TRANSFORMASI Z
(Z - TRANSFORM)

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BARISAN (SEQUENCE), DITULIS DALAM BENTUK:

$\{f(K)\}, \{C_n\}, \{X_n\}$ dst DAPAT TERDIRI DARI BILANGAN REAL ATAU KOMPLEKS

CONTOH : $\{5, 10, 3, -2, 1, \underset{\uparrow}{4}, 6\}$

SIMBOL $\uparrow$ DIGUNAKAN UNTUK MENYATAKAN POSISI NOL, YAITU UNTUK $n = 0$ ATAU $K = 0$.

BILA TIDAK DIBERI SIMBOL $\uparrow$, BERARTI SUKU PALING KIRI (PERTAMA) MERUPAKAN POSISI $n = 0$.

CONTOH :

$f(n) = \frac{1}{2^n}$ $\quad \left\{\ldots, \frac{1}{2^{-3}}, \frac{1}{2^{-2}}, \frac{1}{2^{-1}}, \underset{\uparrow}{1}, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots\right\}$

$f(n) = 2^n$ $\quad \{2^{-4}, 2^{-3}, 2^{-2}, 2^{-1}, 1, 2, 2^2, 2^3\}$
$-4 \leq n \leq 3$

OPERASI PADA TRANSFORMASI Z.

BILA $f(n)$ DAN $g(n)$ MEMPUNYAI JUMLAH SUKU YANG SAMA, MAKA:

$$\{f(n)\} + \{g(n)\} = \{f(n) + g(n)\}$$

$$c\{f(n)\} = \{c \cdot f(n)\}$$

$$a\{f(n)\} + b \cdot \{g(n)\} = \{a \cdot f(n) + b \cdot g(n)\}$$

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① $\{15, 20, \underset{\uparrow}{14}, 12, 9, 7, 5\}$

SUKU KE $n=3$ ADALAH 7 DAN
SUKU KE $n=-2$ ADALAH 15.

(2) TULIS BARISAN $f(n) = 2^n$, $-4 \leq n \leq 3$

$\{2^{-4}, 2^{-3}, 2^{-2}, 2^{-1}, 1, 2, 2^2, 2^3\}$

(3) DARI CONTOH DI ATAS, BILA $2 \cdot f(n) = 2 \cdot 2^n$
$= 2^{n+1}$, MAKA

$\{2^{-3}, 2^{-2}, 2^{-1}, 1, 2, 2^2, 2^3, 2^4\}$

(4) BILA $f(n) = \begin{cases} 0 & , x < 0 \\ \frac{1}{3^n} & , x \geq 0 \end{cases}$

$\{\ldots, 0, 1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \ldots\}$

TRANSFORMASI Z

DEFINISI : TRANSFORMASI Z DARI $\{X_n\}$ ATAU $\{f(k)\}$ ADALAH :

$$\boxed{Z\{X_n\} = \sum_{-\infty}^{\infty} X_n \cdot Z^{-n}}$$

DI MANA :
1) Z SUATU BILANGAN KOMPLEKS
2) Z ADALAH OPERATOR DARI TRANSFORMASI Z
3) $Z(X_n)$ ATAU $X(z)$ ADALAH TRANSFORMASI DARI $\{X_n\}$

NOTASI YANG SERING DIGUNAKAN ADALAH
$Z[\{f(k)\}]$, $F(z)$, $X(z)$, $Z(X_n)$

(5) BILA $\{X_n\} = \{10, 8, \underset{\uparrow}{7}, 1, -1, 2\}$

MAKA $Z\{X_n\} = \{10z^2, 8z, 7z^0, 1 \cdot z^{-1}, -1 \cdot z^{-2}, 2 \cdot z^{-3}\}$
$= \{10z^2, 8z, 7, 1z^{-1}, -1z^{-2}, 2z^{-3}\}$

(6) BILA $\{X_n\} = \{10, 8, 7, 1, -1, 2\}$
MAKA $Z\{X_n\} = \{10z^0 + 8z^{-1} + 7z^{-2} + 1z^{-3} - 1z^{-4} + 2z^{-5}\}$
$= \{10 + 8z^{-1} + 7z^{-2} + 1z^{-3} - 1z^{-4} + 2z^{-5}\}$

(7) TRANSFORMASI Z DARI $X_n = \frac{1}{2^n}$, $-2 \leq n \leq 3$ (3)

$$X(z) = \frac{1}{2^{-2}} z^{-(-2)} + \frac{1}{2^{-1}} z^{-(-1)} + \frac{1}{2^0} z^{-(0)} + \frac{1}{2} z^{-1} + \frac{1}{2^2} z^{-2} + \frac{1}{2^3} z^{-3}$$

$$X(z) = 2^2 z^2 + 2z + \frac{1}{1} z^0 + \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2} + \frac{1}{8} z^{-3}$$

$$X(z) = 4z^2 + 2z + 1 + \frac{1}{2z} + \frac{1}{4z^2} + \frac{1}{8z^3}$$

SIFAT TRANSFORMASI Z

$$Z\left[a\{X_n\} + b\{Y_n\}\right] = a \cdot Z\{X_n\} + b \cdot Z\{Y_n\}$$

TEOREMA :

(1) BILA $Z\{X_n\} = X(z)$, MAKA $Z\{a^n \cdot (X_n)\} = X\left(\frac{z}{a}\right)$

(2) $Z\{X_{n \pm k}\} = z^{\pm k} \cdot X(z)$

(3) $Z\{n(X_n)\} = -z \frac{d}{dz}\{X(z)\}$

(4) $Z\{n^k(X_n)\} = \left(-z \frac{d}{dz}\right)^n \cdot X(z)$

(5) $Z\left\{\frac{X_n}{X}\right\} = -\int^z z^{-1} \cdot X(z)\, dz$

CONTOH .

(8) CARI TRANSFORMASI Z DARI $U_n = 1$, $n \geq 0$

$$Z\{U_n\} = \sum_0^{\infty} 1 \cdot z^{-n}$$

$$= 1 \cdot z^{-(0)} + 1 \cdot z^{-1} + 1 z^{-2} + 1 z^{-3} + z^{-4} + \dots$$

$$= 1 + z^{-1} + z^{-2} + \frac{1}{z^{-3}} + \frac{1}{z^4} + \dots$$

$$= \frac{a}{1-r} = \frac{1}{1-z^{-1}} = \frac{1}{1-\frac{1}{z}} \cdot \frac{z}{z} + \dots$$

$$= \boxed{\frac{z}{z-1}}$$

(9) CARI TRANSFORMASI Z DARI $U_n = 1$, $n \leq 0$

$$Z\{U_n\} = \sum_{-\infty}^{0} U_n . z^{-n} = \sum_{-\infty}^{0} (1).z^{-n} = \sum_{-\infty}^{0} \frac{1}{z^n}$$

$$= \ldots + \frac{1}{z^{-4}} + \frac{1}{z^{-3}} + \frac{1}{z^{-2}} + \frac{1}{z^{-1}} + \frac{1}{z^0}$$

$$= \ldots + z^4 + z^3 + z^2 + z + 1$$

$$= 1 + z + z^2 + z^3 + z^4 + \ldots (a = 1, r = z)$$

$$= \frac{a}{1-r} = \boxed{\frac{1}{1-z}}$$

(10) CARI TRANSFORMASI Z DARI $U_n = 1$, $n < 0$

$\{n = \ldots, -3, -2, -1\}$

$$Z\{U_n\} = \sum_{-\infty}^{-1} U_n . z^{-n} = \sum_{-\infty}^{-1} 1 . \frac{1}{z^n}$$

$$= \sum_{-\infty}^{-1} \frac{1}{z^n} = \ldots + \frac{1}{z^{-4}} + \frac{1}{z^{-3}} + \frac{1}{z^{-2}} + \frac{1}{z^{-1}}$$

$$= \ldots + z^4 + z^3 + z^2 + z^1$$

$$= z + z^2 + z^3 + z^4 + \ldots (a = z, r = z)$$

$$= \frac{a}{1-r} = \boxed{\frac{z}{1-z}}$$

(11) CARI TRANSFORMASI Z DARI $U_n = 1$, $n > 0$

$n = 1, 2, 3, 4, \ldots$

$$Z\{U_n\} = \sum_{n=1}^{\infty} U_n . z^{-n} = \sum_{n=1}^{\infty} 1 . \frac{1}{z^n} = \sum_{n=1}^{\infty} \frac{1}{z^n}$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ldots \quad (a = \tfrac{1}{z}, r = \tfrac{1}{z})$$

$$= \frac{a}{1-r} = \frac{\frac{1}{z}}{1-\frac{1}{z}} . \frac{z}{z} = \boxed{\frac{1}{z-1}}$$

(12) CARI TRANSFORMASI Z DARI $X_n = \frac{1}{2^n}$, $-2 \leq n \leq 3$

$n = \{-2, -1, 0, 1, 2, 3\}$

$$Z\{X_n\} = \sum_{n=-2}^{3} X_n . z^{-n} = \sum_{n=-2}^{3} \frac{1}{2^n} . z^{-n}$$

$$= \frac{z^{-(-2)}}{2^{-2}} + \frac{z^{-(-1)}}{2^{-1}} + \frac{z^{-(0)}}{2^0} + \frac{z^{-(1)}}{2^1} + \frac{z^{-2}}{2^2} + \frac{z^{-3}}{2^3}$$

$$= 4z^2 + 2z + 1 + \frac{1}{2z} + \frac{1}{4z^2} + \frac{1}{8z^3}$$

(13) CARI TRANSFORMASI Z DARI $x_n = a^n$, $n \geq 0$

$n = 0, 1, 2, 3, 4, \ldots$

$$Z\{x_n\} = \sum_{n=0}^{\infty} x_n \cdot z^{-n} = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} (a \cdot z^{-1})^n$$

$$= (a \cdot z^{-1})^0 + (a \cdot z^{-1})^1 + (az^{-1})^2 + (az^{-1})^3 + (az^{-1})^4 \ldots$$

$$= 1 + (az^{-1}) + (az^{-1})^2 + (az^{-1})^3 + (az^{-1})^4 + \ldots$$

$$(a = 1 ,\ r = a \cdot z^{-1})$$

$$= \boxed{\frac{a}{1-r}} = \frac{1}{1 - a \cdot z^{-1}} = \frac{1}{1 - \frac{a}{z}} \cdot \frac{z}{z}$$

$$= \boxed{\frac{z}{z-a}}$$

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(14) TRANSFORMASI Z DARI $x_n = a^{|n|}$, $-\infty < n < \infty$

$b = |n|$

$b^2 = n^2$

$b = \pm\sqrt{n^2}$

$\boxed{b = \pm n}$

$$Z\{x_n\} = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n} = \sum_{n=-\infty}^{\infty} a^{|n|} \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n} z^{-n} + \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (a^{-1} \cdot z^{-1})^n + \sum_{n=0}^{\infty} (a \cdot z^{-1})^n$$

$$= \left[\ldots + (a^{-1}z^{-1})^{-3} + (a^{-1}z^{-1})^{-2} + (a^{-1}z^{-1})^{-1}\right] + \left[(az^{-1})^0 + (az^{-1})^1 + (az^{-1})^2 + \ldots\right]$$

$$= \left[\ldots + a^3z^3 + a^2z^2 + az\right] + \left[1 + (az^{-1}) + (az^{-1})^2 + \ldots\right]$$

$$= \frac{az}{1-az} + \frac{1}{1-az^{-1}}$$

$$= \frac{az}{1-az} + \frac{1}{1-\frac{1}{az}} \cdot \frac{az}{az}$$

$$= \frac{az}{1-az} + \frac{az}{az-1}$$

INVERSE TRANSFORMASI Z

INVERSE DARI TRANSFORMASI Z DINYATAKAN :

$$Z^{-1}\{f(n)\} = \{x_n\}$$

ADA 3 CARA MENCARI INVERSE DARI TRANSF. Z

I. DENGAN PEMBAGIAN LANGSUNG

$$Z\{x_n\} = \frac{1}{z-3}, \quad |z| > 3 \longrightarrow Z^{-1}\{f(n)\} = \{x_n\} \;?$$

$$
\begin{array}{r|l}
 & z^{-1} + 3z^{-2} + 9z^{-3} + 27z^{-4} + \dots \\
z-3 & 1 \\
 & 1 - 3z^{-1} \; - \\
 & 3z^{-1} \\
 & 3z^{-1} - 9z^{-2} \; - \\
 & 9z^{-2} \\
 & 9z^{-2} - 27z^{-3} \; - \\
 & 27z^{-3} \\
 & 27z^{-3} - 81z^{-4} \; - \\
 & 81z^{-4}
\end{array}
$$

$$\sum_{n=1}^{\infty} x_n \cdot z^{-n} \; \cancel{\{x_n\} = z^{-1}\{f(n)\}} = z^{-1} + 3z^{-2} + 9z^{-3} + 27z^{-4} + \dots$$

$$\cancel{\{x_n\} = 3^{0} \cdot z^{0-1} + 3^{1} z}$$

$$\{x_n\} = 1 + 3 + 9 + 27 + \dots$$

$$= 3^0 + 3^1 + 3^2 + 3^3 + \dots$$

$$\boxed{\{x_n\} = 3^{n-1}, \quad n > 0}$$

II. DENGAN BINOMIAL

$$Z\{X_n\} = \frac{1}{z-3}, \quad |z| > 3 \longrightarrow \{X_n\} = ?$$

$$Z\{X_n\} = \frac{1}{z-3} = \frac{1}{z} \cdot \frac{1}{\left(1-\frac{3}{z}\right)}$$

$$= z^{-1} \cdot \left(\frac{1}{1-3 \cdot z^{-1}}\right)$$

$$= z^{-1}\left\{1 + (3z^{-1}) + (3z^{-1})^2 + (3z^{-1})^3 + \cdots\right\}$$

$$\sum_{n=-\infty}^{-1} x_n \cdot z^{-n} = 1 \cdot z^{-1} + 3z^{-2} + 3^2 z^{-3} + 3^3 \cdot z^{-4} + \cdots$$

$$\{X_n\} = 1 + 3 + 3^2 + 3^3 + \cdots$$

$$\{X_n\} = 3^0 + 3^1 + 3^2 + 3^3 + \cdots$$

$$\boxed{\{X_n\} = 3^{n-1}, \quad n > 0}$$

III. DENGAN MENGGUNAKAN RESIDU

YAITU RESIDU DARI $z^{n-1} X(z) = x_n = \left[(z - z_1)\, z^{n-1} X(z)\right]$

UNTUK $z = z_i$

CARI INVERSE DARI TRANSFORMASI Z BERIKUT INI

$$Z\{x_n\} = \frac{1}{(z-\frac{1}{2})(z-\frac{1}{3})} \quad , \quad |z| > \frac{1}{2}$$

$$z - \tfrac{1}{2} = 0 \qquad\qquad z - \tfrac{1}{3} = 0$$

$$\underline{z = \tfrac{1}{2}} \qquad\qquad \underline{z = \tfrac{1}{3}}$$

RESIDU DI $\boxed{z = \frac{1}{2}}$ ADALAH

$$= z^{n-1} \cdot X(z)$$
$$= \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{\left(\frac{1}{2} - \frac{1}{3}\right)}$$
$$= \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{\frac{3}{6} - \frac{2}{6}}$$
$$= \left(\frac{1}{2}\right)^{n-1} \cdot \frac{1}{\frac{1}{6}}$$
$$= \left(\frac{1}{2}\right)^{n-1} \cdot 6$$
$$= \underline{\underline{6 \cdot \left(\frac{1}{2}\right)^{n-1}}}$$

RESIDU DI $\boxed{z = \frac{1}{3}}$ ADALAH

$$= z^{n-1} \cdot X(z)$$
$$= \left(\frac{1}{3}\right)^{n-1} \cdot \frac{1}{\frac{1}{3} - \frac{1}{2}}$$
$$= \left(\frac{1}{3}\right)^{n-1} \cdot \frac{1}{\frac{2}{6} - \frac{3}{6}}$$
$$= \left(\frac{1}{3}\right)^{n-1} \cdot \frac{1}{-\frac{1}{6}}$$
$$= \left(\frac{1}{3}\right)^{n-1} \cdot (-6)$$
$$= \underline{\underline{-6 \left(\frac{1}{3}\right)^{n-1}}}$$

$$x_n = 6\left(\frac{1}{2}\right)^{n-1} + \left\{-6\left(\frac{1}{3}\right)^{n-1}\right\}$$

$$\boxed{x_n = 6\left\{\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{3}\right)^{n-1}\right\} \quad , \quad n \geq 1}$$

LATIHAN SOAL

1. CARI TRANSFORMASI Z DARI : $x_n = \left(\frac{1}{2}\right)^n$

2. CARI TRANSFORMASI Z DARI : $x_n = 3^{n-1}$

3. CARI TRANSFORMASI Z DARI :

$$x_n = 6\left\{\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{3}\right)^{n-1}\right\}, \quad n \geq 1$$

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4. CARI INVERSE DARI TRANSFORMASI Z :

$$Z\{x_n\} = \frac{1}{1-z}$$

5. CARI INVERSE DARI TRANSFORMASI Z :

$$Z\{x_n\} = \frac{z}{z-a}$$
