Penurunan Rumus-Rumus Reduksi

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Intisari

Sampai saat ini masih banyak buku tentang Kalkulus yang belum mengemukakan penurunan rumus-rumus reduksi, oleh karena itu penulis ingin membantu para dosen Matematika dan mahasiswa dalam hal pembuktian rumus-rumus integral reduksi tersebut.

Pendahuluan

Penurunan rumus-rumus reduksi pada buku-buku Kalkulus jarang dibahas, maka penulis ingin menurunkan rumus-rumus reduksi tersebut agar para dosen Matematika dan mahasiswa dapat memanfaatkannya.

<u>Penurunan Rumus</u>:

1. Buktikan:

$$\int \sin^n x \cdot dx = -\frac{1}{n}\sin^{n-1} x \cdot \cos x + \frac{n-1}{n}\int \sin^{n-2} x \cdot dx$$

Bukti:

$$
\begin{aligned}
\int \sin^n x \cdot dx &= \int \sin^{n-1} x \cdot \sin x \cdot dx \\
&= \int \sin^{n-1} x \cdot d(-\cos x) \\
&= \left(\sin^{n-1} x\right)(-\cos x) - \int -\cos x \cdot d\left[\sin^{n-1} x\right] \\
&= -\sin^{n-1} x \cdot \cos x + \int \cos x \cdot (n-1)(\sin x)^{n-1-1} \cdot d(\sin x) \\
&= -\sin^{n-1} x \cdot \cos x + (n-1)\int \cos x \cdot \sin^{n-2} x \cdot \cos x \cdot dx \\
&= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x \cdot \cos^2 x \cdot dx \\
&= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x \cdot \left(1-\sin^2 x\right) \cdot dx \\
\int \sin^n x \cdot dx &= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x \cdot dx - (n-1)\int \sin^n x \cdot dx
\end{aligned}
$$

$$
\begin{aligned}
\int \sin^n x \cdot dx + (n-1)\int \sin^n x \cdot dx &= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x \cdot dx \\
(1+n-1)\int \sin^n x \cdot dx &= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x \cdot dx \\
n\int \sin^n x \cdot dx &= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x \cdot dx \quad \left(\frac{1}{n}\right)
\end{aligned}
$$

Terbukti bahwa:

$$
\boxed{\int \sin^n x \cdot dx = -\frac{1}{n}\sin^{n-1} x \cdot \cos x + \frac{n-1}{n}\int \sin^{n-2} x \cdot dx}
$$

2. Buktikan:

$$
\boxed{\int \cos^n x \cdot dx = \frac{1}{n}\cos^{n-1} x \cdot \sin x + \frac{n-1}{n}\int \cos^{n-2} x \cdot dx}
$$

Bukti:

$$\begin{aligned}
\int \cos^n x \cdot dx &= \int \cos^{n-1} x \cdot \cos x \cdot dx \\
&= \int \cos^{n-1} x \cdot d(\sin x) \\
&= \left(\cos^{n-1} x\right)(\sin x) - \int \sin x \cdot d\left[\cos^{n-1} x\right] \\
&= \cos^{n-1} x \cdot \sin x - \int \sin x \cdot (n-1)(\cos x)^{n-1-1} \cdot d(\cos x) \\
&= \cos^{n-1} x \cdot \sin x - (n-1)\int \sin x \cdot \cos^{n-2} x \cdot (-\sin x) \cdot dx \\
&= \cos^{n-1} x \cdot \sin x + (n-1)\int \cos^{n-2} x \cdot \sin^2 x \cdot dx \\
&= \cos^{n-1} x \cdot \sin x + (n-1)\int \cos^{n-2} x \cdot \left(1-\cos^2 x\right) \cdot dx \\
\int \cos^n x \cdot dx &= \cos^{n-1} x \cdot \sin x + (n-1)\int \cos^{n-2} x \cdot dx - (n-1)\int \cos^n x \cdot dx
\end{aligned}$$

$$\begin{aligned}
\int \cos^n x \cdot dx + (n-1)\int \cos^n x \cdot dx &= \cos^{n-1} x \cdot \sin x + (n-1)\int \cos^{n-2} x \cdot dx \\
(1+n-1)\int \cos^n x \cdot dx &= \cos^{n-1} x \cdot \sin x + (n-1)\int \cos^{n-2} x \cdot dx \\
n\int \cos^n x \cdot dx &= \cos^{n-1} x \cdot \sin x + (n-1)\int \cos^{n-2} x \cdot dx \quad \left(\frac{1}{n}\right)
\end{aligned}$$

Terbukti bahwa

$$\boxed{\int \cos^n x \cdot dx = \frac{1}{n}\cos^{n-1} x \cdot \sin x + \frac{n-1}{n}\int \cos^{n-2} x \cdot dx}$$

3. Buktikan:

$$\int \text{cosec}^n x \cdot dx = -\frac{1}{n-1} \text{cosec}^{n-2} x \cdot \text{cotg}\, x + \frac{n-2}{n-1} \int \text{cosec}^{n-2} x \cdot dx$$

Petunjuk:

$$1 + \text{cotg}^2 x = \text{cosec}^2 x$$
$$\text{cotg}^2 x = \text{cosec}^2 x - 1$$

<u>Bukti</u>:

$$\begin{aligned}
\int \text{cosec}^n x \cdot dx &= \int \text{cosec}^{n-2} x \cdot \text{cosec}^2 x \cdot dx \\
&= \int \text{cosec}^{n-2} x \cdot d(-\text{cotg}\, x) \\
&= (\text{cosec}^{n-2} x)(-\text{cotg}\, x) - \int (-\text{cotg}\, x) \cdot d[\text{cosec}^{n-2} x] \\
&= -\text{cosec}^{n-2} x \cdot \text{cotg}\, x + \int \text{cotg}\, x \cdot (n-2)(\text{cosec}\, x)^{n-2-1} \cdot d(\text{cosec}\, x) \\
&= -\text{cosec}^{n-2} x \cdot \text{cotg}\, x + (n-2) \int \text{cotg}\, x \cdot \text{cosec}^{n-3} x \cdot (-\text{cosec}\, x \cdot \text{cotg}\, x) \cdot dx \\
&= -\text{cosec}^{n-2} x \cdot \text{cotg}\, x - (n-2) \int \text{cosec}^{n-2} x \cdot \text{cotg}^2 x \cdot dx \\
&= -\text{cosec}^{n-2} x \cdot \text{cotg}\, x - (n-2) \int \text{cosec}^{n-2} x \cdot (\text{cosec}^2 x - 1) \cdot dx \\
\int \text{cosec}^n x \cdot dx &= -\text{cosec}^{n-2} x \cdot \text{cotg}\, x - (n-2) \int \text{cosec}^n x \cdot dx + (n-2) \int \text{cosec}^{n-2} x \cdot dx
\end{aligned}$$

$$\begin{aligned}
\int \text{cosec}^n x \cdot dx + (n-2) \int \text{cosec}^n x \cdot dx &= -\text{cosec}^{n-2} x \cdot \text{cotg}\, x + (n-2) \int \text{cosec}^{n-2} x \cdot dx \\
(1 + n - 2) \int \text{cosec}^n x \cdot dx &= -\text{cosec}^{n-2} x \cdot \text{cotg}\, x + (n-2) \int \text{cosec}^{n-2} x \cdot dx \\
(n-1) \int \text{cosec}^n x \cdot dx &= -\text{cosec}^{n-2} x \cdot \text{cotg}\, x + (n-2) \int \text{cosec}^{n-2} x \cdot dx \quad \left(\frac{1}{n-1}\right)
\end{aligned}$$

Terbukti bahwa

$$\int \text{cosec}^n x \cdot dx = -\frac{1}{n-1} \text{cosec}^{n-2} x \cdot \text{cotg}\, x + \frac{n-2}{n-1} \int \text{cosec}^{n-2} x \cdot dx$$

4. Buktikan:

$$\int \sec^n x \cdot dx = \frac{1}{n-1} \sec^{n-2} x \cdot \operatorname{tg} x + \frac{n-2}{n-1} \int \sec^{n-2} x \cdot dx$$

Petunjuk:

$$1 + \operatorname{tg}^2 x = \sec^2 x$$
$$\operatorname{tg}^2 x = \sec^2 x - 1$$

<u>Bukti</u>:

$$\begin{aligned}
\int \sec^n x \cdot dx &= \int \sec^{n-2} x \cdot \sec^2 x \cdot dx \\
&= \int \sec^{n-2} x \cdot d(\operatorname{tg} x) \\
&= \left(\sec^{n-2} x\right)(\operatorname{tg} x) - \int (\operatorname{tg} x) \cdot d\left[\sec^{n-2} x\right] \\
&= \sec^{n-2} x \cdot \operatorname{tg} x - \int \operatorname{tg} x \cdot (n-2) \cdot (\sec x)^{n-2-1} \cdot d[\sec x] \\
&= \sec^{n-2} x \cdot \operatorname{tg} x - (n-2) \int \operatorname{tg} x \cdot \sec^{n-3} x \cdot \sec x \cdot \operatorname{tg} x \cdot dx \\
&= \sec^{n-2} x \cdot \operatorname{tg} x - (n-2) \int \sec^{n-2} x \cdot \operatorname{tg}^2 x \cdot dx \\
&= \sec^{n-2} x \cdot \operatorname{tg} x - (n-2) \int \sec^{n-2} x \cdot \left(\sec^2 x - 1\right) \cdot dx \\
\int \sec^n x \cdot dx &= \sec^{n-2} x \cdot \operatorname{tg} x - (n-2) \int \sec^n x \cdot dx + (n-2) \int \sec^{n-2} x \cdot dx
\end{aligned}$$

$$\begin{aligned}
\int \sec^n x \cdot dx + (n-2) \int \sec^n x \cdot dx &= \sec^{n-2} x \cdot \operatorname{tg} x + (n-2) \int \sec^{n-2} x \cdot dx \\
(1 + n - 2) \int \sec^n x \cdot dx &= \sec^{n-2} x \cdot \operatorname{tg} x + (n-2) \int \sec^{n-2} x \cdot dx \\
\underline{(n-1) \int \sec^n x \cdot dx} &\underline{= \sec^{n-2} x \cdot \operatorname{tg} x + (n-2) \int \sec^{n-2} x \cdot dx} \quad \left(\frac{1}{n-1}\right)
\end{aligned}$$

Terbukti bahwa

$$\int \sec^n x \cdot dx = \frac{1}{n-1} \sec^{n-2} x \cdot \operatorname{tg} x + \frac{n-2}{n-1} \int \sec^{n-2} x \cdot dx$$

5. Buktikan:

$$\int \cot g^{n} x \cdot dx = -\frac{1}{n-1} \cot g^{n-1} x - \int \cot g^{n-2} x \cdot dx$$

Petunjuk:

$$1 + \cot g^{2} x = \operatorname{cosec}^{2} x$$
$$\cot g^{2} x = \operatorname{cosec}^{2} x - 1$$

<u>Bukti</u>:

$$\begin{aligned}
\int \cot g^{n} x \cdot dx &= \int \cot g^{n-2} x \cdot \cot g^{2} x \cdot dx \\
&= \int \cot g^{n-2} x \cdot \left(\operatorname{cosec}^{2} x - 1\right) \cdot dx \\
&= \int \cot g^{n-2} x \cdot \operatorname{cosec}^{2} x \cdot dx - \int \cot g^{n-2} x \cdot dx \\
&= \int \cot g^{n-2} x \cdot d\left(-\cot g\, x\right) - \int \cot g^{n-2} x \cdot dx \\
&= -\int \cot g^{n-2} x \cdot d\left(\cot g\, x\right) - \int \cot g^{n-2} x \cdot dx \\
\int \cot g^{n} x \cdot dx &= -\frac{1}{(n-2)+1} \cot g^{(n-2)+1} x - \int \cot g^{n-2} x \cdot dx
\end{aligned}$$

Terbukti bahwa

$$\int \cot g^{n} x \cdot dx = -\frac{1}{n-1} \cot g^{n-1} x - \int \cot g^{n-2} x \cdot dx$$

6. Buktikan:

$$\boxed{\int tg^{n} x \cdot dx = \frac{1}{n-1} tg^{n-1} x - \int tg^{n-2} x \cdot dx}$$

Petunjuk:

$$\boxed{\begin{aligned} 1 + tg^{2} x &= \sec^{2} x \\ tg^{2} x &= \sec^{2} x - 1 \end{aligned}}$$

<u>Bukti</u>:

$$\begin{aligned}
\int tg^{n} x \cdot dx &= \int tg^{n-2} x \cdot tg^{2} x \cdot dx \\
&= \int tg^{n-2} x \cdot \left(\sec^{2} x - 1\right) \cdot dx \\
&= \int tg^{n-2} x \cdot \sec^{2} x \cdot dx - \int tg^{n-2} x \cdot dx \\
&= \int tg^{n-2} x \cdot d(tg\, x) - \int tg^{n-2} x \cdot dx \\
\int tg^{n} x \cdot dx &= \frac{1}{(n-2)+1} tg^{(n-2)+1} x - \int tg^{n-2} x \cdot dx
\end{aligned}$$

Terbukti bahwa

$$\boxed{\int tg^{n} x \cdot dx = \frac{1}{n-1} tg^{n-1} x - \int tg^{n-2} x \cdot dx}$$

Kesimpulan :

Kesimpulan :

Telah diturunkan rumus-rumus Reduksi di bawah ini.

$$\int \sin^{n} x \cdot dx = -\frac{1}{n} \sin^{n-1} x \cdot \cos x + \frac{n-1}{n} \int \sin^{n-2} x \cdot dx$$

$$\int \cos^{n} x \cdot dx = \frac{1}{n} \cos^{n-1} x \cdot \sin x + \frac{n-1}{n} \int \cos^{n-2} x \cdot dx$$

$$\int \operatorname{cosec}^{n} x \cdot dx = -\frac{1}{n-1} \operatorname{cosec}^{n-2} x \cdot \operatorname{cotg} x + \frac{n-2}{n-1} \int \operatorname{cosec}^{n-2} x \cdot dx$$

$$\int \sec^{n} x \cdot dx = \frac{1}{n-1} \sec^{n-2} x \cdot \operatorname{tg} x + \frac{n-2}{n-1} \int \sec^{n-2} x \cdot dx$$

$$\int \operatorname{cotg}^{n} x \cdot dx = -\frac{1}{n-1} \operatorname{cotg}^{n-1} x - \int \operatorname{cotg}^{n-2} x \cdot dx$$

$$\int \operatorname{tg}^{n} x \cdot dx = \frac{1}{n-1} \operatorname{tg}^{n-1} x - \int \operatorname{tg}^{n-2} x \cdot dx$$

Daftar Pustaka

Susanto,Gunawan. **Soal dan Penyelesaian Analisa I** , edisi 8. Delta Teknik Group Jakarta,1981.

Soemartojo,N. **Kalkulus I** . Karunika Jakarta, 1986.

Varberg, Purcell . **Calculus** , seventh edition. Prentise-Hall,Inc, 1997.