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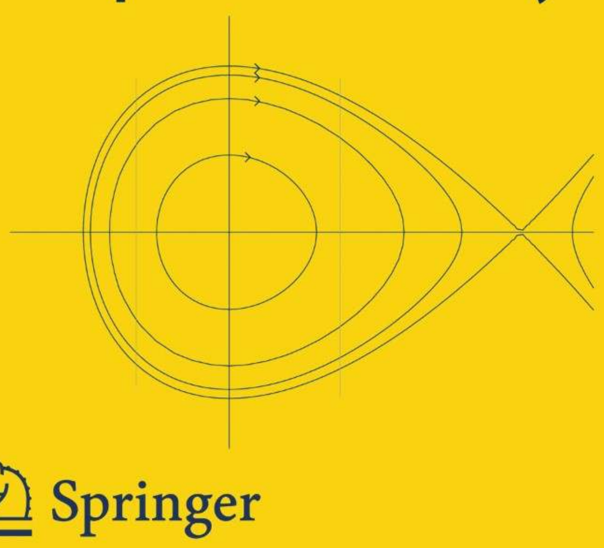
Ferdinand Verhulst

TEXTS IN APPLIED MATHEMATICS

Methods and Applications of Singular Perturbations

Boundary Layers and Multiple Timescale Dynamics



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Methods and Applications of Singular Perturbations

Boundary Layers and Multiple Timescale Dynamics

With 26 Illustrations

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Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series Texts in Applied Mathematics (TAM).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and to encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the Applied Mathematical Sciences (AMS) series, which will focus on advanced textbooks and research-level monographs.

Pasadena, California — J.E. Marsden
Providence, Rhode Island — L. Sirovich
Houston, Texas — M. Golubitsky
College Park, Maryland — S.S. Antman

Preface

Mathematics is more an activity than a theory
(Mathematatik is mehr ein Tun als eine Lehre)
Hermann Weyl, after L.E.J. Brouwer

Perturbation theory is a fundamental topic in mathematics and its applications to the natural and engineering sciences. The obvious reason is that hardly any problem can be solved exactly and that the best we can hope for is the solution of a "neighbouring" problem. The original problem is then a perturbation of the solvable problem, and what we would like is to establish the relation between the solvable and the perturbation problems.

What is a singular perturbation? The traditional idea is a differential equation (plus other conditions) having a small parameter that is multiplying the highest derivatives. This covers a lot of cases but certainly not everything. It refers to boundary layer problems only.

The modern view is to consider a problem with a small parameter ε and solution $x(t, \varepsilon)$. Also defined is an "unperturbed" (neighbouring) problem with solution $x(t, 0)$. If, in an appropriate norm, the difference $\|x(t, \varepsilon) - x(t, 0)\|$ does not tend to zero when ε tends to zero, this is called a singular perturbation problem. The problems in Chapters 1–9 are covered by both the old(fashioned) definition and the new one. Slow-time problems (multiple time dynamics), as will be discussed in later chapters, fall under the new definition. Actually, most perturbation problems in this book are singular by this definition; only in Chapter 10 shall we consider problems where "simple" continuation makes sense.

This book starts each chapter with studying explicit examples and introducing methods without proof. After many years of teaching the subject of singular perturbations, I have found that this is the best way to introduce this particular subject. It tends to be so technical, both in calculations and in theory, that knowledge of basic examples is a must for the student. This view is not only confirmed by my lectures in Utrecht and elsewhere but also by lecturers who used parts of my text in various places. In this respect, Hermann Weyl's quotation which is concerned with the fundamentals of mathematics, gives us the right perspective.

I have stressed that the proposed workbook format is very suitable for singular perturbation problems, but I hope that the added flavour of precise estimates and excursions into the theoretical background makes the book of interest both for people working in the applied sciences and for more theoretically oriented mathematicians.

Let me mention one more important subject of the forthcoming chapters. There will be an extensive discussion of timescales and a priori knowledge of the presence of certain timescales. This is one of the most widely used concepts in slow-time dynamics, and there is a lot of confusion in the literature. I hope to have settled some of the questions arising in choosing timescales.

What about theory and proofs one may ask. To limit the size of the book, those mathematical proofs that are easy to obtain from the literature are listed at the end of each chapter in a section "Guide to the Literature". If they are readily accessible, it usually makes no sense to reproduce them. Exceptions are sometimes cases where the proof contains actual constructions or where a line of reasoning is so prominent that it has to be included. In all cases discussed in this book - except in Chapter 14 - proofs of asymptotic validity are available. Under "Guide to the Literature" one also finds other relevant and recent references.

In a final chapter I collected pieces of theory that are difficult to find in the literature or a summary such as the one on perturbations of matrices or a typical and important type of proof such as the application of maximum principles for elliptic equations. Also, in the epilogue I return to the discussion about "proving and doing".

To give a general introduction to singular perturbations, I have tried to cover as many topics as possible, but of course there are subjects omitted. The first seven chapters contain standard topics from ordinary differential equations and partial differential equations, boundary value problems and problems with initial values within a mathematical framework that is more rigorous in formulation than is usual in perturbation theory. This improves the connection with theory-proof approaches. Also, we use important, but nearly forgotten theorems such as the du Bois-Reymond theorem.

Some topics are missing (such as the homogenisation method) or get a sketchy treatment (such as the WKBJ method). I did not include relaxation oscillations, as an elementary treatment can be found in my book *Nonlinear Differential Equations and Dynamical Systems*. Also there are books available on this topic, such as *Asymptotic Methods for Relaxation Oscillations and Applications* by Johan Grasman.

Perturbation theory is a fascinating topic, not only because of its applications but also because of its many unexpected results. A long time ago, Wiktor Eckhaus taught me the basics of singular perturbation theory, and at about the same time Bob O'Malley introduced me to Tikhonov's theorem and multiple scales.

Many colleagues and students made remarks and gave suggestions. I mention Abadi, Taoufik Bakri, Arjen Doelman, Hans Duistermaat, Wiktor Eckhaus, Johan Grasman, Richard Haberman, Michiel Hochstenbach, James Murdock, Bob O'Malley, Richard Rand, Bob Rink, Thijs Ruijgrok, Theo Tuwankotta, Adriaan van der Burgh. I got most of section 15.5 from Van Harten's (1975) thesis, section 15.9 is based on Buitelaar's (1993) thesis.

The figures in the first nine chapters were produced by Theo Tuwankotta; other figures were obtained from Abadi, Taoufik Bakri and Hartono. Copyeditor Hal Henglein of Springer proposed the addition of thousands of comma's and many layout improvements.

I am grateful to all of them.

Corrections and additions will be posted on
http://www.math.uu.nl/people/verhulst

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