Aplikasi Persamaan Diferensial Linear Orde-2 pada Rangkaian Arus Searah

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ABSTRACT

This paper discusses solution models to differential equations of one-way current circuit of the form $L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}\,\mathrm{I}=\mathrm{E}_0\ \omega\ \cos\omega t$ which will produce solutions of stable current I_p in the one-way current circuit. Analytic solutions to the second order non homogeneous differential equations above are found using inverse operator method, and they can be made faster using Maple program.

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ABSTRAK

Makalah ini membahas tentang model penyelesaian persamaan diferensial Rangkaian Arus Searah dengan bentuk $L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}\,\mathrm{I}=\mathrm{E}_0\ \omega\ \cos\omega t$ yang akan menghasilkan Solusi arus dalam keadaan-mantap I_p di Rangkaian Arus Searah. Penyelesaian analitik PD Linear orde-2 Non Homogen tersebut dapat diperoleh dengan metode operator invers dan jauh lebih cepat penyelesaiannya bila dikerjakan dengan program Maple.

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PENDAHULUAN

Pada rangkaian arus searah yang digambarkan pada diagram di bawah, berlaku :

$E_L = L\,\mathrm{I}' = L\frac{dI}{dt}$ (*drop* pada induktor)

$E_R = R\,\mathrm{I}$ (*drop* pada resistor dengan hukum Ohm)

$E_C = \frac{1}{C}\int I(t)\,dt$ (*drop* pada kapasitor)

Jumlah dari tiga besaran di atas sama dengan besar dari gaya elektromotif $E(t)$. Mengingat $E(t)=E_0\sin\omega\,\mathrm{t}$ didapat persamaan :

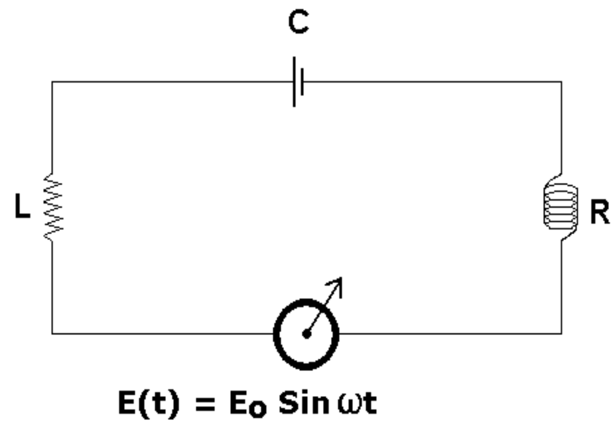
$$E_L+E_R+E_C=E(t)$$

$$L\frac{dI}{dt}+R\,\mathrm{I}+\frac{1}{C}\int I(t)\,dt=E_0\sin\omega\,\mathrm{t}$$

Dengan menurunkan kedua ruas ke t, diperoleh persamaan diferensial :

$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{C}\,\mathrm{I}=\mathrm{E}_0\ \omega\ \cos\omega t$ yang akan menghasilkan Solusi arus dalam keadaan-mantap I_p di rangkaian arus searah.

PEMODELAN MATEMATIKA



Gambar Rangkaian Arus Searah

Di mana : L = Induktans

R = Tahanan

C = Kapasitor

$E_0 =$ Gaya Gerak Listrik (GGL)

Persamaan Diferensial (PD) dari arus dalam Rangkaian Arus Searah adalah :

$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}\mathrm{I} = \mathrm{E}_0\ \omega\ \cos\omega t$,

$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{CL}\mathrm{I} = \frac{\mathrm{E}_0\,\omega}{\mathrm{L}}\cos\omega t$ (PD orde-2 Non Homogen) ,

$\mathrm{D}^2 I + \frac{R}{L}DI + \frac{1}{CL}\mathrm{I} = \frac{\mathrm{E}_0\,\omega}{\mathrm{L}}\cos\omega t$,

$\left(\mathrm{D}^2 + \frac{R}{L}D + \frac{1}{CL}\right)\mathrm{I} = \frac{\mathrm{E}_0\,\omega}{\mathrm{L}}\cos\omega t$, di mana $\mathrm{D}^2 = -\omega^2$.

Misalkan arus dalam keadaan-mantap di Rangkaian Arus Searah = I_p , di mana I_p merupakan Solusi Khusus dari PD di atas.

$$I_p = \frac{\frac{\mathrm{E}_0\,\omega}{\mathrm{L}}\cos\omega\,\mathrm{t}}{\mathrm{D}^2 + \frac{R}{L}D + \frac{1}{CL}} ,$$

$$I_p = \frac{\frac{\mathrm{E}_0\,\omega}{\mathrm{L}}\cos\omega t}{-\omega^2 + \frac{R}{L}D + \frac{1}{CL}} ,$$

$$I_p = \frac{\frac{\mathrm{E}_0\,\omega}{\mathrm{L}}\cos\omega t}{-\omega^2 + \frac{R}{L}D + \frac{1}{CL}} \cdot \frac{CL}{CL} ,$$

$$I_p = \frac{E_0\ \omega\ C \cos \omega t}{-\omega^2 CL + RCD + 1}\ ,$$

$$I_p = \frac{E_0\ \omega\ C \cos \omega t}{RCD - \left(\omega^2 CL - 1\right)} \cdot \frac{RCD + \left(\omega^2 CL - 1\right)}{RCD + \left(\omega^2 CL - 1\right)}\ ,$$

$$I_p = \frac{RCD\left(E_0\ \omega\ C \cos \omega t\right) + \left(\omega^2 CL - 1\right) E_0\ \omega\ C \cos \omega t}{\left(RCD\right)^2 - \left(\omega^2 CL - 1\right)^2}\ ,$$

$$I_p = \frac{-\ E_0\ \omega^2 C^2 R \sin \omega t + \left(\omega^2 CL - 1\right) E_0\ \omega\ C \cos \omega t}{R^2 C^2 D^2 - \left(\omega^2 CL - 1\right)^2}\ ,$$

$$I_p = \frac{-\ E_0\ \omega^2 C^2 R \sin \omega t + \left(\omega^2 CL - 1\right) E_0\ \omega\ C \cos \omega t}{-R^2 C^2 \omega^2 - \left(\omega^2 CL - 1\right)^2} \cdot \frac{(-1)}{(-1)}\ ,$$

$$I_p = \frac{E_0\ \omega^2 C^2 R \sin \omega t - \left(\omega^2 CL - 1\right) E_0\ \omega\ C \cos \omega t}{R^2 C^2 \omega^2 + \left(\omega^2 CL - 1\right)^2}\ ,$$

$$I_p = \frac{\dfrac{E_0\ \omega^2 C^2 R \sin \omega t}{\omega^2 C^2} - \dfrac{\left(\omega^2 CL - 1\right) E_0\ \omega\ C \cos \omega t}{\omega^2 C^2}}{\dfrac{R^2 C^2 \omega^2}{\omega^2 C^2} + \dfrac{\left(\omega^2 CL - 1\right)^2}{\omega^2 C^2}}\ ,$$

$$I_p = \frac{E_0\ R \sin \omega t - \dfrac{\left(\omega^2 CL - 1\right) E_0 \cos \omega t}{\omega C}}{R^2 + \dfrac{\left(\omega^2 CL - 1\right)^2}{(\omega C)^2}}\ ,$$

$$I_p = \frac{E_0\ R \sin \omega t - E_0\left(\dfrac{\omega^2 CL - 1}{\omega C}\right)\cos \omega t}{R^2 + \left(\dfrac{\omega^2 CL - 1}{\omega C}\right)^2}\ ,$$

$$I_p = \frac{E_0\ R \sin \omega t - E_0\left(\omega L - \dfrac{1}{\omega C}\right)\cos \omega t}{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}\ .$$

Ambil $\omega L - \dfrac{1}{\omega C} = S$, sehingga $I_p = \dfrac{E_0\ R \sin \omega t - E_0\ S \cos \omega t}{R^2 + S^2}$,

maka $I_p = \dfrac{E_0\ R}{R^2 + S^2} \sin \omega t - \dfrac{E_0\ S}{R^2 + S^2} \cos \omega t$,

$I_p = -\frac{E_0\, S}{R^2 + S^2} \cos \omega t + \frac{E_0\, R}{R^2 + S^2} \sin \omega t$ adalah arus dalam keadaan-mantap di Rangkaian Arus Searah.

$$I_p = -\frac{E_0\, S}{R^2 + S^2} \cos \omega t + \frac{E_0\, R}{R^2 + S^2} \sin \omega t \;, \tag{1}$$

$$I_p = \frac{E_0\, R}{R^2 + S^2} \left[\frac{\frac{E_0\, S}{R^2 + S^2}}{\frac{E_0\, R}{R^2 + S^2}} (\text{-} \cos \omega t) + \sin \omega t \right] ,$$

$$I_p = \frac{E_0\, R}{R^2 + S^2} \left[\frac{S}{R} (\text{-} \cos \omega t) + \sin \omega t \right] , \tag{2}$$

Ambil $\frac{S}{R} = tg\theta$ (3)

Substitusi (3) pada (2) :

$$I_p = \frac{E_0\, R}{R^2 + S^2} \left[tg\theta \cdot (\text{-} \cos \omega t) + \sin \omega t \right] ,$$

$$I_p = \frac{E_0\, R}{R^2 + S^2} \left[\frac{\sin\theta}{\cos\theta} \cdot (\text{-} \cos \omega t) + \sin \omega t \right] ,$$

$$I_p = \frac{E_0\, R}{R^2 + S^2} \left[\frac{\text{-} \cos \omega t \sin\theta + \sin \omega t \cos\theta}{\cos\theta} \right] ,$$

$$I_p = \frac{E_0\, R}{R^2 + S^2} \cdot \frac{1}{\cos\theta} \cdot \left[\sin \omega t \cos\theta \,\text{-}\, \cos \omega t \sin\theta \right] ,$$

$$I_p = \frac{E_0\, R}{R^2 + S^2} \cdot \sec\theta \cdot \sin(\omega t - \theta) \;, \tag{4}$$

di mana $\sec^2\theta = 1 + tg^2\theta$,

$$\sec^2\theta = 1 + \left(\tfrac{S}{R}\right)^2 \;,$$

$$\sec^2\theta = 1 + \tfrac{S^2}{R^2} \;,$$

$$\sec^2\theta = \tfrac{R^2 + S^2}{R^2} \;,$$

$$\sec\theta = \sqrt{\tfrac{R^2 + S^2}{R^2}} = \tfrac{\sqrt{R^2 + S^2}}{R} \;. \tag{5}$$

Substitusi (5) pada (4) :

$$I_p = \frac{E_0\, R}{R^2 + S^2} \cdot \frac{\sqrt{R^2 + S^2}}{R} \cdot \sin(\omega t - \theta) \;,$$

$$I_p = \frac{\mathrm{E_0\ R}}{\left(\sqrt{R^2+S^2}\right)^2} \cdot \frac{\sqrt{R^2+S^2}}{R} \cdot \sin(\omega t - \theta) \ ,$$

$$I_p = \frac{\mathrm{E_0}}{\sqrt{R^2+S^2}} \cdot \sin(\omega t - \theta) \ , \qquad (6)$$

di mana $\frac{\mathrm{E_0}}{\sqrt{R^2+S^2}} = I_0$,

sehingga (5) menjadi :

$I_p = I_0 \cdot \sin(\omega t - \theta)$ merupakan **arus keadaan-mantap** di Rangkaian Arus Searah.

Contoh aplikasi pada Rangkaian Arus Searah :

Tentukan arus keadaan-mantap di Rangkaian Arus Searah, apabila R = 50 Ohm, L = 25 Henry, C = 0,01 Farad, E = 500 Volt, dan $\omega = 3$!

Penyelesaian dengan PD 0rde-2 non homogen,

P D arus adalah :

$$L \tfrac{d^2I}{dt^2} + R \tfrac{dI}{dt} + \tfrac{1}{C}\mathrm{I} = \mathrm{E_0}\ \omega\ \cos \omega t \ ,$$

$$25 \tfrac{d^2I}{dt^2} + 50 \tfrac{dI}{dt} + \tfrac{1}{0{,}01}\mathrm{I} = (500)(3)\cos 3\mathrm{t} \ ,$$

$$25 \tfrac{d^2I}{dt^2} + 50 \tfrac{dI}{dt} + 100\ \mathrm{I} = 1500 \cos 3\mathrm{t} \ ,$$

$\tfrac{d^2I}{dt^2} + 2 \tfrac{dI}{dt} + 4\ \mathrm{I} = 60 \cos 3\mathrm{t}$ (PD linear orde-2 non homogen) ,

$$\mathrm{D}^2 I + 2DI + 4I = 60 \cos 3\mathrm{t} \ ,$$

$(\mathrm{D}^2 + 2D + 4)I = 60 \cos 3\mathrm{t}$, di mana $\mathrm{D}^2 = -(3)^2 = -9$,

Solusi Khusus (Partikulir) PD :

$$I_p = \frac{60 \cos 3\mathrm{t}}{\mathrm{D}^2 + 2D + 4} \ ,$$

$$I_p = \frac{60 \cos 3\mathrm{t}}{-9 + 2D + 4} \ ,$$

$$I_p = \frac{60 \cos 3\mathrm{t}}{2D - 5} \cdot \frac{2D+5}{2D+5} \ ,$$

$$I_p = \frac{2D(60 \cos 3\mathrm{t}) + 5(60 \cos 3\mathrm{t})}{(2D)^2 - (5)^2} \ ,$$

$$I_p = \frac{120(-\sin 3\mathrm{t})(3) + 300(\cos 3\mathrm{t})}{4D^2 - 25} \ ,$$

$$I_p = \frac{-360 \sin 3t + 300 \cos 3t}{4(-9) - 25} \ ,$$

$$I_p = \frac{300 \cos 3t - 360 \sin 3t}{-36 - 25} \ ,$$

$$I_p = \frac{300 \cos 3t - 360 \sin 3t}{-61} \cdot \frac{(-1)}{(-1)} \ ,$$

$$I_p = \frac{-300 \cos 3t + 360 \sin 3t}{61} \ , \qquad (7)$$

$$I_p = \frac{60}{61}(-5 \cos 3t + 6 \sin 3t) \ ,$$

$$I_p = \frac{60}{61} \cdot 6 \left[\frac{5}{6}(-\cos 3t) + \sin 3t \right] ,$$

$$I_p = \frac{360}{61} \left[\frac{5}{6}(-\cos 3t) + \sin 3t \right] ,$$

Ambil $\frac{5}{6} = tg\theta$,

maka $I_p = \frac{360}{61}\left[tg\theta \cdot (-\cos 3t) + \sin 3t\right]$,

$$I_p = \frac{360}{61} \left[\frac{\sin\theta}{\cos\theta} \cdot (-\cos 3t) + \sin 3t \right] ,$$

$$I_p = \frac{360}{61} \left[\frac{-\cos 3t \cdot \sin\theta + \sin 3t \cdot \cos\theta}{\cos\theta} \right] ,$$

$$I_p = \frac{360}{61} \cdot \frac{1}{\cos\theta} \left[\sin 3t \cdot \cos\theta - \cos 3t \cdot \sin\theta \right] ,$$

$$I_p = \frac{360}{61} \cdot \sec\theta \cdot \sin(3t - \theta) \ . \qquad (8)$$

di mana $\sec^2\theta = 1 + tg^2\theta$,

$$\sec^2\theta = 1 + \left(\tfrac{5}{6}\right)^2 \ ,$$

$$\sec^2\theta = 1 + \tfrac{25}{36} \ ,$$

$$\sec^2\theta = \tfrac{61}{36} \ ,$$

$$\sec\theta = \sqrt{\tfrac{61}{36}} = \tfrac{\sqrt{61}}{6} \ . \qquad (9)$$

Substitusi (9) pada (8) :

$$I_p = \frac{360}{61} \cdot \frac{\sqrt{61}}{6} \cdot \sin(3t - \theta) \ ,$$

$$I_p = \frac{60}{\left(\sqrt{61}\right)^2} \cdot \frac{\sqrt{61}}{1} \cdot \sin(3t - \theta) \ ,$$

$I_p = \frac{60}{\sqrt{61}} \cdot \sin(3t - \theta)$ merupakan **arus keadaan-mantap** di Rangkaian Arus Searah.

Penyelesaian dengan cara yang lebih singkat menggunakan rumus :

Tentukan arus keadaan-mantap di Rangkaian Arus Searah, apabila R = 50 Ohm, L = 25 Henry, C = 0,01 Farad, E = 500 Volt, dan $\omega = 3$!

Solusi :

$$S = \omega \mathrm{L} - \frac{1}{\omega \mathrm{C}} \ ,$$

$$S = 3(25) - \frac{1}{3(0{,}01)} \ ,$$

$$S = 75 - \frac{100}{3} = \frac{225 - 100}{3} = \frac{125}{3} \ .$$

$$I_0 = \frac{E_0}{\sqrt{R^2 + S^2}} \ ,$$

$$I_0 = \frac{500}{\sqrt{(50)^2 + \left(\frac{125}{3}\right)^2}} = \frac{500}{\sqrt{2500 + \frac{15625}{9}}} = \frac{500}{\sqrt{\frac{22500 + 15625}{9}}} \ ,$$

$$I_0 = \frac{500}{\sqrt{\frac{38125}{9}}} = \frac{500}{\frac{25\sqrt{61}}{3}} = \frac{1500}{25\sqrt{61}} = \frac{60}{\sqrt{61}} \ .$$

Arus keadaan-mantap di Rangkaian Arus Searah :

$I_p = I_0 \cdot \sin(\omega t - \theta)$ di mana $tg\theta = \frac{5}{6}$, sehingga $\theta = arc\ tg\ \frac{5}{6}$.

$I_p = \frac{60}{\sqrt{61}} \sin(3t - \theta)$.

Dengan program Maple dapat diperoleh hasilnya dalam waktu yang singkat.

$\frac{d^2 I}{dt^2} + 2\frac{dI}{dt} + 4\,\mathrm{I} = 60\cos 3\mathrm{t}$ (PD orde-2 non homogen),

```
> restart:
> ode[1] := diff(diff(y(t),t),t) = -2*diff(y(t),t)-4*y(t)+60*cos(3*t);
```

$$ode_1 := \frac{d^2}{dt^2}\,y(t) = -2\left(\frac{d}{dt}\,y(t)\right) - 4\,y(t) + 60\cos(3\,t)$$

```
> ans[1] := dsolve(ode[1]);
```

$$ans_1 := y(t) = \mathbf{e}^{(-t)}\sin(\sqrt{3}\,t)\,\_C2 + \mathbf{e}^{(-t)}\cos(\sqrt{3}\,t)\,\_C1 + \frac{360}{61}\sin(3\,t) - \frac{300}{61}\cos(3\,t)$$

di mana :

Solusi umum PD Linear orde-2 Homogen :

$$I_h = C_2 e^{-t}\sin\left(\sqrt{3}\cdot t\right) + C_1 e^{-t}\cos\left(\sqrt{3}\cdot t\right)$$

Solusi Khusus (Partikulir) PD :

$I_p = \frac{360}{61}\sin 3t - \frac{300}{61}\cos 3t$ (**arus keadaan-mantap** di Rangkaian Arus Searah yang jawabannya sama dengan persamaan (7).

Simpulan

Penerapan PD Linear orde-2 pada Rangkaian Arus Searah dari $L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}\,\mathrm{I} = \mathrm{E}_0\,\omega\,\cos\omega t$ menghasilkan Solusi arus dalam keadaan-mantap di Rangkaian Arus Searah :

$I_p = I_0\cdot\sin(\omega t - \theta)$ di mana : $I_0 = \frac{\mathrm{E}_0}{\sqrt{R^2 + s^2}}$ dan $\theta = arc\ \mathrm{tg}\,\frac{s}{\mathrm{R}}$.

Penyelesaian masalah Rangkaian Arus Searah dapat menggunakan PD Linear orde-2 Non Homogen atau penyelesaiannya mengunakan rumus dan jauh lebih cepat menggunakan program Maple.

DAFTAR PUSTAKA

Edward,C.H.,Penney,D.E. 2002, *Calculus 6e* , New Jersey, Prentice-Hall,Inc.
Kartono.2001, *Maple untuk Persamaan Diferensial* , Yogyakarta, J & J Learning
Kreyszig,Erwin.1999, *Advanced Engineering Mathematics* , New York, John Wiley and Sons,Inc.
Nababan,S.M. 1997, *Pendahuluan Persamaan Diferensial Biasa* , Jakarta, Karunika-UT