HOMEWORK

2006.12.7

4. Solve the *Ornstein-Uhlenbeck* equation:

$$dX_t = \mu X_t dt + \sigma dB_t,\ t \geq 0; X_0 = x_0$$

where μ, σ and x_0 are constants. And then, find $\mathbf{E}[X_t]$ and $\mathrm{var}(X_t)$ for each $t \geq 0$.
Solution:

$$dX_t = (b_1(t)X_t + b_2(t))dt + \sigma_2(t)dB_t \tag{2.3}$$

$$\Phi_t = \exp\{\int_0^t b_1(s)ds\}$$

then,

$$X_t = \Phi_t \cdot (\xi + \int_0^t b_2(s)\Phi_s^{-1}ds + \int_0^t \sigma_2(s)\Phi_s^{-1}dB_s)$$

$b_1(t) = \mu$, $b_2(t) = 0$, $\sigma_2(t) = \sigma$, satisfies (2.3)

$$\Phi_t = \exp\{\int_0^t \mu ds\} = e^{\mu t}$$

Hence,

$$X_t = e^{\mu t}(x_0 + \int_0^t 0 \cdot e^{-\mu s}ds + \int_0^t \sigma e^{-\mu s}dB_s) = e^{\mu t}(x_0 + \int_0^t \sigma e^{-\mu s}dB_s)$$

By Theorem 6.2.3. linear SDE(2.3) satisfy,
let $m(t) = \mathbf{E}[X_t]$, and $p(t) = \mathbf{E}[X_t^2]$, with the following system ODE.

$$\begin{cases} dm(t) = (b(t)m(t) + b_2(t))dt \\ dp(t) = 2b_1(t)p(t)dt + 2m(t)b_2(t)dt + \sigma_2^2(t)dt \end{cases}$$

$dm(t) = \mu \cdot m(t)dt \Rightarrow m^{-1}(t)dm(t) = \mu dt \Rightarrow \ln(m(t)) = \mu \int dt = \mu t + \ln(m_0(t))$
$\Rightarrow m(t) = m_0(t) \cdot e^{\mu t} = x_0 e^{\mu t}$
where $m_0(t) = \mathbf{E}(X_0) = \mathbf{E}(x_0) = x_0$, hence $\mathbf{E}[X_t] = x_0 e^{\mu X_t}$
$dp(t) = \mu \cdot p(t)dt + 2x_0 e^{\mu t} \cdot 0dt + \sigma^2 dt$
$\Rightarrow dp(t) = (\mu \cdot p(t) + \sigma^2)dt$
$\Rightarrow (\mu \cdot p(t) + \sigma^2)^{-1}dp(t) = dt$
$\Rightarrow \ln(\mu \cdot p(t) + \sigma) = t + \ln(\mu \cdot p_0(t) + \sigma^2)$
$\Rightarrow \mu \cdot p(t) + \sigma^2 = (\mu \cdot p_0(t) + \sigma^2) \cdot e^t$
$\Rightarrow p(t) = \mu^{-1} \cdot ((\mu \cdot x_0 + \sigma^2) \cdot e^t - \sigma^2)$
Therefore, $\mathbf{E}[X_t^2] = \mu^{-1} \cdot ((\mu \cdot x_0^2 + \sigma^2)e^{X_t} - \sigma^2)$,
Hence, $\mathrm{var}(X_t) = \mathbf{E}[X_t^2] - \mathbf{E}[X_t]^2 = [((\mu x_0^2 + \sigma^2)e^{X_t} - \sigma^2) \cdot \mu^{-1} - x_0^2 e^{2\mu X_t}]^{\frac{1}{2}}$.

7. Solve the reducible SDE:

$$dX_t = \frac{1}{2}a(a-1)X_t^{1-\frac{2}{a}}dt + aX_t^{1-\frac{1}{a}}dB_t,\ t \geq 0; X_0 = x_0.$$

Solution:

$$dX_t = \frac{1}{2}\sigma(X_t)\sigma'(X_t)dt + \sigma(X_t)dB_t \tag{3.3}$$

let

$$h(x) = \int_a^x \frac{1}{\sigma(u)}du;\ \forall x \in \mathrm{Dom}(\sigma)$$

then,

$$X_t = h^{-1}(B_t + h(x_0))$$

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let $\sigma(X_t) = aX_t^{1-\frac{1}{a}}$, then, $\sigma'(X_t) = a(1-\frac{1}{a})X_t^{-\frac{1}{a}} = (a-1)X_t^{-\frac{1}{a}}$ and $\sigma(X_t)\sigma'(X_t) = a(a-1)X_t^{1-\frac{2}{a}}$, satisfies (3.3), so that

$$h(x) = \int_a^x \frac{1}{\sigma(u)}du = \frac{1}{a}\int_a^x u^{\frac{1}{a}-1}du = \frac{1}{a}\cdot a\cdot x^{\frac{1}{a}}$$

$$h^{-1}(x) = x^a$$

Hence,

$$X_t = h^{-1}(B_t + h(x_0)) = h^{-1}(B_t + x_0^{\frac{1}{a}}) = (B_t + x_0^{\frac{1}{a}})^a.$$

8. Solve the reducible SDE:

$$dX_t = \frac{1}{2}(\ln a)^2 X_t dt + (\ln a)X_t dB_t,\ t \geq 0;\ X_0 = x_0$$

Solution:

$$dX_t = \frac{1}{2}(\ln a)^2 X_t dt + (\ln a)X_t dB_t$$

$$X_t^{-1}dX_t = \frac{1}{2}(\ln a)^2 dt + (\ln a)dB_t$$

$$\ln(X_t) = \ln(X_0) + \frac{1}{2}(\ln a)^2\int dt + (\ln a)\int dB_t$$

$$X_t = \exp\{\ln(x_0) + \frac{1}{2}(\ln a)^2\cdot t + (\ln a)\cdot B_t\}$$

$$X_t = \exp\{\ln(x_0) + \frac{1}{2}t(\ln a)\cdot(\ln a) + (\ln a)\cdot B_t\}$$

$$X_t = x_0 a^{\frac{1}{2}t\ln a + B_t}$$

9. Solve the reducible SDE:

$$dX_t = -\frac{1}{2\ln b}b^{-2X_t}dt + \frac{1}{\ln b}b^{-X_t}dB_t,\ t \geq 0;\ X_0 = x_0$$

Solution: let $\sigma(X_t) = \frac{1}{\ln b}b^{-X_t}$, then, $\sigma'(X_t) = \frac{1}{\ln b}b^{-X_t}\cdot(\ln b)\cdot(-1) = (-1)b^{X_t}$, and $\frac{1}{2}\sigma(X_t)\sigma'(X_t) = -\frac{1}{2\ln b}b^{-2X_t}$, satisfies (3.3), so that

$$h(x) = \int_a^x \frac{1}{\sigma(u)}du = \int_a^x \ln b\cdot b^u du = b^x$$

$$h^{-1}(x) = \frac{\ln x}{\ln b}$$

Hence,

$$X_t = h^{-1}(B_t + h(x_0)) = h_{-1}(B_t + b^{x_0}) = \frac{1}{\ln b} \cdot \ln(B_t + b^{x_0}),\ \forall 0 \le t < \tau$$

where $\tau = \inf\{t \ge 0 | B_t + b^{x_0} \le 0\}$

10. Solve the reducible SDE:

$$dX_t = -\frac{1}{2}a^2 X_t dt + a\sqrt{1 - X_t^2} dB_t$$

Solution: let $\sigma(X_t) = a\sqrt{1 - X_t^2}$, then, $\sigma'(X_t) = a\cdot(-\frac{1}{2})(\sqrt{1 - X_t^2})^{-1}\cdot(2X_t) = (-1)aX_t(\sqrt{1 - X_t^2})^{-1}$ and $\sigma(X_t\sigma'(X_t = (-1)a^2 X_t$, satisfies (3.3), so that

$$h(x) = \int_a^t \frac{1}{\sigma(u)} du = \frac{1}{a} \int_a^x \frac{1}{\sqrt{1 - u^2}} du = \frac{1}{a} \arcsin x$$

$$h^{-1}(x) = \sin(ax)$$

Hence,

$$X_t = h^{-1}(B_t + h(x_0)) = h^{-1}(B_t + \frac{1}{a} \arcsin x_0) = \sin(aB_t + a \cdot \frac{1}{a} \arcsin x_0) = \sin(aB_t + \arcsin x_0).$$

11. Solve the reducible SDE:

$$dX_t = a^2 X_t(1 + X_t^2)dt + a(1 + X_t^2)dB_t,\ t \ge 0;\ X_0 = x_0$$

Solution: let $\sigma(X_t) = a(1 + X_t^2)$, then, $\sigma'(X_t) = a \cdot 2X_t = 2aX_t$
and $\frac{1}{2}\sigma(X_t)\sigma'(X_t) = \frac{1}{2}a(1 + X_t^2) \cdot 2aX_t = a^2 X_t(1 + X_t^2)$, satisfies (3.3), so that

$$h(x) = \int_a^x = \frac{1}{\sigma(u)} du = \frac{1}{a} \int_a^x \frac{a}{1 + u^2} du = \frac{1}{a} \arctan x$$

$$h^{-1}(x) = \tan(ax)$$

Hence,

$$X_t = h^{-1}(B_t + h(x_0)) = h^{-1}(B_t + \frac{1}{a} \arctan x_0) = \tan(aB_t + \arctan x_0),\ \forall 0 \le t < \tau$$

where $\tau = \inf\{t \ge 0 || aB_t + \arctan x_0| \le \frac{\pi}{2}\}$

12. Solve the reducible SDE:

$$dX_t = -\frac{1}{2}a^2 X_t dt - a\sqrt{1 - X_t^2} dB_t$$

Solution: let $\sigma(X_t) = -a\sqrt{1 - X_t^2}$, then, $\sigma'(X_t) = -a\cdot(-\frac{1}{2})(\sqrt{1 - X_t^2})^{-1}\cdot(2X_t) = aX_t(\sqrt{1 - X_t^2})^{-1}$ and $\sigma(X_t\sigma'(X_t = (-1)a^2 X_t$, satisfies (3.3), so that

$$h(x) = \int_a^t \frac{1}{\sigma(u)} du = \frac{-1}{a} \int_a^x \frac{1}{\sqrt{1 - u^2}} du = \frac{-1}{a} \arcsin x$$

$$h^{-1}(x) = \sin(-ax)$$

Hence,

$$X_t = h^{-1}(B_t + h(x_0)) = h^{-1}(B_t - \frac{1}{a} \arcsin x_0) = \sin(-aB_t + a \cdot \frac{1}{a} \arcsin x_0) = \sin(-aB_t + \arcsin x_0).$$

14. Solve the *stochastic Verhulst* equation:

$$dX_t = (\lambda X_t - X_t^2)dt + \sigma X_t dB_t,\ t \geq 0;\ X_0 = x_0$$

Solution

$$dX_t = (\alpha X_t^n + \beta X_t)dt + \sigma X_t dB_t;\ X_0 = x_0 \neq 0 \tag{3.13}$$

let,

$$\Theta_t = \exp\{(\beta - \frac{1}{2}\sigma^2)t + \sigma B_t\}$$

then,

$$X_t = \Theta_t \cdot (x_0^{1-n} + (1-n)\alpha \int_0^t \Theta_s^{n-1} ds)^{\frac{1}{1-n}}$$

Since $\alpha = 1$, $\beta = \lambda$, $\sigma = \sigma$, $n = 2$, satisfies (3.13)

$$\Theta_t = \exp\{(\lambda - \frac{1}{2}\sigma^2)t + \sigma B_t\} = e^{(\lambda - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

Hence,

$$X_t = \exp\{(\lambda - \frac{1}{2}\sigma^2)t + \sigma B_t\}(x_0^{-1} + (-1)(-1)\int_0^t \exp\{(\lambda - \frac{1}{2}\sigma^2)t + \sigma B_t\} ds)^{-1}$$

$$X_t = e^{(\lambda - \frac{1}{2}\sigma^2)t + \sigma B_t}(x_0^{-1} + \int_0^t e^{(\lambda - \frac{1}{2}\sigma^2)t + \sigma B_t} ds)^{-1}.$$

15. Solve the *stochastic Ginzbury-Landau* equation:

$$dX_t = [-X_t^3 + (\alpha + \frac{1}{2}\sigma^2)X_t]dt + \sigma X_t dB_t,\ t \geq 0;\ X_0 = x_0$$

Solution: Since $\alpha = -1$, $\beta = \alpha + \frac{1}{2}\sigma^2$, $\sigma = \sigma$, satisfies (3.13), so that

$$\Theta_t = \exp\{(\alpha + \frac{1}{2}\sigma^2 - \frac{1}{2}\sigma^2)t + \sigma B_t\} = e^{\alpha t + \sigma B_t}$$

Hence,

$$X_t = e^{\alpha t + \sigma B_t}(x_0^{-2} + (-2)(-1)\int_0^t e^{2(\alpha s + \sigma B_s)} ds)^{-\frac{1}{2}}$$

$$X_t = e^{\alpha t + \sigma B_t}(x_0^{-2} + 2\int_0^t e^{2(\alpha s + \sigma B_s)} ds)^{-\frac{1}{2}}$$

17. Solve the reducible SDE:

$$dX_t = \frac{1}{2}a^2 m X_t^{2m-1} dt + a X_t^m dB_t,\ m \neq 1,\ t \geq 0;\ X_0 = x_0$$

Solution: let $\sigma(X_t) = aX_t^m$, then, $\sigma'(X_t) = amX_t^{m-1}$ and $\frac{1}{2}\sigma(X_t)\sigma'(X_t) = \frac{1}{2}a^2 m X_t^{2m-1}$, satifies (3.3), so that

$$h(x) = \int_a^x \frac{1}{\sigma(u)} du = \frac{1}{2}\int_a^x u^{-m} du = \frac{1}{a} \cdot \frac{1}{1-m} x^{1-m} = \frac{1}{a(1-m)} x^{1-m}$$

$$h^{-1}(x) = a(1-m)x^{\frac{1}{(1-m)}}$$

Hence,

$$X_t = h^{-1}(B_t + h(x_0)) = h^{-1}(B_t + \frac{1}{a(1-m)} x_0^{1-m}) = a(1-m)(B_t + \frac{1}{a(1-m)} x_0^{1-m})^{\frac{1}{(1-m)}}.$$

18. Solve the reducible SDE:

$$dX_t = \frac{1}{3} X_t^{\frac{1}{3}} dt + X_t^{\frac{2}{3}} dB_t,\ t \geq 0;\ X_0 = x_0$$

Solution: let $\sigma(X_t) = X_t^{\frac{2}{3}}$, $\sigma'(X_t) = \frac{2}{3} X_t^{-\frac{1}{3}}$, and $\frac{1}{2}\sigma(X_t)\sigma'(X_t) = \frac{1}{2} \cdot X_t^{\frac{2}{3}} \cdot \frac{2}{3} X^{-\frac{1}{3}} = \frac{1}{3} X^{\frac{1}{3}}$, satisfies (3.3), so that

$$h(x) = \int_a^x \frac{1}{\sigma(u)} du = \int_a^x u^{-\frac{2}{3}} du = 3X^{\frac{1}{3}}$$

$$h^{-1}(x) = \frac{x^3}{27}$$

Hence,

$$X_t = h^{-1}(B_t + h(x_0)) = h^{-1}(B_t + 3x_0^{\frac{1}{3}}) = \frac{1}{27}(B_t + 3x_0^{\frac{1}{3}})^3 = (\frac{1}{3} B_t + x_0^{\frac{1}{3}})^3$$