APLIKASI TRANSFORMASI LAPLACE PADA STATISTIKA MATEMATIKA (TEORI STATISTIK)

SEBARAN GAMMA

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \cdot \beta^{\alpha}} & \text{, UNTUK } 0 \le x < \infty \\ 0 & \text{, UNTUK } x \text{ LAINNYA } (-\infty < x < 0) \end{cases}$$

BUKTIKAN : $\boxed{\mu = \alpha\beta \quad , \quad \sigma^2 = \alpha\beta^2}$

BUKTI

$$\mu = E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_{-\infty}^{0} x \cdot (0)\, dx + \int_{0}^{\infty} x \left(\frac{x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \cdot \beta^{\alpha}} \right) dx$$

$$= 0 + \frac{1}{\Gamma(\alpha) \cdot \beta^{\alpha}} \int_{0}^{\infty} e^{-\frac{1}{\beta}x} \cdot (x^{\alpha})\, dx$$

MISALKAN : $\boxed{\frac{1}{\beta} = s}$ DAN $x = t \rightarrow \frac{dx}{dt} = 1$

$\boxed{dx = dt}$

$$\mu = \frac{1}{\Gamma(\alpha) \cdot \beta^{\alpha}} \int_{0}^{\infty} e^{-st} (t^{\alpha})\, dt$$

$$= \frac{1}{\Gamma(\alpha) \cdot \beta^{\alpha}} \cdot L\{t^{\alpha}\} = \frac{1}{\Gamma(\alpha) \cdot \beta^{\alpha}} \cdot \frac{\alpha!}{s^{\alpha+1}}$$

$$= \frac{1}{\cancel{\Gamma(\alpha)} \cdot \beta^{\alpha}} \cdot \frac{\alpha \cdot \cancel{\Gamma(\alpha)}}{\left(\frac{1}{\beta}\right)^{\alpha+1}}$$

$\boxed{n! = n\,\Gamma(n)}$

$$= \frac{\alpha}{\beta^{\alpha} \cdot (\beta^{-1})^{\alpha+1}} = \frac{\alpha}{\underbrace{\beta^{\alpha} \cdot \beta^{-\alpha}}_{1} \cdot \beta^{-1}}$$

$$\boxed{\mu = \alpha\beta}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

$$= \int_{-\infty}^{0} x^2 .(0)\, dx + \int_{0}^{\infty} x^2 . \left(\frac{x^{\alpha-1} . e^{-\frac{x}{\beta}}}{\Gamma(\alpha) . \beta^{\alpha}} \right) dx$$

$$= 0 + \frac{1}{\Gamma(\alpha) . \beta^{\alpha}} \int_{0}^{\infty} e^{-\left(\frac{1}{\beta}\right)x} . x^{\alpha+1} . dx$$

$$= \frac{1}{\Gamma(\alpha) . \beta^{\alpha}} \int_{0}^{\infty} e^{-st} . \left(t^{\alpha+1}\right) dt$$

$$= \frac{1}{\Gamma(\alpha) . \beta^{\alpha}} \cdot L\{t^{\alpha+1}\}$$

$$= \frac{1}{\Gamma(\alpha) . \beta^{\alpha}} \cdot \frac{(\alpha+1)!}{s^{(\alpha+1)+1}}$$

$$= \frac{1}{\Gamma(\alpha) . \beta^{\alpha}} \cdot \frac{(\alpha+1) . \alpha!}{s^{\alpha+2}}$$

$$= \frac{1}{\Gamma(\alpha) . \beta^{\alpha}} \cdot \frac{(\alpha+1) . \alpha!}{\left(\frac{1}{\beta}\right)^{\alpha+2}}$$

$$= \frac{1}{\cancel{\Gamma(\alpha)} . \beta^{\alpha}} \cdot \frac{(\alpha+1) . \alpha \cancel{\Gamma(\alpha)}}{(\beta^{-1})^{\alpha+2}}$$

$$= \frac{\alpha(\alpha+1)}{\beta^{\alpha} . \beta^{-\alpha} . \beta^{-2}} = (\alpha^2+\alpha) . \beta^2$$

$$\boxed{E(x^2) = \alpha^2\beta^2 + \alpha\beta^2}$$

$$\sigma^2 = E(x^2) - \{E(x)\}^2$$

$$= E(x^2) - (\mu)^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2 - (\alpha\beta)^2$$

$$= \cancel{\alpha^2\beta^2} + \alpha\beta^2 - \cancel{\alpha^2\beta^2}$$

$$\boxed{\sigma^2 = \alpha\beta^2}$$