Experiments in relation to adaptive decomposition of Signals into mono-components

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Abstract. In my study of signals: We known that find $f(t) = \rho(t)e^{i\theta(t)}$ such that $Hf = -if$, $\rho(t) \geq 0$ and $\theta'(t) \geq 0$, almost everywhere. where H is Hilbert transform. Functions satisfying the above conditions are called mono-components; if $f \in L^p$ and $0 \leq p \leq \infty$, then satisfy $Hf = -if$; if f is starlke function, then satisfy $\theta'(t) \geq 0$; in fact, any input signal

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Chapter 2 Theory on R

1. Möbius transforms and finite Blaschke product

Definition 1.1
Let a, b, c, d be complex numbers satisfying $ad - bc \neq 0$. We define a Mobius transformation from C(complex numbers) to C as

$$f(z) = \frac{az + b}{cz + d}$$

form the definition it is clear that c and d cannot both be zero, and we see that if $ad = bc$ the resulting transformation is constant.

Let $a = c = 1$,$b = -i$ and $d = i$, the Mobius transform become to the Cayley transform, given by

$$f(z) = \frac{z - i}{z + i}$$

Definition 1.2
Finite Blaschke products can be characterized (as analytic functions on the unit disc) in the following: Assume the f is an analytic function on the open unit disc with the following three properties:
(1) f maps the open unit disc to itself, i.e. $|f(z)| < 1$ if $|z| < 1$.
(2) f has only finitely many zeros $a_1, a_2, \cdots, a_n$ inside the unit disc.
(3) f can be extended to a continuous function on the close unit disc $\bar{D} = \{z \in C||z| \leq 1\}$, which maps the unit circle to itself.
Then f is equal to a finite Blaschke product

$$B(z) = \zeta \prod_{i=1}^{n} \left(\frac{z - a_i}{1 - \bar{a_i} z} \right)^{m_i}$$

where ζ lies on the unit circle and m_i is multiplicity of the zero a_i, $|a_i| < 1$.

2. Simplest singular inner functions and products of them

if $f : D \to C$ is an analytic function on the unit disc, we denote by $f^*(e^{i\theta})$ the radial limit of f where it exists, that is

$$f^*(e^{i\theta}) := \lim_{r \to 1, r < 1} f(re^{i\theta}).$$

A bounded analytic function on the disc will have radial limits almost everywhere(with respect to the Lebesgue measure on the ∂D)
Definition2.1
A bounded analytic function f is called an *inner function* if $|f^*(e^{i\theta})| = 1$ almost everywhere. if f has no zeros on the unit disc, then f is called a *singular inner function.*

3. Starlike functions

Definition3.1
A domain Ω is said starlike, if $0 \in \Omega$, and $tz \in \Omega$, $0 < t < 1$, whenever $z \in \Omega$. A univalent a holomorphic function $f : (D) \to f(D)$ is said to be *starlike function*, if $f(D)$ is starlike and $f(0) = 0$.

4. p-valent starlike functions

Definition4.1
The function $f(z)$, regular in $|z| < 1$, is said to be in the class $S(p, m)$, where p and m are positive integers with$p \geq m$ if and only if
(1) there exits a positive $\rho < 1$ such that

$$\mathbf{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \rho < |z| < 1$$

(2)$f(z) = z^m + a_{m+1}z^{m+1} + \cdots$, for $|z| < 1$
(3)$\int_0^{2\pi} \mathbf{Re}\{\frac{zf'(z)}{f(z)}\}d\theta = 2\pi p$, for $z = re^{i\theta}$, $\rho < r < 1$
(i.e. $f(z)$ is *p-valent starlike function* in $|z| < 1$)

The p-valent starlike function can be decomposition as following theorem, and the proof in [4], but it is not unique way to decomposition of function in $S(p, m)$.

Theorem4.2
if $1 < m < p$, then

$$\begin{aligned} S(p,m) &= S(m-1, m-1)S(p-m+1, 1) \\ &= [S(1,1)]^{m-1}S(p-m+1, 1) \end{aligned}$$

5. outer functions

Definition5.1
Let

$$f(z) := \exp\left(\int \frac{e^{i\theta}+z}{e^{i\theta}-z}h(e^{i\theta})dm(e^{i\theta})\right),$$

where h is real valued Lebesgue integrable function on the unit circle and m is the Lebesgue measure. Then f is called an *outer function.*

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