Experiments in relation to adaptive decomposition of Signals into mono-components

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Abstract. In my study of signals: We known that find $f(t) = \rho(t)e^{i\theta(t)}$ such that $Hf = -if$, $ho(t) \geq 0$ and $\theta'(t) \geq 0$, almost everywhere. where $H$ is Hilbert transform. Functions satisfying the above conditions are called mono-components; if $f \in L^p$ and $0 \leq p \leq \infty$, then satisfy $Hf = -if$; if $f$ is starlike function, then satisfy $\theta'(t) \geq 0$; in fact, any input signal
Index

Chapter 1 Theory on $D$
(I) mono-components including
Möbius transforms and finite Blaschke product
Simplest singular inner functions and products of them
Starlike functions
$p$-valent starlike functions
outer functions
(II) adaptive decomposition of signals

Chapter 2 Theory on $R$
1. Möbius transforms and finite Blaschke product

**Definition 1.1**
Let $a, b, c, d$ be complex numbers satisfying $ad - bc \neq 0$. We define a Möbius transformation from $\mathbb{C}$ (complex numbers) to $\mathbb{C}$ as

$$f(z) = \frac{az + b}{cz + d}$$

form the definition it is clear that $c$ and $d$ cannot both be zero, and we see that if $ad = bc$ the resulting transformation is constant.

Let $a = c = 1, b = -i$ and $d = i$, the Möbius transform become to the Cayley transform, given by

$$f(z) = \frac{z - i}{z + i}$$

**Definition 1.2**
Finite Blaschke products can be characterized (as analytic functions on the unit disc) in the following:
Assume the $f$ is an analytic function on the open unit disc with the following three properties:
(1) $f$ maps the open unit disc to itself, i.e. $|f(z)| < 1$ if $|z| < 1$.
(2) $f$ has only finitely many zeros $a_1, a_2, \cdots, a_n$ inside the unit disc.
(3) $f$ can be extended to a continuous function on the close unit disc $\bar{D} = \{z \in \mathbb{C}||z| \leq 1\}$, which maps the unit circle to itself.

Then $f$ is equal to a finite Blaschke product

$$B(z) = \zeta \prod_{i=1}^{n} \left(\frac{z - a_i}{1 - \bar{a}_i z}\right)^{m_i}$$

where $\zeta$ lies on the unit circle and $m_i$ is multiplicity of the zero $a_i, |a_i| < 1$.

2. Simplest singular inner functions and products of them

if $f : D \rightarrow \mathbb{C}$ is an analytic function on the unit disc, we denote by $f^*(e^{i\theta})$ the radial limit of $f$ where it exists, that is

$$f^*(e^{i\theta}) := \lim_{r \to 1, r < 1} f(re^{i\theta}).$$

A bounded analytic function on the disc will have radial limits almost everywhere(with respect to the Lebesgue measure on the $\partial D$)

**Definition 2.1**
A bounded analytic function $f$ is called an inner function if $|f^*(e^{i\theta})| = 1$ almost everywhere. if $f$ has no zeros on the unit disc, then $f$ is called a singular inner function.

3. Starlike functions

**Definition 3.1**
A domain $\Omega$ is said starlike, if $0 \in \Omega$, and $tz \in \Omega, 0 < t < 1$, whenever $z \in \Omega$. A univalent a holomorphic function $f : (D) \rightarrow f(D)$ is said to be starlike function, if $f(D)$ is starlike and $f(0) = 0$.

4. $p$-valent starlike functions

**Definition 4.1**
The function $f(z)$, regular in $|z| < 1$, is said to be in the class $S(p,m)$, where $p$ and $m$ are positive integers with $p \geq m$ if and only if
(1) there exits a positive $\rho < 1$ such that

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \rho < |z| < 1$$
(2) \( f(z) = z^m + a_{m+1}z^{m+1} + \cdots \), for \( |z| < 1 \)
(3) \( \int_0^{2\pi} \text{Re}\{ \frac{zf'(z)}{f(z)} \} d\theta = 2\pi p \), for \( z = re^{i\theta}, \rho < r < 1 \)
(i.e. \( f(z) \) is \( p \)-valent starlike function in \( |z| < 1 \))

The \( p \)-valent starlike function can be decomposition as following theorem, and the proof in [4], but it is not unique way to decomposition of function in \( S(p, m) \).

**Theorem 4.2**
if \( 1 < m < p \), then
\[
S(p, m) = S(m - 1, m - 1)S(p - m + 1, 1) = [S(1, 1)]^{m-1}S(p - m + 1, 1)
\]

**5. outer functions**

**Definition 5.1**
Let
\[
f(z) := \exp \left( \int \frac{e^{i\theta} + z}{e^{i\theta} - z} h(e^{i\theta}) dm(e^{i\theta}) \right),
\]
where \( h \) is real valued Lebesgue integrable function on the unit circle and \( m \) is the Lebesgue measure. Then \( f \) is called an **outer function**.
References


