Experiments in Relation to Adaptive Decomposition of Signals into Mono-components

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Dec 11, 2008

Outline

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Problem

For the real-valued 2π-periodic signal f, we want to construct a mono-component decomposition.

Hilbert Transform

Definition

Let f be a real-valued Lebesgue-measurable function defined on $\mathbb{R}$. The Hilbert transform of f is defined by

$$Hf(t) = p.v.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{t-s}ds, \tag{2.1}$$

where p.v. stands for the principal value of the integral, viz

$$p.v.\int_{-\infty}^{\infty}\frac{f(s)}{t-s}ds = \lim_{\varepsilon\to 0^+}\int_{|s-t|\geq\varepsilon}\frac{f(s)}{t-s}ds.$$

Hilbert Transform

We also rewrite Hilbert transform of f in the convolution form

$$Hf(t) = \mathrm{p.v.}(\frac{1}{\pi(\cdot)} * f)(t) \tag{2.2}$$

One can check that $H^2 = -I$, where I is the identity transform.

Hilbert Transform

If f is square-integrable, then f is given by a Fourier series in the L^2-convergence sense, i.e. $f(t) = \sum_{n\in\mathbb{Z}} c_k e^{ikt}$, where $(c_k)_{k\in\mathbb{Z}}$ is a sequence of complex numbers, then the Hilbert transform $H(f)$ of f is given by

$$H(f)(t) = \sum_{n\in\mathbb{Z}} -i\,\mathrm{sgn}(k) c_k e^{ikt}.$$

Here, we won't discuss the convergence of the series, though this issue can be easily settled in the appropriate setting.

Circular Hilbert Transform

Definition

The Circular Hilbert transform of a Lebesgue-measurable function defined on the unit circle $\mathbb{S}^1$ *is defined by*

$$\widetilde{H}f(t) = p.v.\frac{1}{2\pi}\int_{-\pi}^{\pi}\cot\left(\frac{t-s}{2}\right)f(s)ds. \tag{2.3}$$

where $\mathbb{S}^1 = \{\ e^{it} \in \mathbb{C} \mid 0 \le t \le 2\pi\ \}$.

One can check that $\widetilde{H}^2 = -I$, where I is the identity transform.

Analytic signal

Definition

If f is real-valued signal and $f(t) \in L^p(\mathbb{R})$, define $A(f) = f + iHf$ to be analytic signal associated with f.

Definition

Let $f(t)$ be a measurbale function defined on $\mathbb{S}^1$, define $\widetilde{A}(f) = f + i\,\widetilde{H}f$ to be analytic signal associated with f, where $\mathbb{S}^1 = \{\ e^{it} \in \mathbb{C} \mid 0 \leq t \leq 2\pi\ \}$.

Fourier Series

Definition

If $f \in L^1([-\pi, \pi])$, then the Fourier series of $f(x)$ is given by

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{2.4}$$

where the n-th Fourier coefficients a_n and b_n of f are given by

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx, & n = 1, 2, 3, \cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx & n = 1, 2, 3, \cdots \end{aligned} \tag{2.5}$$

Fourier Series

Theorem

If $f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt))$, *then*

(a) $\widetilde{H}f(t) = \sum_{n=1}^{\infty}(a_n \sin(nt) - b_n \cos(nt))$, *where* $a_n, b_n \in \mathbb{R}$;

(b) $\widetilde{A}(f) = a_0 + \sum_{n=1}^{\infty}(a_n - ib_n)z^n$, *where* $z = e^{it}$;

(c) If $f \in L^2([-\pi, \pi])$, *then both* $\widetilde{H}f$ *and* $\widetilde{A}(f)$ *are in* $L^2([-\pi, \pi])$.

Sufficient Conditions for Convergence

Theorem

Let f and f' be piecewise continuous on the interval $(-\pi, \pi)$; that is, let f and f' be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then the Fourier series of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity the Fourier series converges to the average

$$\frac{f(x^+) + f(x^-)}{2}$$

where $f(x^+)$ and $f(x^-)$ denote the limits of f at x from the right and from the left, respectively.

Trigonometric polynomial of order M

Definition

A function of the form

$$f_M(t) = a_0 + \sum_{k=1}^{M} (a_k \cos kt + b_k \sin kt) \tag{2.6}$$

is called a trigonometric polynomial of order M.

Discrete Fourier Series

Theorem

Let N and M be two positive integers, f be a 2π-periodic function defined on $\mathbb{R}$, and define two sequences in $\mathbb{R}$:

$$x_k = -\pi + k\frac{2\pi}{N} \text{ and } y_k = f(x_k) \text{ for } 0 \leq k \leq N.$$

If $N > 2M$, then there exists a trigonometric polynomial f_M of order M, which minimizes the quantity $\sum_{k=1}^{N}(f(x_k) - f_M(x_k))^2$ among all trigonometric polynomials of order M.

Discrete Fourier Series

Theorem

In this case, the coefficients a_j and b_j of the polynomial $f_M(x)$ are given as follows:

$$a_j = \frac{2}{N} \sum_{k=1}^{N} f(x_k) \cos(jx_k) \text{ for } j = 0, 1, \ldots, M, \tag{2.7}$$

and

$$b_j = \frac{2}{N} \sum_{k=1}^{N} f(x_k) \cos(jx_k) \text{ for } j = 1, \ldots, M. \tag{2.8}$$

Complex Fourier Series

let $f(t)$ be a 2π-periodic complex-valued function defined on $\mathbb{R}$, then one can write down the complex Fourier Series of f as

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt}, \tag{2.9}$$

where the complex Fourier coefficients c_k $(k \in \mathbb{Z})$ is given by

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt. \tag{2.10}$$

Then in terms of the convergent Fourier series, one can rewrite the Circular Hilbert transform as follows

$$\widetilde{H}f(t) = \sum_{k=-\infty}^{\infty} -i\mathrm{sgn}(k)c_k e^{ikt}, \tag{2.11}$$

where c_k are complex Fourier coefficients of $f(t)$.

The relation of complex Fourier coefficients and real Fourier coefficients are given as follows

$$c_0 = a_0, \quad c_k = \frac{1}{2}(a_k - ib_k), \quad c_{-k} = \frac{1}{2}(a_k + ib_k).$$

And

$$e^{ikt} = \cos kt + i \sin kt,$$

for $k = 1, 2, 3, \cdots$, then

$$\widetilde{H}f(t) = \sum_{k=1}^{\infty}(-b_k)\cos kt + (a_k)\sin kt.$$

Numerical Circular Hilbert transform

Simlarly, for any positive integer M and a 2π-periodic function f defined on $\mathbb{R}$, the Numerical Circular Hilbert Transform $\widetilde{H}_M$ of order M is defined, in terms of trigonometric polynomial, to be

$$\widetilde{H}_M f(t) = \sum_{k=1}^{M} (-b_k) \cos kt + (a_k) \sin kt, \qquad (2.12)$$

where (a_k) and (b_k) are defined by (2.5).

Numerical Circular Hilbert transform

Similarly to discrete Fourier series

Definition

Let N and M be two positive integers, f be a 2π-periodic function defined on $\mathbb{R}$, and define two sequences in $\mathbb{R}$:

$$x_k = -\pi + k\frac{2\pi}{N} \text{ and } y_k = f(x_k) \text{ for } 0 \leq k \leq N.$$

If $N > 2M$, then there exists a trigonometric polynomial $\widetilde{H}_M f(t)$ of order M, where (a_k) and (b_k) are defined by (2.7) and (2.8).

Starlike function

Definition

A domain Ω is said starlike on $\mathbb{D}$, if the following two conditions are satisfied:
(1) $0 \in \Omega$, and
(2) $tz \in \Omega$ for all real number, $0 < t < 1$, and for all $z \in \Omega$.

Definition

A univalent (injective) holomorphic function $f : \mathbb{D} \to \mathbb{C}$, called a starlike function, if
(1) $f(0) = 0$ and
(2)the image $f(\mathbb{D})$ of f is starlike domain in $\mathbb{C}$.

Starlike function

Theorem

(a) $f(z)$ is starlike function on $\mathbb{D}$ if and only if

$$Re\{\frac{zf'(z)}{f(z)}\} \geq 0.$$

(b) $f(z)$ is starlike function on $\mathbb{D}$ if and only if

$$\frac{\partial}{\partial t} \, arg\{f(re^{it})\} = \frac{\partial \theta_r}{\partial t} \geq 0.$$

Starlike function

Theorem

Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function on $\mathbb{D}$, given by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

such that the following series

$$\sum_{j=2}^{\infty} j|a_j| \leq 1,$$

then f is a starlike on $\mathbb{D}$, and is univalent on $\mathbb{D}$.

Proof. For $|z| < 1$, it follows from the triangle inequality and the given condition $\sum_{j\geq 2} j|a_j| \leq 1$ that

$$\begin{aligned} |zf'(z) - f(z)| &= \left|\sum_{j=0}^{\infty}(ja_j - a_j)z^j\right| \leq \sum_{j=2}^{\infty}(j-1)|a_j||z|^j \\ &\leq |z| - \sum_{j=2}^{\infty}|a_j||z|^j = |z| - \sum_{j=2}^{\infty}|a_j z^j| \\ &\leq |z| - \left|\sum_{j=2}^{\infty} a_j z^j\right| \leq |f(z)|. \end{aligned}$$

Thus $\left|z\dfrac{f'(z)}{f(z)} - 1\right| \leq 1$, it follows that $\mathrm{Re}\{\dfrac{zf'(z)}{f(z)}\} \geq 0$ on $\mathbb{D}$, and hence $f(z)$ is starlike on $\mathbb{D}$.

Theorem

Let $g_n : \mathbb{D} \to \mathbb{C}$ be a sequence of analytic functions defined on $\mathbb{D}$, given by

$$g_n(z) = z + \sum_{j=2}^{\infty} a_j^{(n)} z^j,$$

such that the series

$$\sum_{j=1}^{\infty} j \, |a_j^{(n)}| \leq 1,$$

where $n = 1, 2, \cdots, N$, $a_j^{(n)}$ is a constant, then $g_n(z)$ is starlike function on $\mathbb{D}$ for all $n = 1, 2, \ldots, N$, and $\sum_{n=1}^{N} g_n$ is also starlike on $\mathbb{D}$.

p-valent starlike function

Definition

Let p and m be positive integers with $p \geq m$. Let f be an analytic function defined on the open unit disc $\mathbb{D}$. f is said to be in the class $S(p, m)$, or p-valent starlike on $\mathbb{D}$, if
(1) there exits a positive $\rho < 1$ such that

$$\mathbf{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad \rho < |z| < 1$$

(2) $f(z) = z^m + a_{m+1}z^{m+1} + \cdots$, for $|z| < 1$
(3) $\int_0^{2\pi} \mathbf{Re}\{\frac{zf'(z)}{f(z)}\}d\theta = 2\pi p$, for $z = re^{i\theta}$, $\rho < r < 1$.

Empirical mode decomposition

Empirical mode decomposition(EMD) in [3] and [11], in which a signal is decomposed into certain type of functions, called *intrinsic mode functions* (IMFs). Though IMFs satisfy two conditions:
(1) in the whole data set, the number of extrema and the number of zero crossings must either equal or differ at most by one, and
(2) at any point, the mean value of the envelope defined by the local maxima and the envelope defined by the local minima is zero.

Empirical mode decomposition

For input signal $X(t)$ can be decomposed into a sum of IMFs ($c_i(t)$, for $i = 1, 2, ..., n$) plus $r_n(t)$, where $r_n(t)$ can be either the mean trend or a constant. i.e. following equation:

$$X(t) = \sum_{i=1}^{n} c_i(t) + r_n(t)$$

For more details, one can consult [9] and the reference in there. The sifting process see [11], is shown in the following figure (2.1.1) where envelope of c_i, for $i = 1, 2, ..., n$ are symmetry.

Empirical mode decomposition

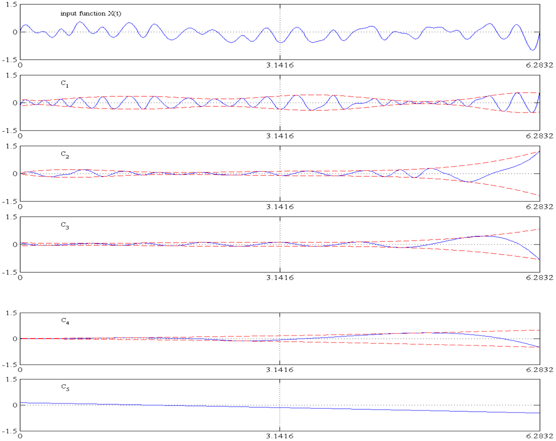


Figure (2.1.1)

Circular mono-components

Definition

Let f be a complex-valued function defined on $\mathbb{R}$, given by $f(t) = \rho(t)e^{i\theta(t)}$ in polar form for $t \in [0, 2\pi]$, where the function ρ is non-negative on $\mathbb{R}$, and θ is 2π-period function on $\mathbb{R}$. f is called circular mono-components, if it satisfies the following two conditions

(1) $\widetilde{H}f = -if$, where $\widetilde{H}$ is the circular Hilbert transform, and

(2) $\theta'(t) \geq 0$.

Similarly, complex valued function f, defined by $f(t) = \rho(t)e^{i\theta(t)}$ where $0 \leq t \leq 2\pi$, is called dual mono-component if

(1) $\widetilde{H}f = if$ and

(2) $\theta'(t) \leq 0$.

Theorem

$\rho(t)e^{i\theta(t)}$ is a (circular) mono-component if and only if $\rho(t)e^{-i\theta(t)}$ is (circular) dual mono-component.

Process of our's mono-component decomposition

First step:

Given any continuous real-valued function $f(t) \in L^2([-\pi, \pi])$, one can think of it as an input signal,then we form a sequence (f_N) of functions in L^2-approximations, given by trigonometric polynomials, where f_N is a trigonometric polynomial of degree N.

Process of our's mono-component decomposition

Second step:

For fixed N, we have an trigonometric polynomial $f_N(t)$, we first form the associated analytic signal $\widetilde{A}_N f$ of f_N by $\widetilde{A}_N f = f_N + i\widetilde{H}_N f$, which is complex-coefficient polynomial in e^{it}, and hence is analytic function in the complex variable $z = e^{it}$.

$$\begin{aligned}\widetilde{A}_N f(t) &= a_0 + \sum_{n=1}^{N} a_n(\cos nt + i \sin nt) - i \sum_{n=1}^{N} b_n(\cos nt + i \sin nt)\\ &= a_0 + \sum_{n=1}^{N} (a_n - ib_n)e^{int},\end{aligned}$$

where a_n and b_n are real fourier coefficients of $f(t)$.

Process of our's mono-component decomposition

Third step:

Polynomial $\widetilde{A}_N(f)$ is written as a function

$$\widetilde{A}_N f(z) = \sum_{n=0}^{N} c_n z^n,$$

of a complex variable $z = e^{it}$ as follows

$$c_0 = a_0, \text{ and } c_n = a_n - ib_n \text{ for } n \geq 1.$$

Process of our's mono-component decomposition

Fourth step:
We want to rewrite the polynomial $\widetilde{A}_N f$

$$\sum_{n=0}^{N} c_n z^n = c_0 + c_1 z + S_1(z) + \widetilde{S}_2(z)$$

as a sum of 4 starlike functions

$$\widetilde{S}_0(z) = c_0 \quad (4.13)$$

$$S_0(z) = c_1 z \quad (4.14)$$

$$S_1(z) = \frac{W_1}{2}\left(z + \sum_{n=2}^{N} \frac{c_n}{W_1} z^n\right) \quad (4.15)$$

$$\widetilde{S}_1(z) = \frac{W_1}{2}\left(-z + \sum_{n=2}^{N} \frac{c_n}{W_1} z^n\right), \quad (4.16)$$

Process of our's mono-component decomposition

where the real number W_1 can be chosen such that the following starlike condition of theorem 2.15 is satisfied:

$$W_1 \geq \sum_{n=2}^{N} n|c_n|. \tag{4.17}$$

In fact, we have $\sum_{n=2}^{N} n|\frac{c_n}{W_1}| = \frac{1}{W_1}\sum_{n=2}^{N} n|c_n| \leq \frac{1}{W_1}W_1 = 1.$
Furthermore, it is also easy to check all $S_0(z)$, $\widetilde{S}_0(z)$ $S_1(z)$ and $\widetilde{S}_2(z)$ are univalent on $\mathbb{D}$ for sufficiently large W_1, and hence they are starlike functions on $\mathbb{D}$.

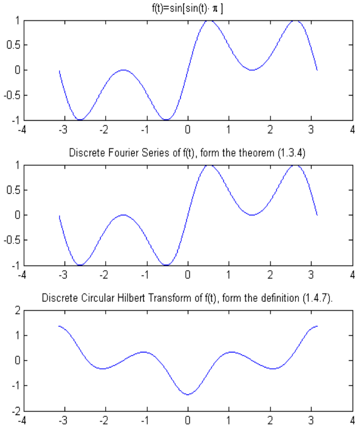
Example of our's mono-component decomposition

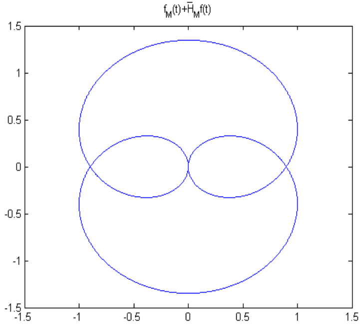
Let

$$f(t) = sin[sin(t) \cdot \pi],$$

then following figures are $f(t)$, $f_M(t)$ from the definition (2.6) and $\widetilde{H}_M f(t)$ with $M = 50$, $N = 6284$ from the definition (2.12), and then draw $f_M(t) + i\widetilde{H}_M f(t)$ in complex plane.

Example of our's mono-component decomposition





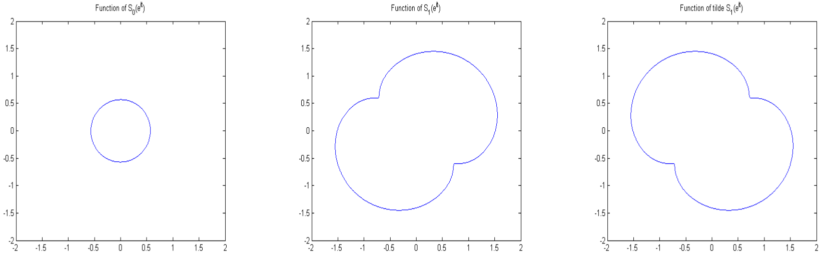
Clearly, in this example, $f_M(t) + i\widetilde{H}_M f(t)$ is not a starlike function.

Example of our's mono-component decomposition

$$\widetilde{A}_N f(z) = \sum_{n=0}^{N} c_n z^n,$$

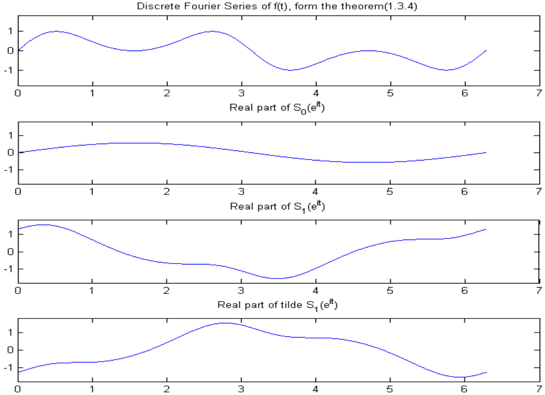
using above method to decompose three mono-components S_0, $\widetilde{S}_1$, S_1, since $\widetilde{S}_0 = 0$, as following figures.

Example of our's mono-component decomposition



Example of our's mono-component decomposition

Following figures are the $f_M(t)$, and real parts of $S_0(z), S_1(z), \widetilde{S}_1(z)$, with $z = e^{it}$



Main Theorem I

Let $f_N(t) = a_0 + \sum_{n=1}^{N}(a_n \cos nt + b_n \sin nt)$ be a trigonometric polynomial of order N, then the associated analytic signal $\widetilde{A}(f_N)$ of f_N, defined by $\widetilde{A}(f_N) = f_N + \widetilde{H}(f_N) = f_N + \widetilde{H}_N(f_N)$, can be decomposed into sum of 4 starlike functions S_0, $\widetilde{S}_0$, S_1 and $\widetilde{S}_1$ given by (4.13)-(4.16), where W_1 is chosen to satisfy the condition (4.17).

In particular, in this decomposition, all these components S_0, $\widetilde{S}_0, S_1$ and $\widetilde{S}_1$ are circular mono-component.

Lemma II

Let g be a continuous function on $[0, 2\pi]$ with $g(0) = g(2\pi)$. Suppose g has a bounded piecewise continuous derivative. Then the Fourier series for g converges uniformly.

Corollary

If g is a complex-valued function satisfying the same condition in Lemma II, and its Fourier series is $\sum_{n\geq 0} a_n e^{inx}$. Then the associated series $\sum_{n\geq 0} a_n z^n$ converges uniformly to a continuous function defined on the closed unit disc $\overline{\mathbb{D}}$, and the limit function is analytic on the unit open disc $\mathbb{D}$.

Main Theorem III

Let $f(z)$ be a complex-valued function defined on the closed unit disc $\overline{\mathbb{D}}$, such that it is analytic on $\mathbb{D}$ and that the function g, defined by $g(t) = f(e^{it})$ for all $t \in [0, 2\pi]$, satisfies the condition of Lemma II. Then for any given $\varepsilon_n \to 0$, there exists a sequence of starlike (analytic) functions: $(S_k)_{0 \le k \le n}$ and $(\widetilde{S}_k)_{0 \le k \le n}$ defined on $\overline{\mathbb{D}}$, such that

$$\left| f(z) - \sum_{k=0}^{n} \left(S_k(z) + \widetilde{S}_k(z) \right) \right| \le \varepsilon_n$$

holds for any $|z| \le 1$.

Corollary

Let f(z) be a complex-valued function defined on the closed unit disc $\overline{\mathbb{D}}$, *such that it is analytic on* $\mathbb{D}$ *and that the function* g, *defined by* $g(t) = f(e^{it})$ *for all* $t \in [0, 2\pi]$, *satisfies the condition of theorem (2). Then for any given* $\varepsilon > 0$, *there exist four mono-components* $S_0(z) = c_1 z$, $\widetilde{S}_0 = c_0$, $S_1(z)$ *and* $\widetilde{S}_1(z)$ *such that*

$$\left| f - (S_0(z) + \widetilde{S}_0(z) + S_1(z) + \widetilde{S}_1(z)) \right| \leq \varepsilon$$

holds for any $|z| \leq 1$.

Main Theorem IV

If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is analytic on $\mathbb{D}$, such that

$$\sum_{k=2}^{\infty} k|a_k| \leq M$$

for $M > 0$. Then f can be decomposed as a sum of three 1-valent starlike functions on $\mathbb{D}$.

Thank You!

- T.Qian, *Mono-componets for decomposition of signals*, Mathematical Methods in the Applied Sciences, 2006; **29**:1187-1198.
- T.Qian, *Analytic Signals and Harmonic Measures*, J.Math.Anal.Appl., 314 (2006) 526-532.
- Tao Qian,Liming Zhang,Hong Li, *Mono-components vs. IMFs in signal decomposition*, accepted to appear in International Journal of Wavelets, Multiresolution and Information Processing.
- A.W.Goodman,J.A.Hummel, *A Note on Multivalent Starlike Functions*, Proceedings of American Mathematical Society, Vol.16,No.2.(Apr.,1965), 284-288.
- M.S.Robertson *The partial sums of multivalently star-like functions*, Ann. of Math. (2) **42** (1941), 829-838.

- Bedrosian E., *A product theorem for Hilbert transforms*, Proceeding of the IEEE 1963; **51**:868-869.
- Stefan L. Hahn, *Hilbert Transforms in Signal Processing*.
- Ian Grabam,Hidetaka Hamada,Gabriela Kohr, *Radius problems for holomorphic mappings on the unit ball in* C^n, Math. Nachr. **279**, No. 13-14,1474-1490 (2006).
- John H.Mathews, Kurtis D.Fink *Numerical Methods Using MATLAB,Fourth Edition.*
- Tao Qian,Qiuhui Chen,Luoqing Li, *Analytic unit quadrature signals with nonlinear phase*, Physica D **203** (2005) 80-87.

- Norden E. Hunag,Zheng Seng,Steven R. Long,Manli C.Wu,Hsing H.Shih,Quanan Zheng,Nai-Chyuan Yen,Chi Chao Tung,Henry H. Liu, *The empirical mode decomposition and the Hilbert spectrum for nonlinear and non-stationary time series analysis*, Proc.R.Soc.Lond. A (1998)**454**,903-995.
- Dennis G.Zill,Michael R.Cullen *Differential Equations with Boundary Problems*.
- *Univalent Functions*
- T. Qian, I.T. Ho, Y.B. Wang and I.T. Leong, *Adaptive Decomposition of Functions Into Generalized Fourier Series,* preprint.
- Barry Simon, *Uniform Convergence of Fourier Series*, The American Mathematical Monthly, Vol. 76, No. 1 (Jan., 1969), pp. 55-56