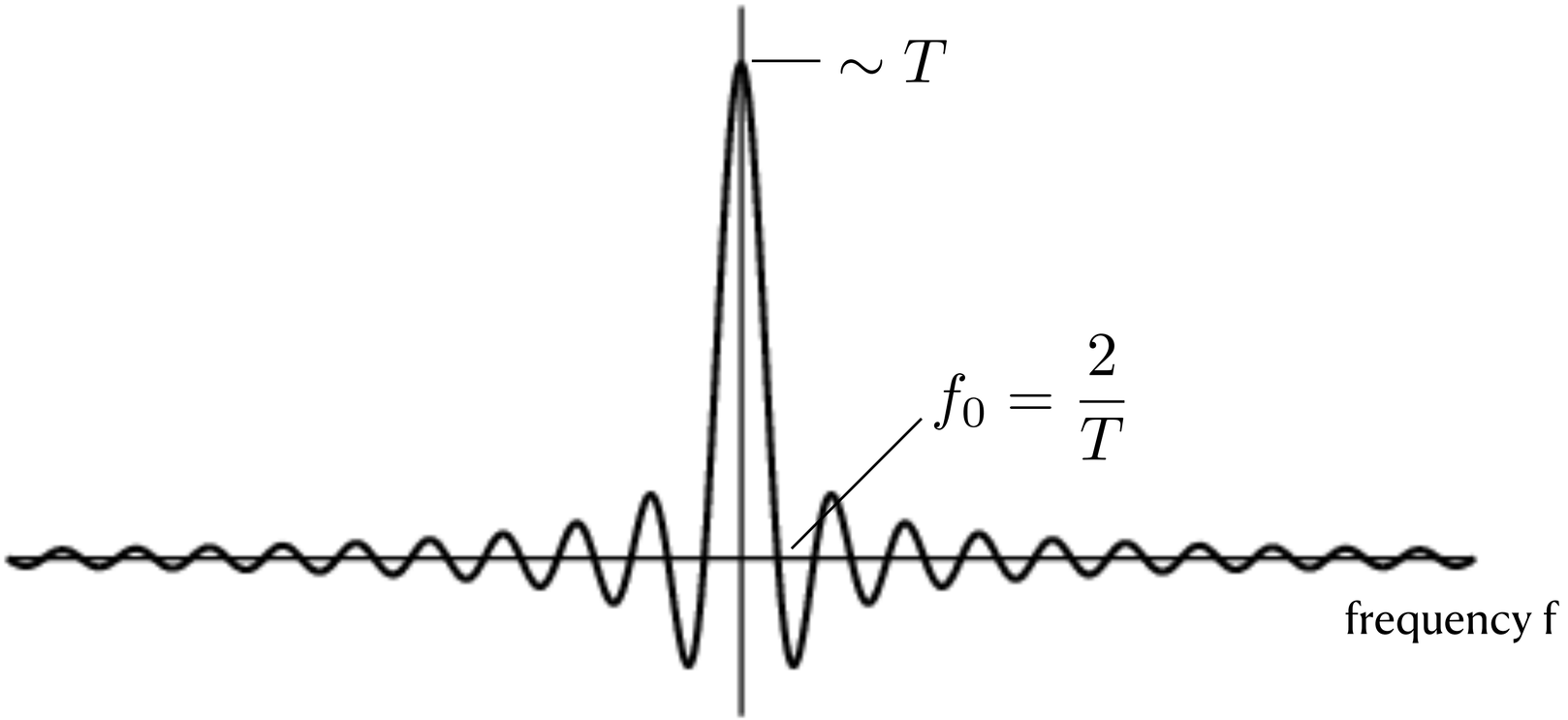
$$d_m = \sum_{k=1}^{N} w(t_k) e^{-i2\pi nk/N}$$



spectrum is convoluted with window Fourier transform

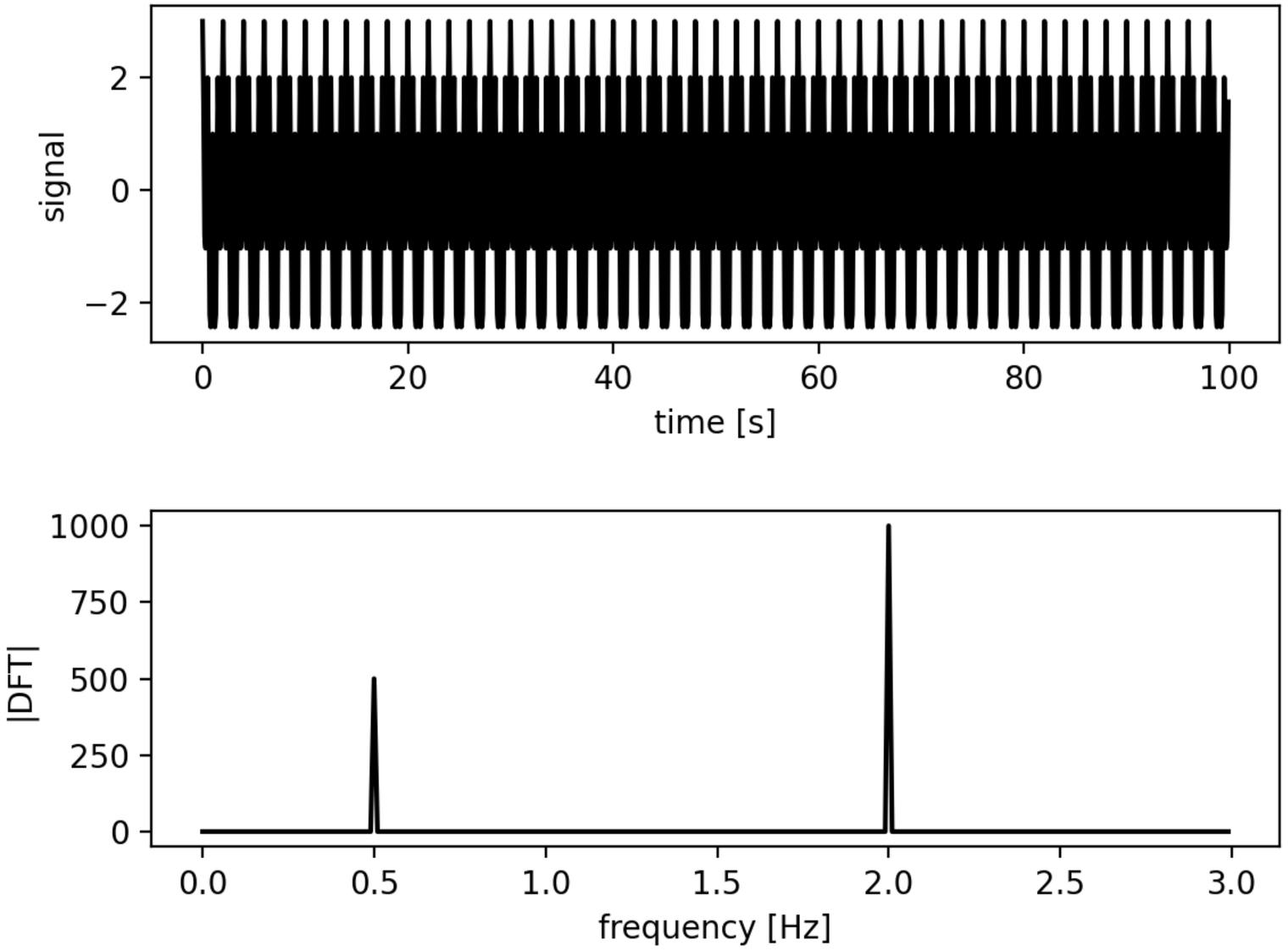
$$d_k \sim \frac{\sin(\pi f_k T)}{f_k}$$

Fourier transform of window function



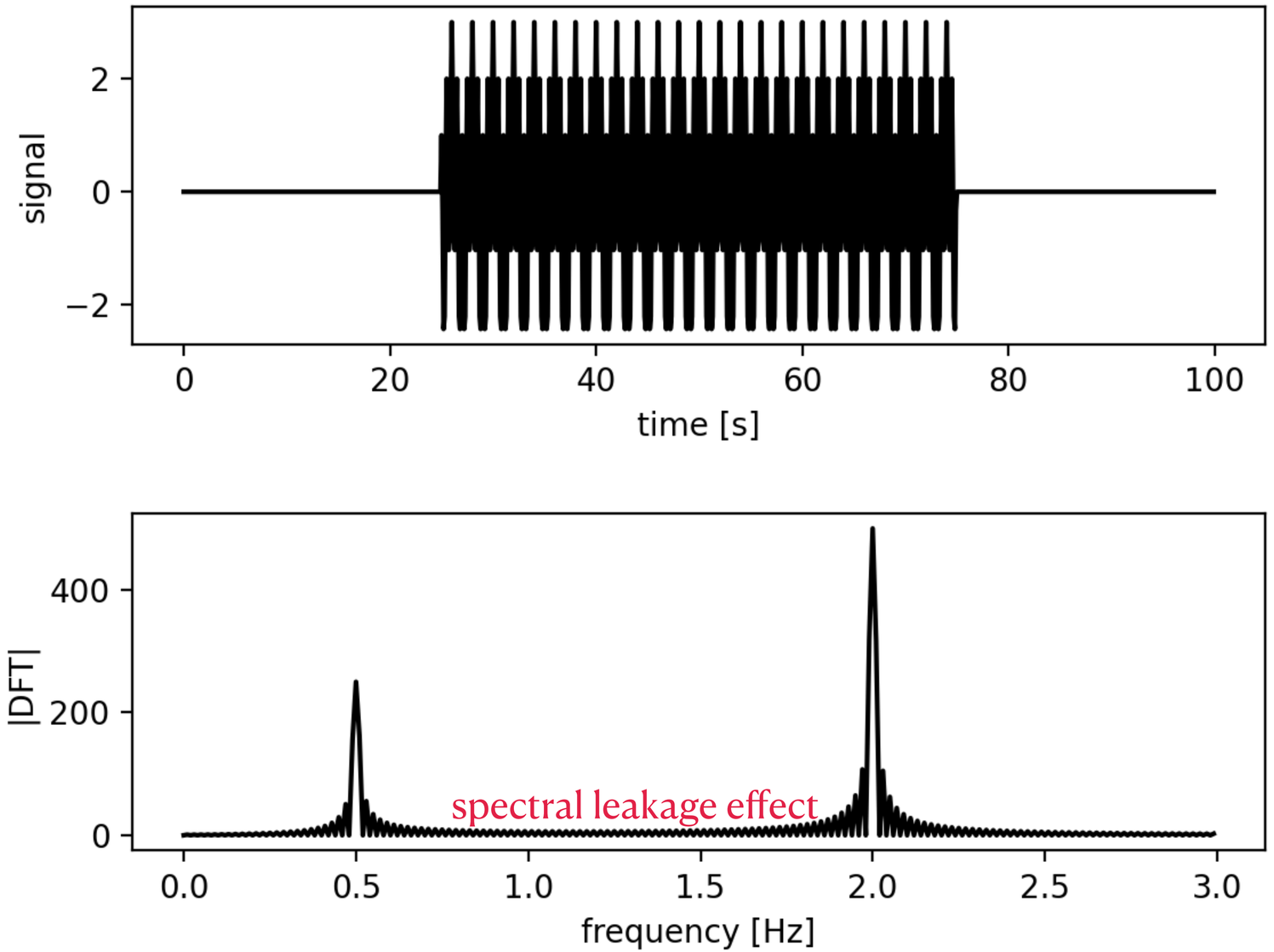
spectral leakage effect

Example: sinusoidal oscillation



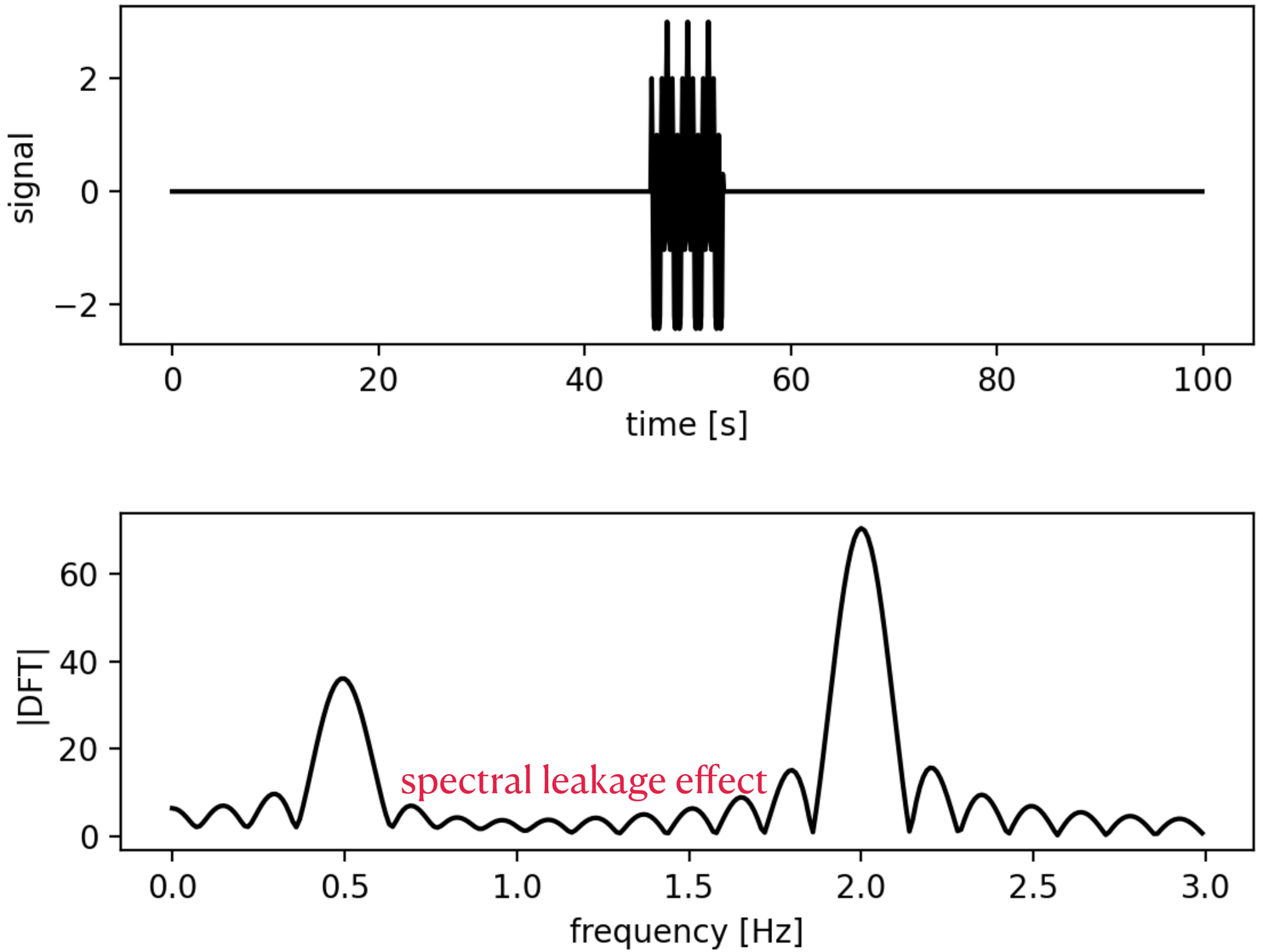
SamplingError_3.py

Example: sinusoidal oscillation



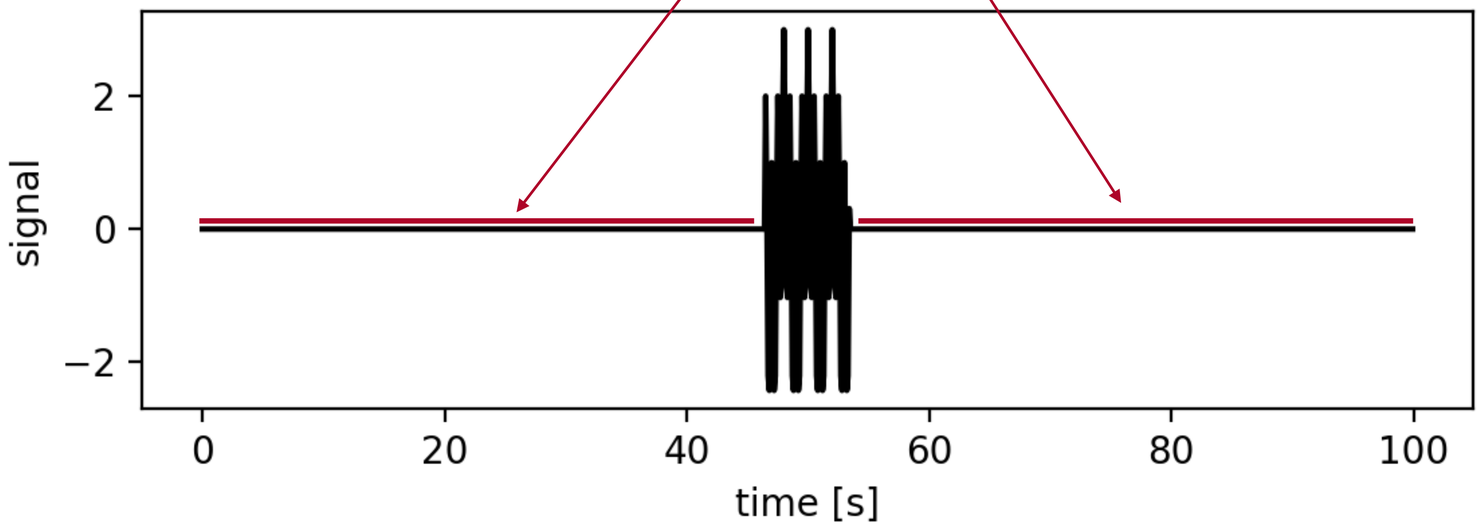
SamplingError_3.py

Example: sinusoidal oscillation

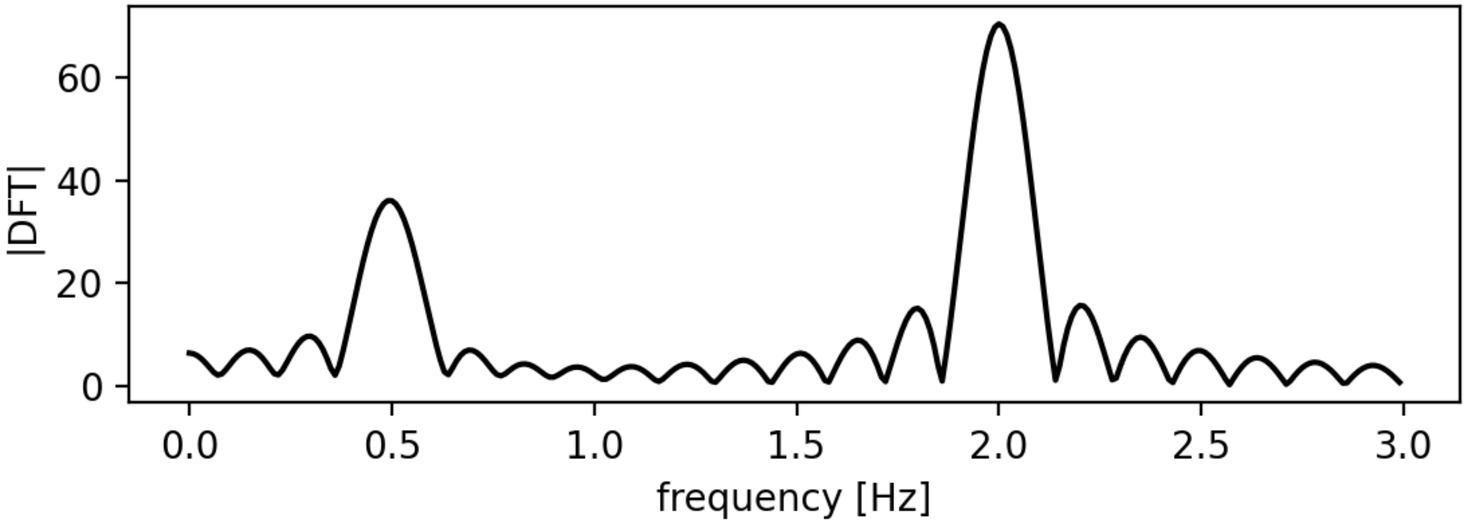


SamplingError_3.py

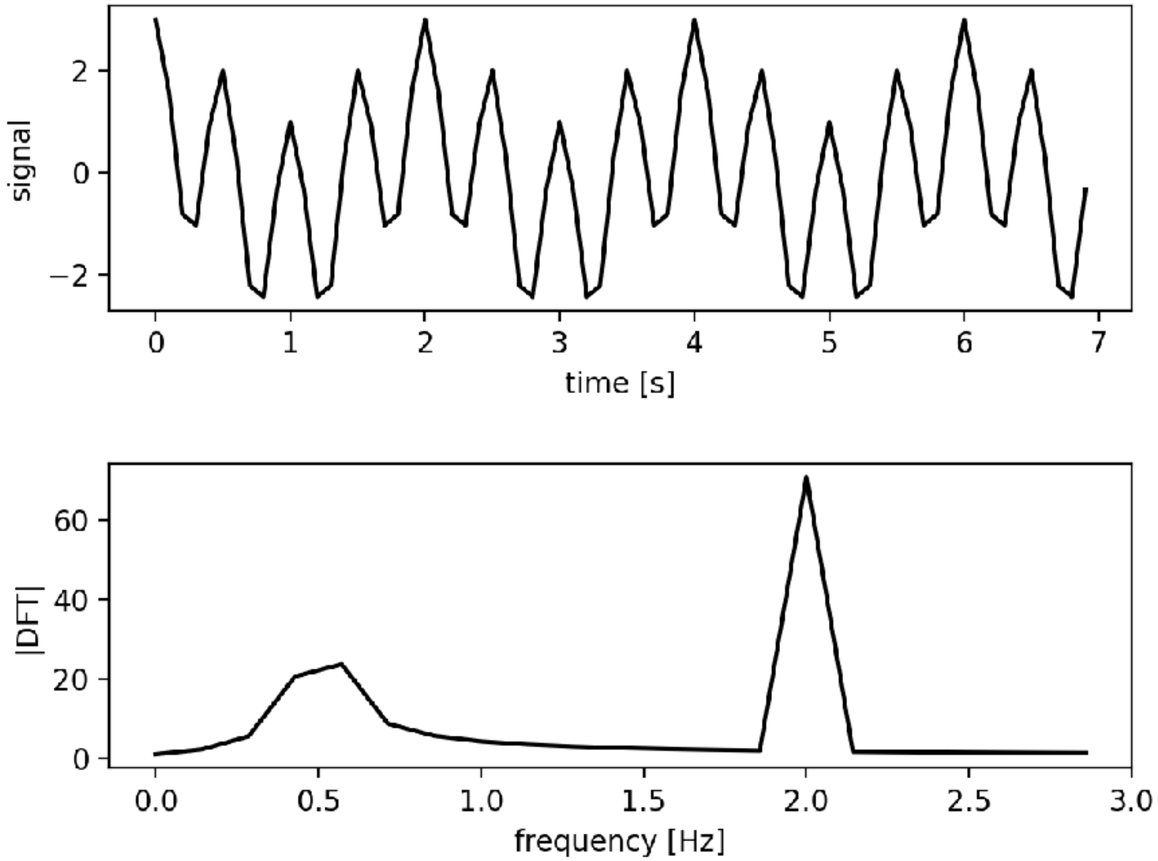
zero padding



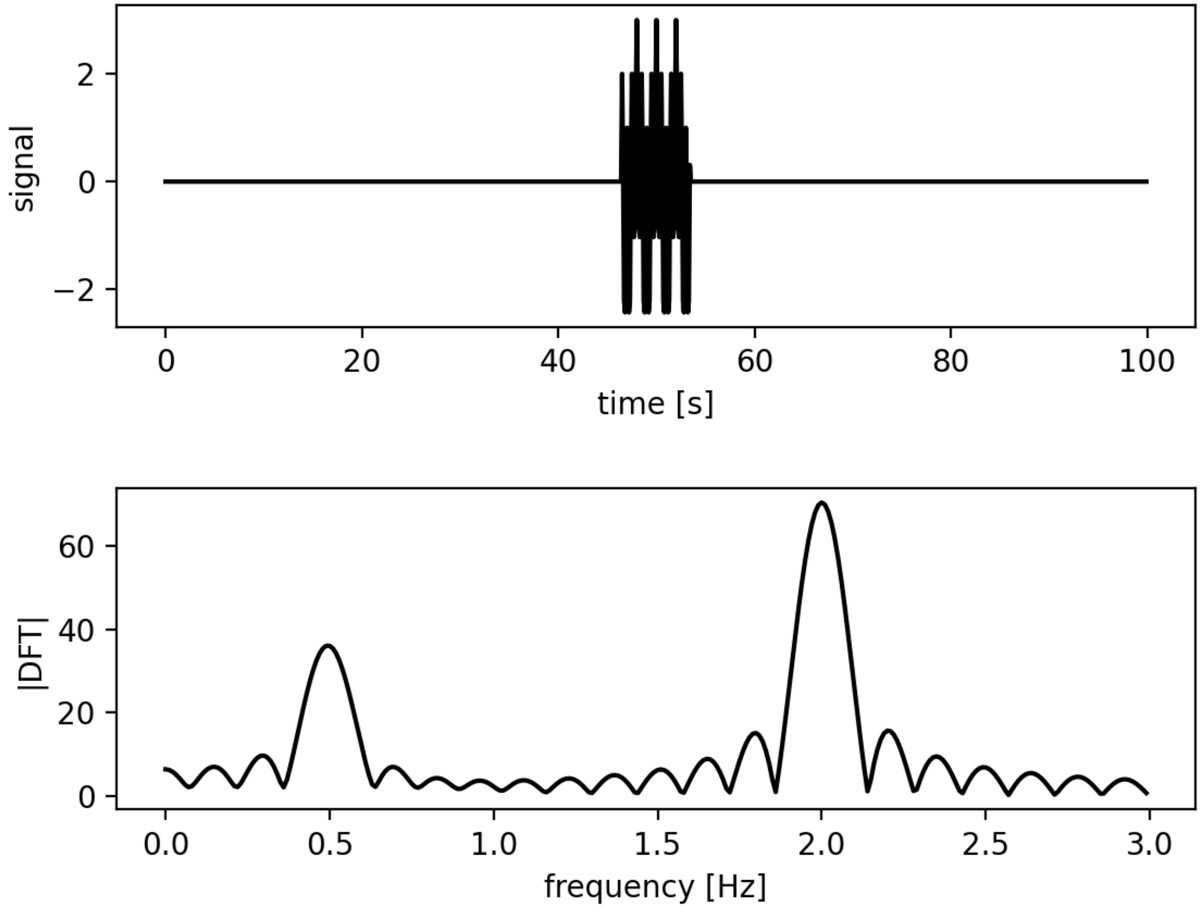
SamplingError_3.py



short signal, no zero padding

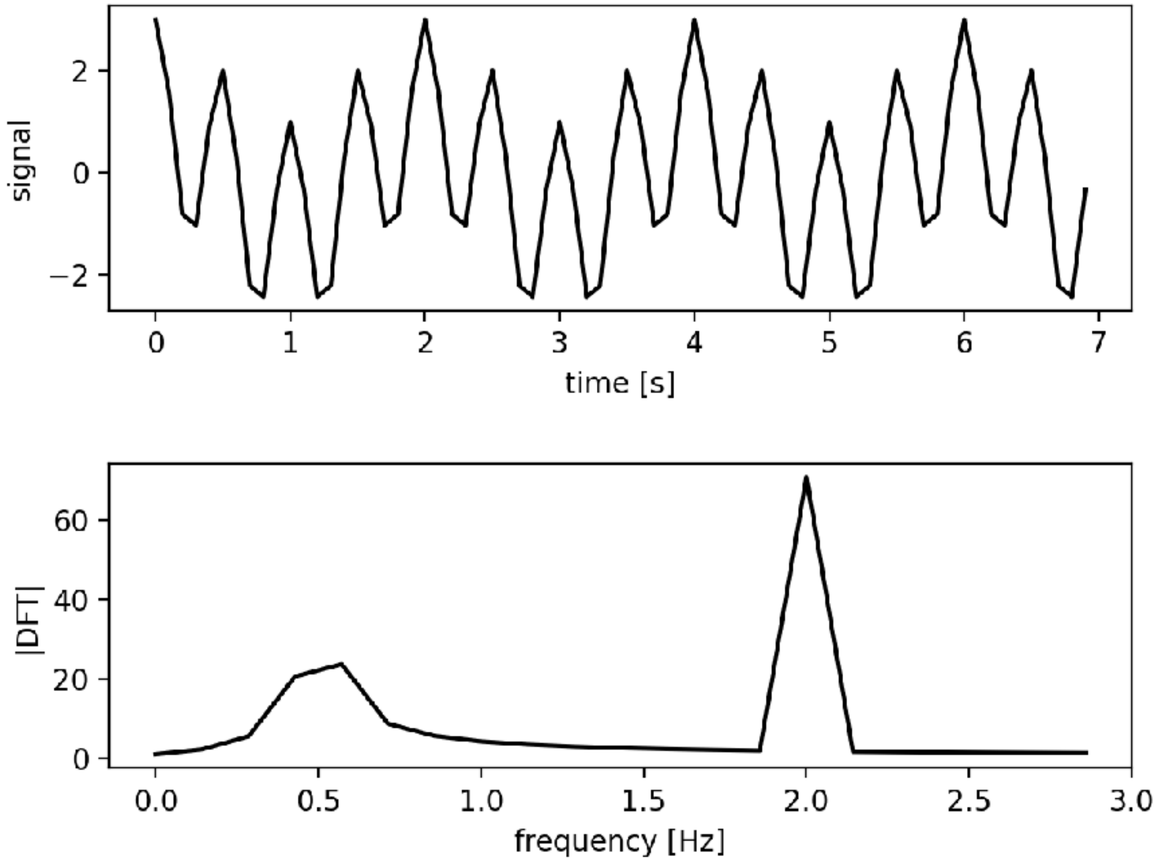


long signal, with zero padding

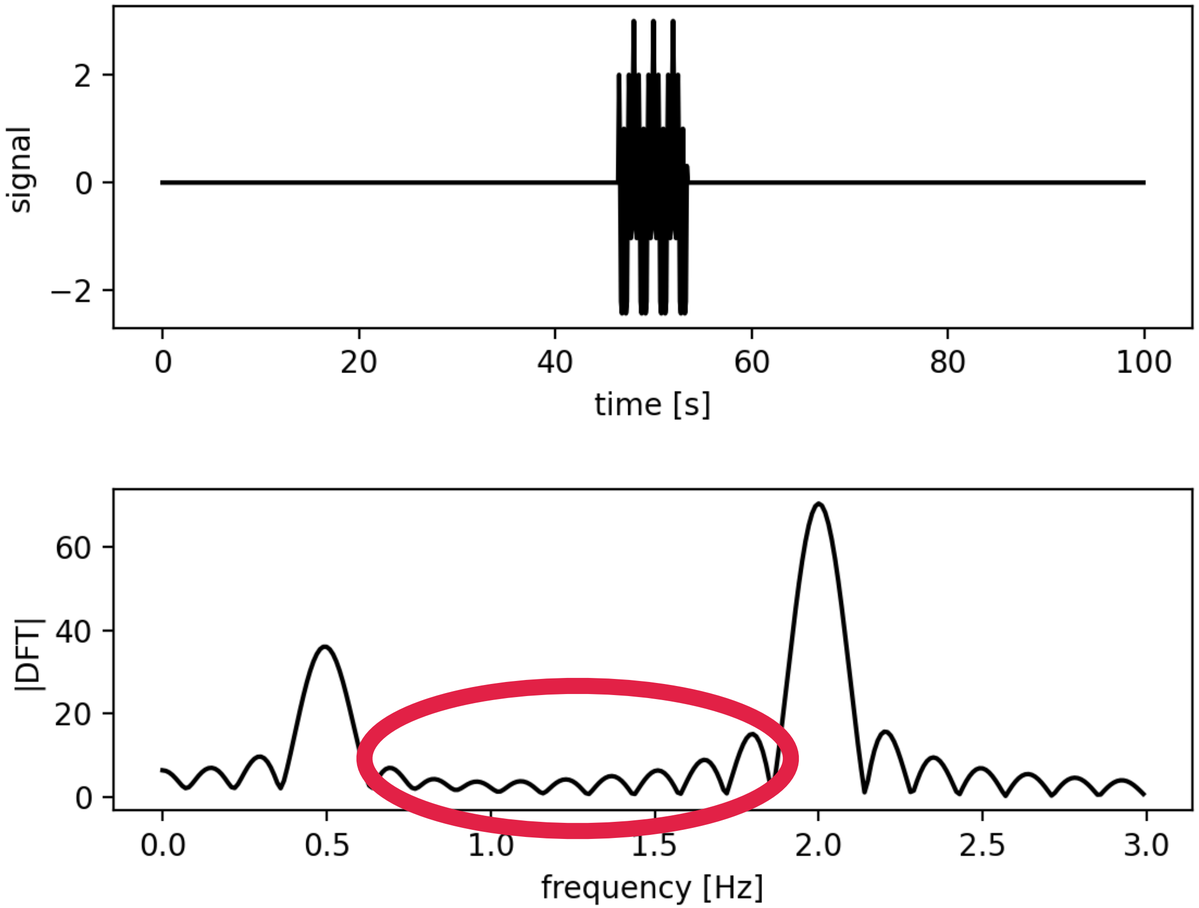


spectral leakage effect by zero-padding

short signal, no zero padding

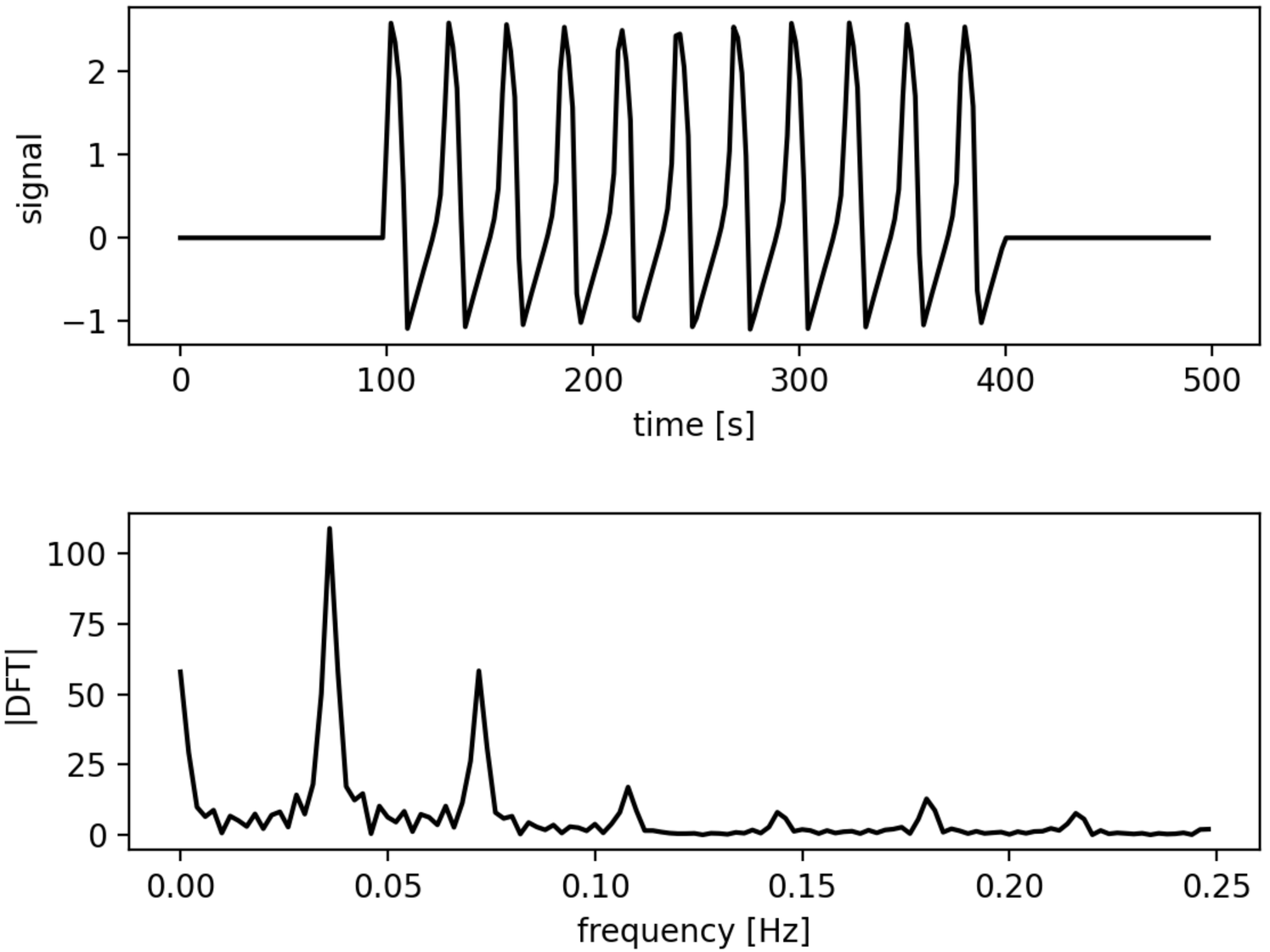


long signal, with zero padding

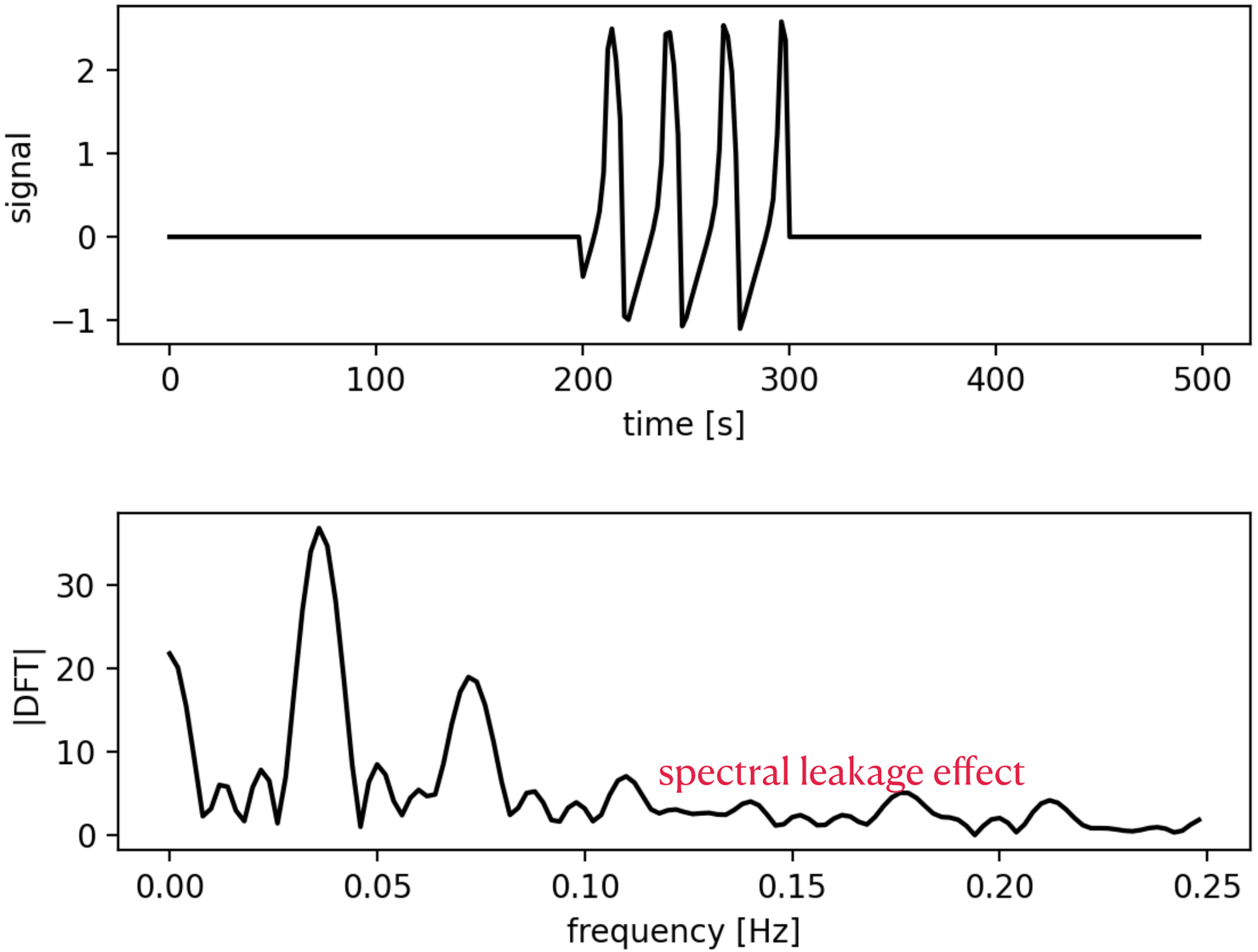


spectral leakage effect by zero-padding

Example: FHN-model

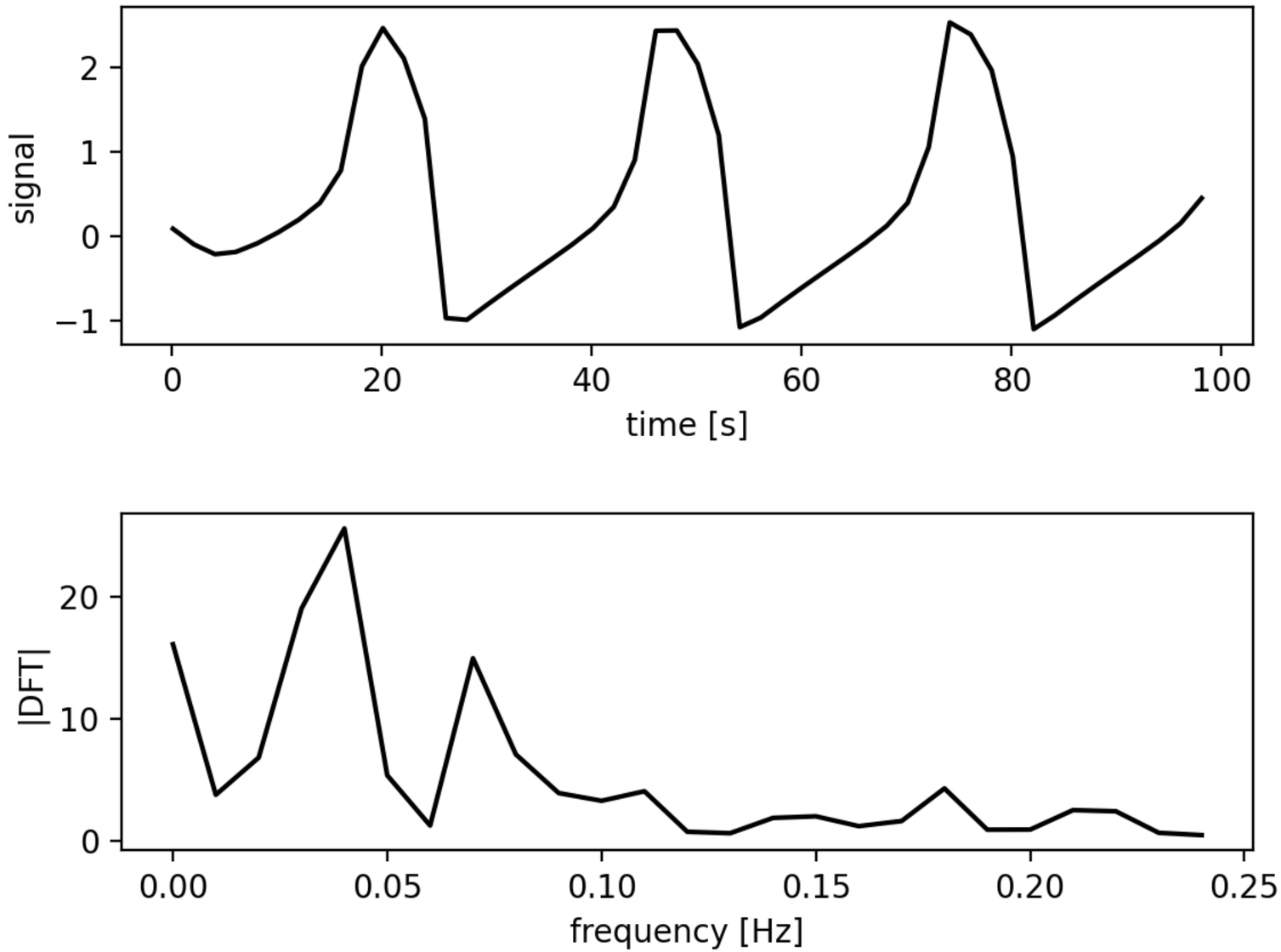


Example: FHN-model



Example: FHN-model

short signal for comparison

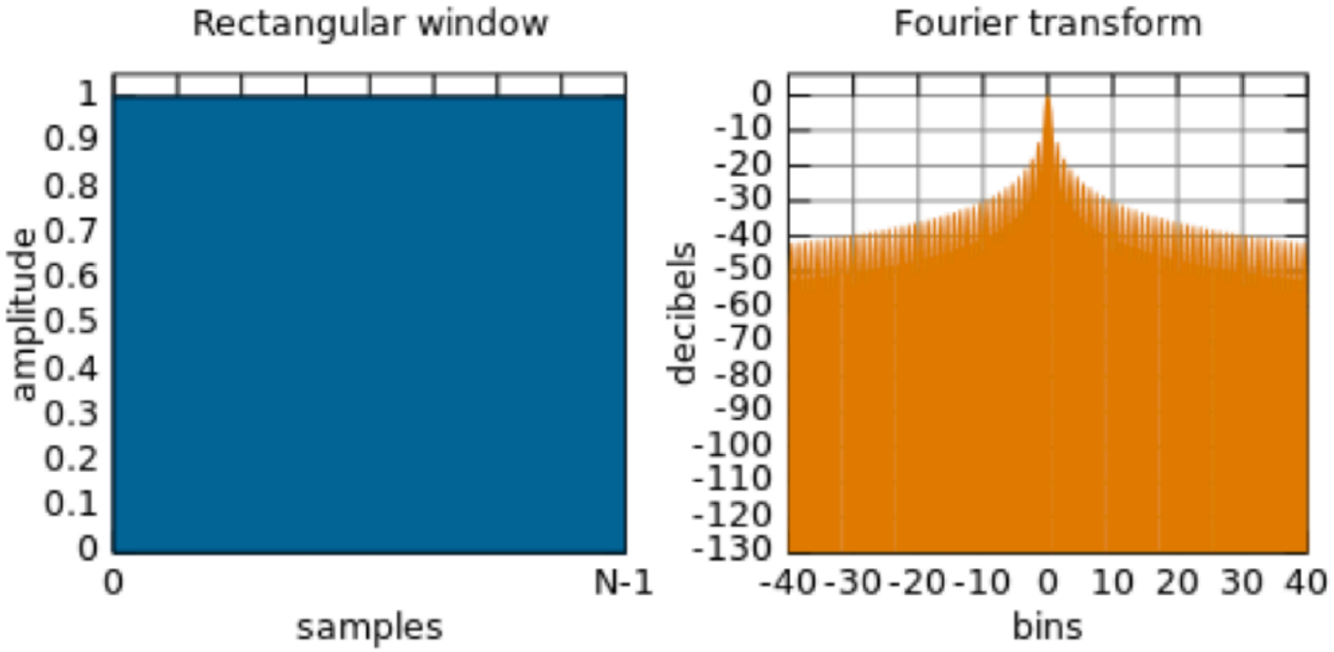


how to reduce spectral leakage:

- choose long time series
- avoid zero-padding

how to reduce spectral leakage:

- choose long time series
- avoid zero-padding
- choose better window function



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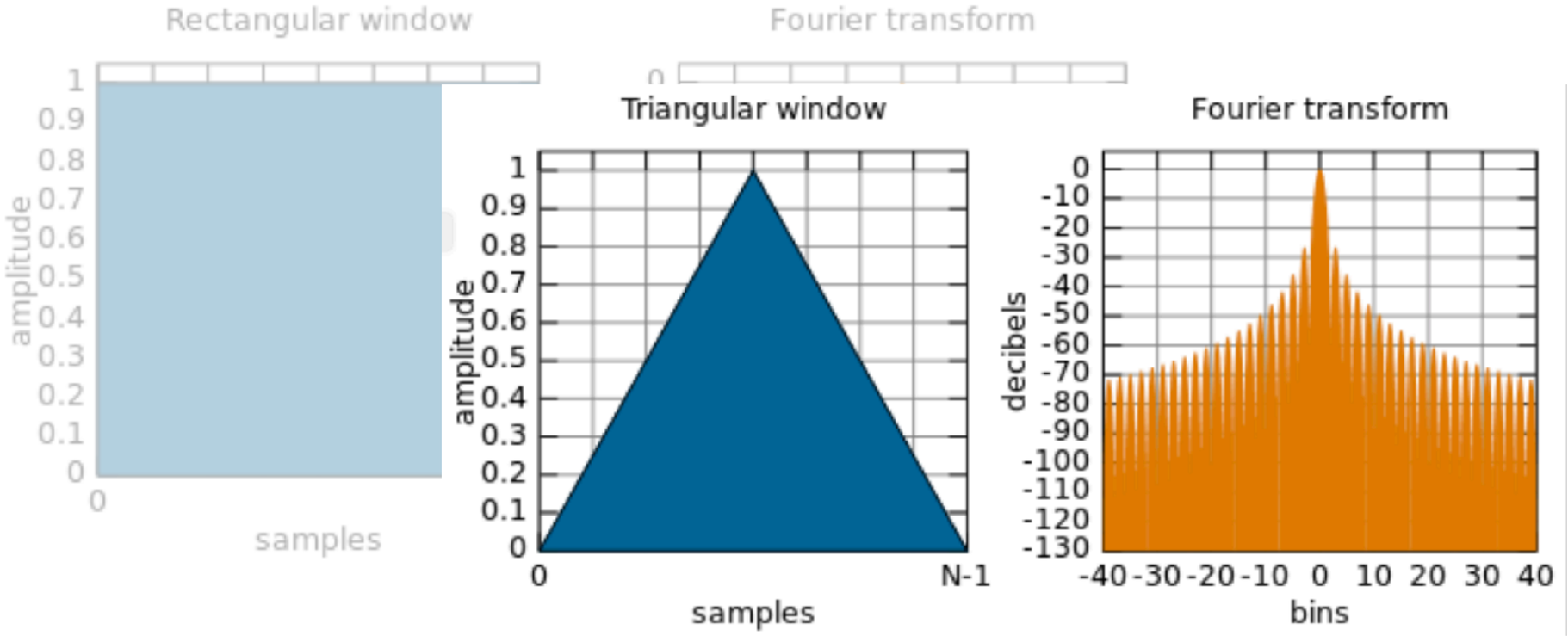

how to reduce spectral leakage:

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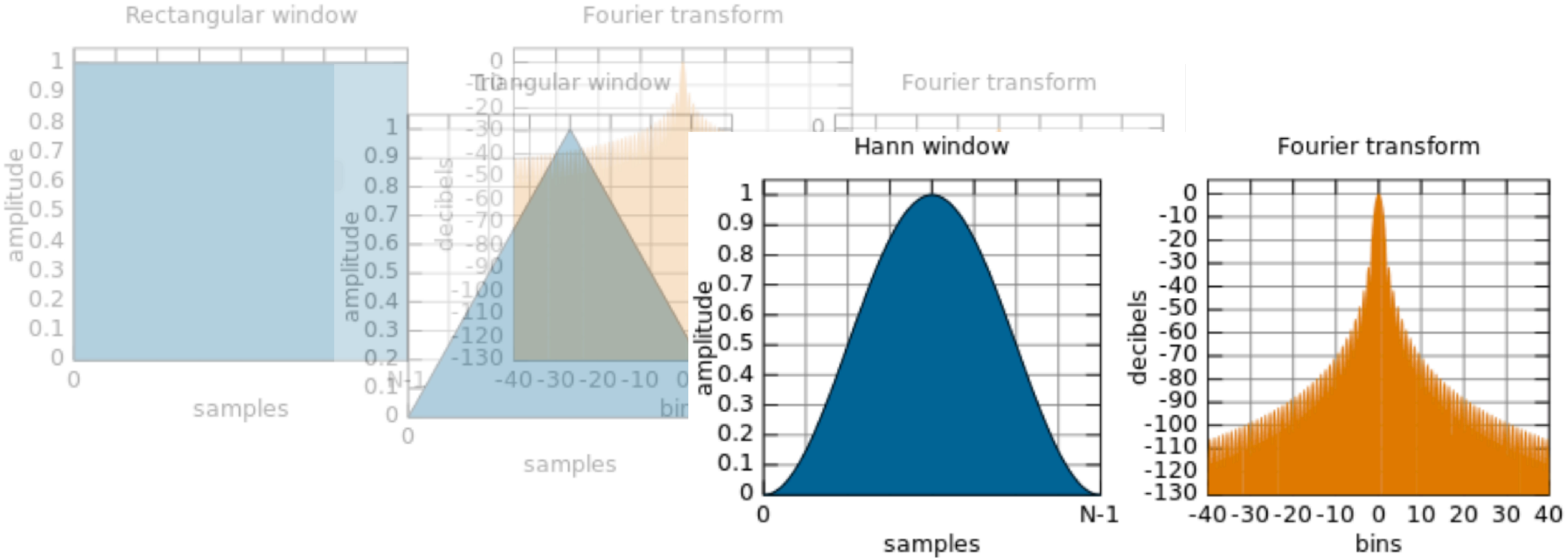
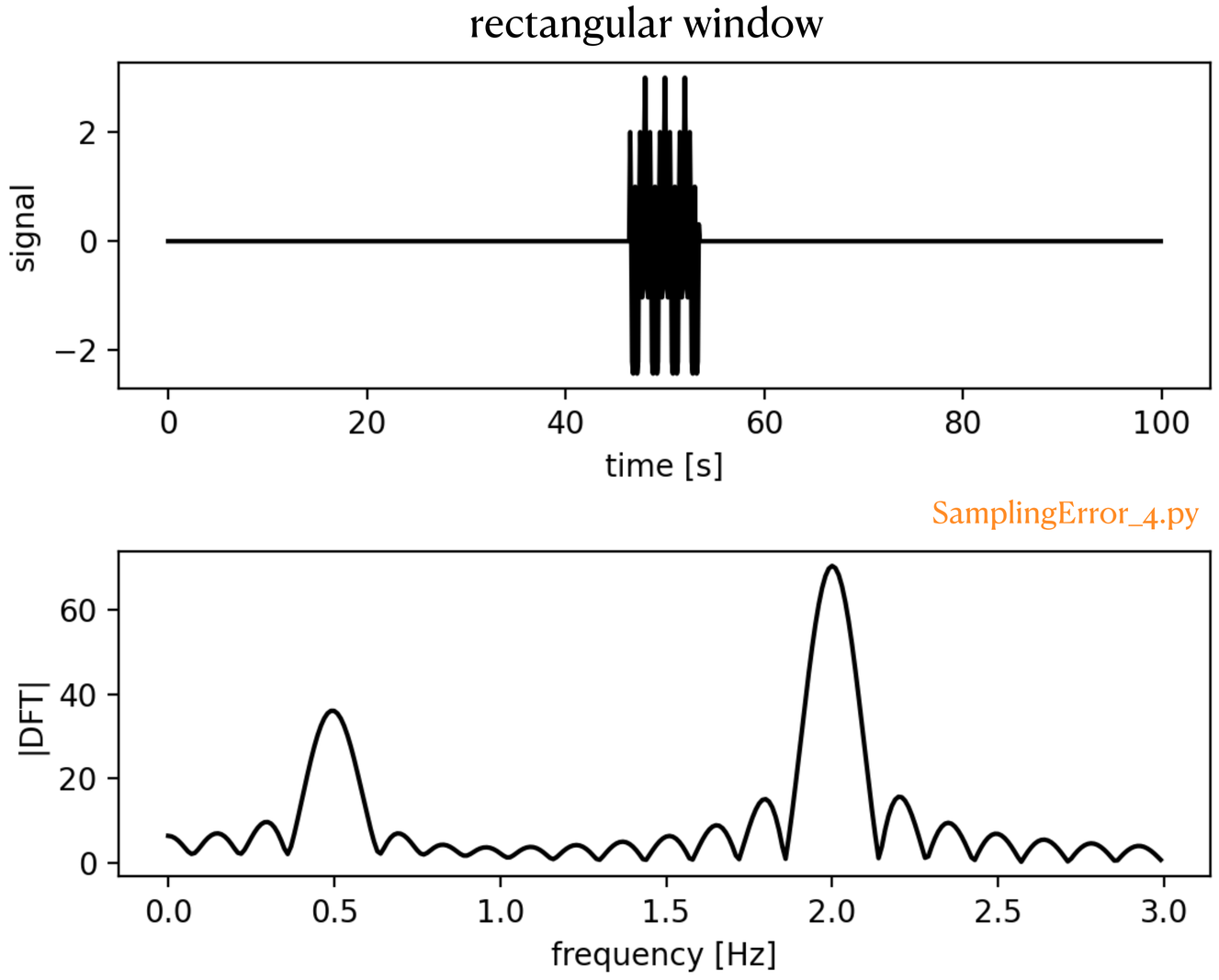
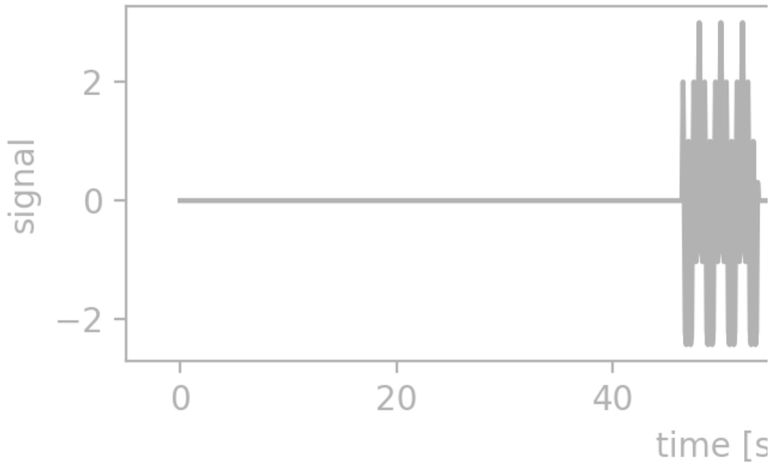
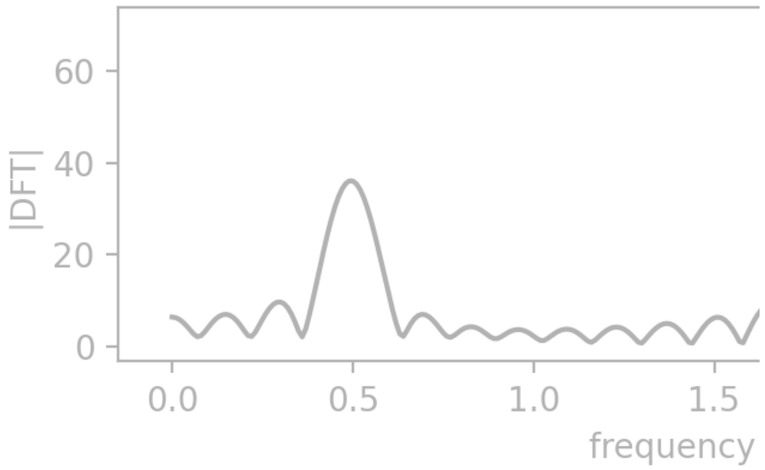
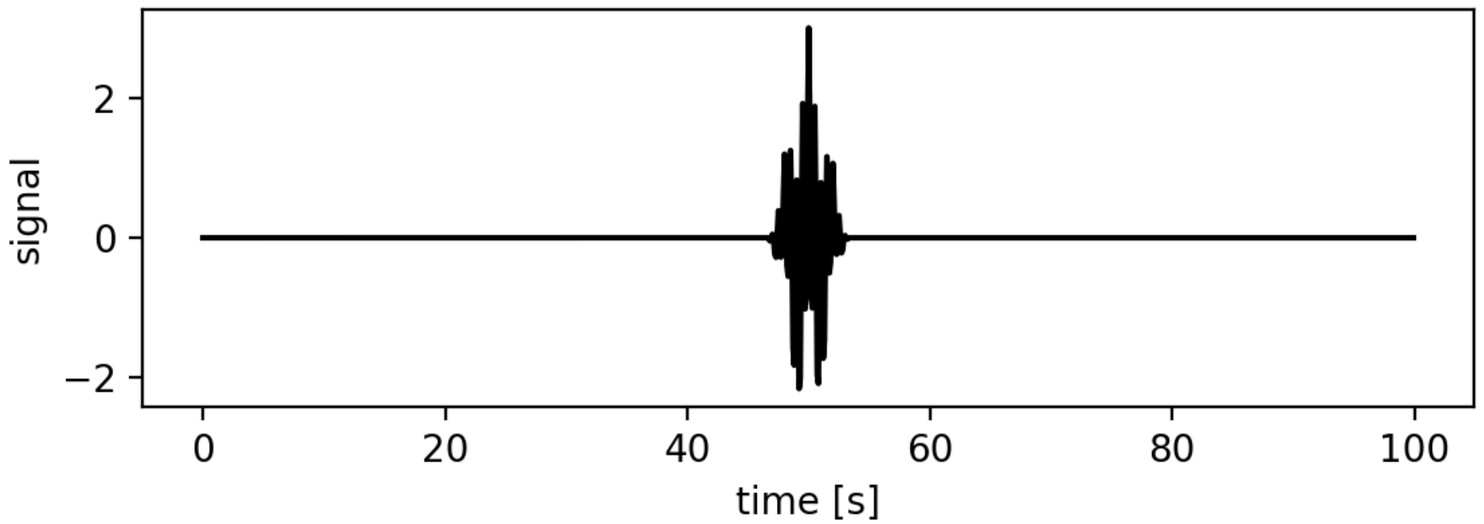



rectangular window
signal
2
0
−2
0
20
40
60
80
100
time [s]
SamplingError_4.py
|DFT|
60
40
20
0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
frequency [Hz]

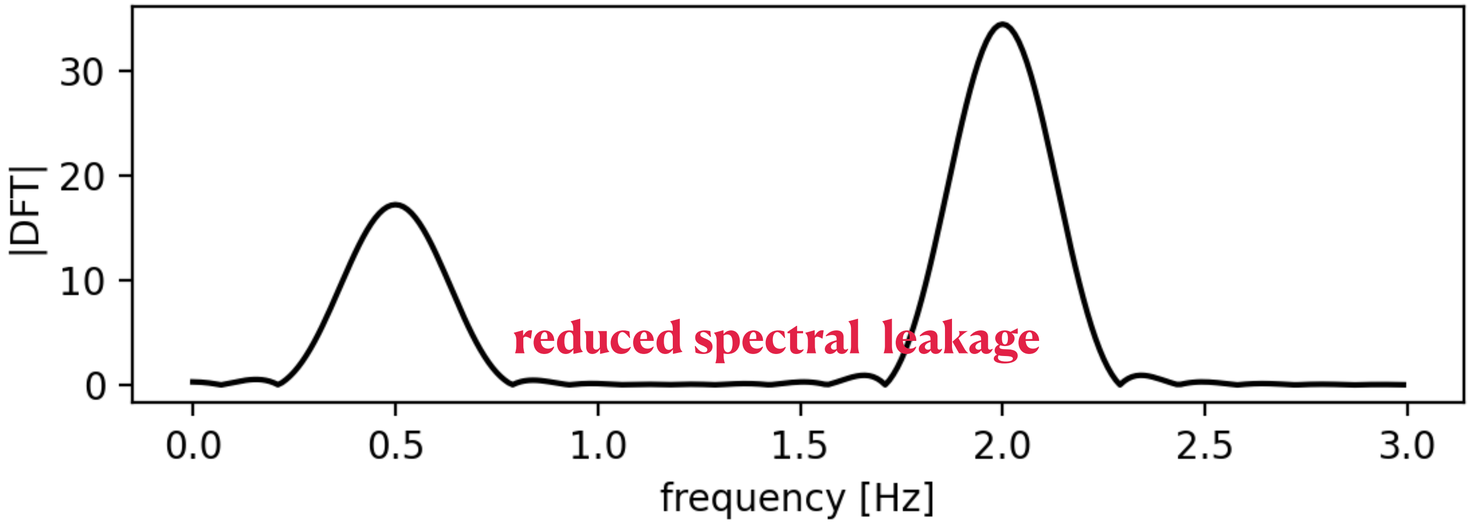




Hann window



SamplingError_4.py



summary:

if your signal is too short, then prolong with zeros (zero-padding)

and apply a specific window function (e.g. Hann window)

data sampling

Fourier analysis

errors in analysis aliasing spectral leakage **spectral power**

linear filters

time-frequency analysis