Homework

2006.11.18

In each of Problem 1 through 4, find the mean $\mathbf{E}[X]$ and the variance $\text{var}(X)$ of X, which is a random variable defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$.
4.X has a normal distribution with parameters μ and $\sigma > 0$.
Solution: its probability density is

$$n(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

$$\mathbf{E}[X] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx$$

let $v = \frac{x-\mu}{\sigma}$, $x = \sigma v + \mu$, $dv = \frac{1}{\sigma}dx$, then

$$\mathbf{E}[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma v + \mu) e^{-\frac{1}{2}v^2} dv = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v e^{\frac{1}{2}v^2} dv + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2} dv$$

since $ve^{-\frac{1}{2}v^2}$ is odd function, $e^{-\frac{1}{2}v^2}$ is even function, and integraling interval is symmetric, hence $\int_{-\infty}^{\infty} ve^{-\frac{1}{2}v^2} dv = 0$, and then

$$\mathbf{E}[X] = \frac{2\mu}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}v^2} dv$$

Only calcaulate $\int_0^{\infty} e^{-\frac{1}{2}v^2} dv$, let $I = \int_0^{\infty} e^{-\frac{1}{2}v^2} dv$

$$I^2 = \int_0^{\infty} e^{-\frac{1}{2}x^2} dx \cdot \int_0^{\infty} e^{-\frac{1}{2}y^2} dy = \int_0^{\infty} \int_0^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dxdy$$

change of variable $x = r\sin\theta, y = r\cos\theta$

$$I^2 = \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-\frac{1}{2}r^2} r dr d\theta = \int_0^{\frac{\pi}{2}} \int_0^{\infty} (-1) e^{-\frac{1}{2}r^2} d(e^{-\frac{1}{2}r^2}) d\theta$$

$$I^2 = \int_0^{\frac{\pi}{2}} (-1) e^{-\frac{1}{2}r^2} |_0^{\infty} d\theta = \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2} \Rightarrow I^2 = \frac{\pi}{2} \Rightarrow I = \frac{\sqrt{2\pi}}{2}$$

Therefore, the mean $\mathbf{E}[X] = \mu$

$$\mathbf{E}[X^2] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx$$

let $v = \frac{x-\mu}{\sigma}$, $x = \sigma v + \mu$, $dv = \frac{1}{\sigma}dx$, then

$$\mathbf{E}[X^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma^2 v^2 + 2\sigma\mu v + \mu^2) e^{-\frac{1}{2}v^2} dv = \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} v^2 e^{-\frac{1}{2}v^2} dv + \frac{2\sigma\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v e^{-\frac{1}{2}v^2} dv + \frac{2\mu^2}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}v^2} dv$$

since $ve^{-\frac{1}{2}v^2}$ is odd function, $v^2 e^{-\frac{1}{2}v^2}, e^{-\frac{1}{2}v^2}$ are even function, and integraling interval is also symmetric, hence $\int_{-\infty}^{\infty} ve - \frac{1}{2}v^2 dv = 0$, we only calculate $\int_0^{\infty} v^2 e^{-\frac{1}{2}v^2} dv$

$$\int_0^{\infty} v^2 e^{-\frac{1}{2}v^2} dv = \int_0^{\infty} (-1) v d(e^{-\frac{1}{2}v^2}) = (-1) v e^{-\frac{1}{2}v^2} |_0^{\infty} + \int_0^{\infty} e^{-\frac{1}{2}v^2} dv = 0 + \frac{\sqrt{2\pi}}{2} = \frac{\sqrt{2\pi}}{2}$$

so that we have

$$\mathbf{E}[X^2] = \frac{2\sigma^2}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{2} + \frac{2\mu^2}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{2} = \sigma^2 + \mu^2$$

Therefore, the variance $\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$.

In each of Problem 5 through 10. X and Y are random varibles defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

5.Assume $\mathbf{E}[X^2] < \infty$. Prove Chebyshev inequality:

$$\mathbf{P}(X \geq \varepsilon) \leq \frac{1}{\varepsilon^2}\mathbf{E}[X^2], \forall \varepsilon > 0.$$

Proof: using notes P.11 properties (5) basic inequalities

$$\mathbf{P}(X \geq \alpha) \leq (g(\alpha))^{-1}\mathbf{E}[g(X)]$$

let $g(x) = x^2$, then $g(\cdot)$ be nonnegative and Borel measurable, $g(\cdot)$ is non-decreasing. hence

$$\mathbf{P}(X \geq \varepsilon) \leq (g(\varepsilon))^{-1}\mathbf{E}[g(X)] = (\varepsilon^2)^{-1}\mathbf{E}[X^2] = \frac{1}{\varepsilon^2}\mathbf{E}[X^2]$$

Therefore, we have $\mathbf{P}(X \geq \varepsilon) \leq \frac{1}{\varepsilon^2}\mathbf{E}[X^2], \forall \varepsilon > 0$.

7.Show that, if X has an exponential distribution with parameter $\theta > 0$, then X satisfies the *forgetfulness property* that

$$\mathbf{P}(X > x + y | X > y) = \mathbf{P}(X > x), \forall x, y \in (0, \infty).$$

Proof:

$$L.H.S = \frac{\mathbf{P}((X > x + y) \bigcap (X > y))}{\mathbf{P}(X > y)} = \frac{\mathbf{P}(X > x + y)}{\mathbf{P}(X > y)} \text{ and } R.H.S = \mathbf{P}(X > x).$$

And exponential distribution probability density is

$$g(x; \theta) = \begin{cases} \frac{1}{\theta}e^{-\frac{x}{\theta}}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{P}(X > x + y) = \int_{x+y}^{\infty} \frac{1}{\theta}e^{-\frac{v}{\theta}}dv = \int_{x+y}^{\infty} (-1)d(e^{-\frac{v}{\theta}}) = (-1)e^{-\frac{v}{\theta}}|_{x+y}^{\infty} = e^{-\frac{x+y}{\theta}}$$

similar, $\mathbf{P}(X > y) = e^{-\frac{y}{\theta}}$, $\mathbf{P}(X > x) = e^{-\frac{x}{\theta}}$, then

$$L.H.S = \frac{e^{-\frac{x+y}{\theta}}}{e^{-\frac{y}{\theta}}} = e^{-\frac{x}{\theta}} = R.H.S$$

Therefore, $\mathbf{P}(X > x + y | X > y) = \mathbf{P}(X > x)$.

8.Suppose that X and Y are integrable, and a and b are real numbers. Show

$$\mathbf{E}[aX + bY | \mathcal{G}] = a\mathbf{E}[X|\mathcal{G}] + b\mathbf{E}[Y|\mathcal{G}], \mathbf{P}\text{-a.s.}$$

Proof:

$$L.H.S = \int_B aX + bY d\mathbf{P} = a\int_B X d\mathbf{P} + b\int_B Y d\mathbf{P} = a\mathbf{E}[X|\mathcal{G}] + b\mathbf{E}[Y|\mathcal{G}] = R.H.S$$

10.Show that $\mathbf{E}[1|\mathcal{G}] = 1$.

Proof:both side take the mean

L.H.S= $|E[\mathbf{E}[1|\mathcal{G}]] = \mathbf{E}[1] = 1$
R.H.S= $\mathbf{E}[1] = 1$
Hence, $\mathbf{E}[1|\mathcal{G}] = 1$

11.Show that, in $[0, \infty)$, the Poisson process $N.$ is both mean-square continuous and continuous in probability, but is not continuous with probability one.
Proof: first, we want to prove it is mean-continuous continuous,
Poisson distribrtion

$$p(x; y) = \frac{\lambda^x e^\lambda}{x!} \text{ for } x = 0, 1, 2. \cdots$$

with momet-generating function is

$$M_{\mathrm{X}}(t) = e^{\lambda(e^t - 1)},\ M'_{\mathrm{X}}(t) = \lambda e^t e^{\lambda(e^t-1)},\ M''_{\mathrm{X}}(t) = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)}$$

and then, $\mathbf{E}[X^2] = M''_{\mathrm{X}}(0) = \lambda + \lambda^2 < \infty$

$$\lim_{s \to t, s \in T} \mathbf{E}[|X_s - X_t|^2] =$$

second, we want to prove it is continuous in probability in$[0, \infty)$,

$$\lim_{s \to t, s \in T} \mathbf{P}(\{\omega : |X_s(\omega) - X_t(\omega)| \geq \varepsilon\}) =$$

third, we want to prove it is not continuous with probability one,
Remake:
[1] $X.$ is said to *mean-square continuous* if, for each $t \in T$, it satisfies that $\mathbf{E}[X_t^2] < \infty$ and $\lim_{s\to t, s\in T} \mathbf{E}[|X_s - X_t|^2] = 0$;
[2] $X.$ is said to be *continuous in probability* if, for every $\varepsilon > 0$ and for each $t \in T$,
$\lim_{s \to t, s\in T} \mathbf{P}(\{\omega : |X_s(\omega) - X_t(\omega)| \geq \varepsilon\}) = 0$

12.Prove that the standard Brownian motion $B.$ is continuous in $[0, \infty)$ with probability one.
Proof: for $\forall 0 \leq s < t$, $(B_t - B_s) \sim N(0, \sqrt{t-s})$. According to the exercises 16 result, we know that for $\forall k > 0$

$$\mathbf{E}[(B_t - B_s)^{2k}] = \frac{(2k)!}{2^k k!}(t-s)^k$$

By Kolmogorov's Criterion Theorem,
Hence, the standard Brownian motion $B.$ is continuous in $[0, \infty)$ with probability one.
Remark:**Kolmogorov's Criterion Therem.**
$X.$ is a stochastic process, if exists sone oisutuve constants α, β, C and δ, such that $|t - s| < \delta$ and

$$\mathbf{E}[|X_t - X_s|^\alpha] \leq C|t-s|^{1+\beta}.$$

then $X.$ is continuous in $[0, \infty)$ with probability one.
Since, in this exercise, $\alpha = 2k$, $\beta = k - 1$, $C = \frac{(2k)!}{2^k k!}$, $|t - s| < \delta$, satify all condition, and then $B.$ is continuous in $[0, \infty)$ with probability one.

13.Show the linearity of stochastic integral: for any $X., Y \in \mathcal{L}_{2.T}$ and any constants a and b,

$$\int_0^T (aX_s + bY_s)dB_s = a\int_0^T X_s dB_s + b\int_0^T Y_s dB_s.$$

Solution: using P.61 theorem 4.2.3

$$\int_0^T X_s dB_{S=s} = \lim_{|\Delta| \to 0} \sum_{i=1}^n X_{t_{i-1}^n}(B_{t_i^n} - B_{t_{i-1}^n})$$

where limit is in $\mathbf{L}^2(\Omega, \mathcal{F}_t, \mathbf{P})$, and $\Delta : 0 = t_0 < t_1 < t_2 \cdots < t_n = T$

$$L.H.S = \lim_{|\Delta|\to 0} \sum_{i=1}^{n} (aX_{t_{i-1}^n} + bY_{t_{i-1}^n})(B_{t_i^n} - B_{t_{i-1}^n}) = a \lim_{|\Delta|\to 0} \sum_{i=1}^{n} X_{t_{i-1}^n}(B_{t_i^n} - B_{t_{i-1}^n}) + b \lim_{|\Delta|\to 0} \sum_{i=1}^{n} Y_{t_{i-1}^n}(B_{t_i^n} - B_{t_{i-1}^n})$$

$$= a \int_0^T X_s dB_s + b \int_0^T Y_s dB_s = R.H.S.$$

14.Use the Itô formula to show

$$\int_0^t B_s^{2n-1} dB_s = \frac{1}{2n} B_t^{2n} - \int_0^t \frac{2n-1}{2} B_s^{2n-2} ds, \forall t \geq 0.$$

Solution: Let $F(t, B_t) = \frac{1}{2n} B_t^{2n}$, by using Itô formula

$$dF(t, B_t) = \frac{\partial F(t, B_t)}{\partial t} dt + \frac{\partial F(t, B_t)}{\partial x} dB_t + \frac{1}{2} \frac{\partial^2 F(t, B_t)}{\partial x^2} (dB_t \cdot dB_t).$$

$$dF(t, B_t) = 0 \cdot dt + B_t^{2n-1} dB_t + \frac{1}{2}(2n-1) B_t^{2n-1} dt$$

$$dF(t, B_t) = B_t^{2n-1} dB_t + \frac{(2n-1)}{2} B_t^{2n-1} dt$$

$$F(t, B_t) = F(0, B_0) + \int_0^t B_s^{2n-1} dB_s + \int_0^t \frac{2n-1}{2} B_s^{2n-2} ds$$

since $F(0, B_0) = \frac{1}{2n} B_0 = 0$, hence

$$\frac{1}{2n} B_t^{2n} = \int_0^t B_s^{2n-1} dB_s + \int_0^t \frac{2n-1}{2} B_s^{2n-2} ds$$

$$\text{i.e. } \int_0^t B_s^{2n-1} dB_s = \frac{1}{2n} B_t^{2n} - \int_0^t \frac{2n-1}{2} B_s^{2n-2} ds.$$

15.Show that, if $f(\cdot)$ a determinstic functio having a continuous and bounded derivative $f'(\cdot)$ in $[0, T]$, then

$$\int_0^t f(s) dB_s = f(t) B_t - \int_0^t f'(s) B_s ds.$$

Solution: Let $F(t, B_t) = f(t) B_t$, by using Itô formula

$$dF(t, B_t) = \frac{\partial F(t, B_t)}{\partial t} dt + \frac{\partial F(t, B_t)}{\partial x} dB_t + \frac{1}{2} \frac{\partial^2 F(t, B_t)}{\partial x^2} (dB_t \cdot dB_t).$$

$$dF(t, B_t) = f'(t) B_t dt + f(t) dB_t + \frac{1}{2} \cdot 0 \cdot (dB_t \cdot dB_t)$$

$$F(t, B_t) = F(0, B_0) + \int_0^t f'(s) B_s ds + \int_0^t f(s) dB_s$$

since $F(0, B_0) = f(0) \cdot B_0 = f(0) \cdot 0 = 0$, hence

$$f(t) B_t = \int_0^t f'(s) B_s ds + \int_0^t f(s) dB_s$$

$$\text{i.e. } \int_0^t f(s) dB_s = f(t) B_t - \int_0^t f'(s) B_s ds \text{ is hold.}$$

16. Let $B.$ be a standard $(\mathcal{F}_t)$-Brownian motion. Prove that, for every $k = 1, 2, \cdots$, and any $0 \le s < t$,

$$\mathbf{E}[(B_t - B_s)^{2k-1}] = 0, \text{ and } \mathbf{E}[(B_t - B_s)^{2k}] = \frac{(2k)!}{2^k \cdot k!}(t-s)^k.$$

Proof: we know that normal distribrtion

$$n(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

And $(B_t - B_s) \sim N(0, \sqrt{t-s})$, so that

$$n(x;0,\sqrt{t-s}) = \frac{1}{\sqrt{t-s}\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x}{\sqrt{t-s}})^2}$$

$$\mathbf{E}[(B_t - B_s)^{2k-1}] = \frac{1}{\sqrt{t-s}\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2k-1} e^{-\frac{1}{2}(\frac{x}{\sqrt{t-s}})^2} dx$$

since $x^{2k-1} e^{-\frac{1}{2}(\frac{x}{\sqrt{t-s}})^2}$ is odd function, and integraling interval is also symmetric, hence $\mathbf{E}[(B_t - B_s)^{2k-1}] = 0$.

$$\mathbf{E}[(B_t - B_s)^{2k}] = \frac{1}{\sqrt{t-s}\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2k} e^{-\frac{1}{2}(\frac{x}{\sqrt{t-s}})^2} dx$$

since $x^{2k} e^{-\frac{1}{2}(\frac{x}{\sqrt{t-s}})^2}$ is even function, and integraling interval is also symmetric, hence

$$\mathbf{E}[(B_t - B_s)^{2k}] = \frac{2}{\sqrt{t-s}\sqrt{2\pi}} \int_{0}^{\infty} x^{2k} e^{-\frac{1}{2}(\frac{x}{\sqrt{t-s}})^2} dx$$

Just to calculate $\int_0^\infty x^{2k} e^{-\frac{1}{2}(\frac{x}{\sqrt{t-s}})^2} dx$,
using change of variable and Integration by parts,
let $v = e^{-\frac{1}{2}(\frac{x}{\sqrt{t-s}})^2}$, then $dv = e^{-\frac{1}{2}(\frac{x}{\sqrt{t-s}})^2} \cdot (-\frac{1}{2}) \cdot \frac{2x}{t-s} dx$,
i.e. $dv = v \cdot (-1)\frac{x}{t-s} dx \Leftrightarrow (-1)\frac{t-s}{x} dv = v dx$

$$\int_0^\infty x^{2k} e^{-\frac{1}{2}(\frac{x}{\sqrt{t-s}})^2} dx = \int_0^\infty x^{2k} v dx \overset{\text{chg.v}}{=} \int_{x=0}^{x=\infty} x^{2k} \cdot (-1) \cdot \frac{t-s}{x} dv = (t-s) \int_{x=0}^{x=\infty} (-1) x^{2k-1} dv$$

$$\overset{\text{by part}}{=} (-1)(t-s) x^{2k-1} v|_{x=0}^{x=\infty} + (t-s) \int_{x=0}^{x=\infty} v d(x^{2k-1}) = (t-s)(2k-1) \int_0^\infty x^{2k-2} \cdot v dx$$

using this method by k step, become

$$(t-s)^k (2k-1)(2k-3) \cdots \cdots 3 \cdot 1 \cdot \int_0^\infty v dx = (t-s)^k \frac{2k(2k-1)(2k-2)(2k-3)\cdots 3 \cdot 2 \cdot 1}{2k(2k-2)\cdots 4 \cdot 2} \cdot \int_0^\infty v dx$$

$$= (t-s)^k \frac{(2k)!}{2^k \cdot k(k-1)\cdots 2 \cdot 1} \cdot \int_0^\infty v dx = \frac{(2k)!}{2^k \cdot k!}(t-s)^k \int_0^\infty v dx$$

since $\int_0^\infty v dx = \frac{\sqrt{t-s}\sqrt{2\pi}}{2}$, so $\mathbf{E}[(B_t - B_s)^{2k}] = \frac{(2k!)}{2^k \cdot k!}(t-s)^k$.

17.Use the Itô formula to show

$$\int_0^t \sin B_s dB_s = 1 - \cos B_t - \int_0^t \frac{1}{2} \cos B_s ds, \ t \ge 0.$$

Solution: Let $F(t, B_t) = l - \cos B_t$, by using Itô formula

$$dF(t, B_t) = \frac{\partial F(t, B_t)}{\partial t} dt + \frac{\partial F(t, B_t)}{\partial x} dB_t + \frac{1}{2} \frac{\partial^2 F(t, B_t)}{\partial x^2} (dB_t \cdot dB_t).$$

$$dF(t,B_t) = 0 \cdot dt + \sin B_t dB_t + \frac{1}{2}\cos B_t (dB_t \cdot dB_t)$$

$$dF(t,B_t) = \sin B_t + \frac{1}{2}\cos B_t dt$$

$$F(t,B_t) = F(0,B_0) + \int_0^t \sin B_s dB_s + \frac{1}{2}\int_0^t \cos B_s ds$$

since $F(0,B_0) = 1 - \cos B_0 = 1 - \cos 0^o = 1 - 1 = 0$, hence

$$1 - \cos B_t = \int_0^t \sin B_s dB_s + \frac{1}{2}\int_0^t \cos B_s ds$$

$$\text{i.e.} \quad \int_0^t \sin B_s dB_s = 1 - \cos B_t - \int_0^t \frac{1}{2}\cos B_s ds.$$

18.Use the Itô formula to show

$$\int_0^t \cos B_s dB_s = \sin B_t + \int_0^t \frac{1}{2}\sin B_s ds, \ t \geq 0.$$

Solution: Let $F(t.B_t) = \sin B_t$, by using Itô formula

$$dF(t,B_t) = \frac{\partial F(t,B_t)}{\partial t}dt + \frac{\partial F(t,B_t)}{\partial x}dB_t + \frac{1}{2}\frac{\partial^2 F(t,B_t)}{\partial x^2}(dB_t \cdot dB_t).$$

$$dF(t,B_t) = 0 \cdot dt + \cos B_t dB_t + \frac{1}{2}(-\sin B_t)(dB_t \cdot dB_t)$$

$$dF(t,B_t) = \cos B_t dB_t - \frac{1}{2}\sin B_t dt$$

$$F(t,B_t) = F(0,B_0) + \int_0^t \cos B_s dB_s - \frac{1}{2}\int_0^t \sin B_s ds$$

since $F(0,B_0) = \sin B_0 = \sin 0^o = 0$, hence

$$\sin B_t = \int_0^t \cos B_s dB_s - \frac{1}{2}\int_0^t \sin B_s ds$$

$$\text{i.e.} \quad \int_0^t \cos B_s dB_s = \sin B_t + \int_0^t \frac{1}{2}\sin B_s ds.$$

19.Use the Itô formula to rewrite the diffusion $X_t = B_t^2, t \geq 0$, in the standard form:

$$X_t = X_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s, \ t \geq 0.$$

Solution: Let $F(t.B_t) = B_t^2 = X_t$, by using itô formula

$$dF(t,B_t) = \frac{\partial F(t,B_t)}{\partial t}dt + \frac{\partial F(t,B_t)}{\partial x}dB_t + \frac{1}{2}\frac{\partial^2 F(t,B_t)}{\partial x^2}(dB_t \cdot dB_t).$$

$$dF(t,B_t) = 0 \cdot dt + 2B_t dB_t + \frac{1}{2} \cdot 2 \cdot dt$$

$$dF(t,B_t) = dt + 2B_t dB_t$$

$$F(t,B_t) = F(0,B_0) + \int_0^t 1 ds + \int_0^t 2B_s dB_s$$

since $F(0, B_0) = B_0^2 = 0$, hence

$$X_t = F(t, B_t) = t + \int_0^t 2B_s dB_s = 0 + \int_0^t 1 ds + \int_0^t 2B_s dB_s.$$

(i.e. $b(s) = 1$, $X_0 = 0$, $\sigma(s) = 2B_s$)

20.Use the Itô formula to rewrite the diffusion $X_t = 2 + t + e^{B_t}, t \geq 0$, in the standard form:

$$X_t = X_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s, \ t \geq 0.$$

Solution: Let $F(t, B_t) = 2 + t + e^{B_t} = X_t$, by using itô formula

$$dF(t, B_t) = \frac{\partial F(t, B_t)}{\partial t}dt + \frac{\partial F(t, B_t)}{\partial x}dB_t + \frac{1}{2}\frac{\partial^2 F(t, B_t)}{\partial x^2}(dB_t \cdot dB_t).$$

$$dF(t, B_t) = 1 \cdot dt + e^{B_t}dB_t + \frac{1}{2} \cdot e^{B_t} \cdot dt$$

$$dF(t, B_t) = (1 + \frac{1}{2}e^{B_t})dt + e^{B_t}dB_t$$

$$F(t, B_t) = F(0, B_0) + \int_0^t (1 + \frac{1}{2}e^{B_s})ds + \int_0^t e^{B_s}dB_s$$

since $F(0, B_)) = 2 + 0 + e^{B_0} = 2 + 0 + e^0 = 2 + 1 = 3$, hence

$$X_t = F(t, B_t) = 3 + \int_0^t (1 + \frac{1}{2}e^{B_s})ds + \int_0^t e^{B_s}dB_s$$

(i.e. $X_0 = 3$, $b(s) = 1 + \frac{1}{2}e^{B_s}$, $\sigma(s) = e^{B_s}$)

21.Prove directly from the definition of stochastic integral that

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$$

Solution:For any fixed time $t > 0$ and any natural number $n \geq 2$, setting $t_k^n = kt/n$ for all $k = 0, 1, \cdots, n$, and define $(B_s^2)^{(n)}(\omega) = \sum_{i=1}^n B_{t_{i-1}^n}^2 1_{[t_{i-1}^n, t_i^n)}(s), (s, \omega) \in \mathcal{B}(\mathbf{R}) \times \Omega$. Then, by the definition of Itô integral, we have that

$$\int_0^t (B_s^2)^{(n)} dB_s = \sum_{i=1}^n B_{t_{i-1}^n}^2 (B_{t_i^n} - B_{t_{i-1}^n})$$

And since we have that as $n \to \infty$, and so that $\int_0^t B_s^2 dB_s = \int_0^t (B_s^2)^{(n)} dB_s$
And we have have that

$$B_t^3 = \sum_{i=1}^n B_{t_i^n}^3 - B_{t_{i-1}^n}^3$$

by the above equation, we have that

$$B_t^3 - 3\int_0^t B_s^2 dB_s = \sum_{i=1}^n (B_{t_i^n}^3 - B_{t_{i-1}^n}^3) - 3\sum_{i=1}^n B_{t_{i-1}^n}^2 (B_{t_i^n} - B_{t_{i-1}^n}) = \sum_{i=1}^n (B_{t_i^n}^3 - B_{t_{i-1}^n}^3 - 3B_{t_{i-1}^n}^2 \cdot B_{t_i^n} + 3B_{t_{i-1}^n}^3)$$

let $A = B_{t_i^n}, D = B_{t_{i-1}^n}$,
and $A^3 - 3AD^2 + 2^3 = (A^3 - 2A^2D + AD^2) + (2A^2D - 4AD^2 + 2D^3) = A(A - D)^2 + 2D(A - D)^2$, then

$$B_t^3 - 3\int_0^t B_s^2 dB_s = \sum_{i=1}^n (B_{t_i^n}^2 - 3B_{t_i^n} \cdot B_{t_{i-1}^n}^2 + 2B_{t_{i-1}^n}^3) = \sum_{i=1}^n B_{t_i^n}(B_{t_i^n} - B_{t_{i-1}^n})^2 + 2\sum_{i=1}^n B_{t_{i-1}^n}(B_{t_i^n} - B_{t_{i-1}^n})^2$$

$$= \sum_{i=1}^{n} B_{t_i^n}(t_i^n - t_{i-1}^n) + 2\sum_{i=1}^{n} B_{t_{i-1}^n}(t_i^n - t_{i-1}^n)$$

since we have that $\sum_{i=1}^{n} B_{t_i^n}(t_i^n - t_{i-1}^n) \to \int_0^t B_s ds$ and $\sum_{i=1}^{n} B_{t_{i-1}^n}(t_i^n - t_{i-1}^n) \to \int_0^t B_s ds$ as $n \to \infty$ then

$$B_t^3 - 3\int_0^t B_s^2 dB_s = \int_0^t B_s ds + 2\int_0^t B_s ds = 3\int_0^t B_s ds$$

Therefore, we have

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$$