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Exercises

1. In each case, show that the singular point of the funcition are poles. Determine the order m of each pole, and find the corresponding residue B.

(a) $\frac{z^2+2}{z-1}$;

Solution:

let $f(z) = \frac{\phi(z)}{z-1}$, and $\phi(z) = z^2 + 2$

Thus

order $m = 1$, $\mathrm{Res}_{z=1} f(z) = \phi(1) = 3$

(b) $\left(\frac{z}{2z+1}\right)^3$

Solution:

let $f(z) = \frac{\phi(z)}{\left(z+\frac{1}{2}\right)^3}$, and $\phi(z) = \frac{1}{8}z^3$

Thus

order $m = 3$, $\mathrm{Res}_{z=-1/2} f(z) = \frac{\phi^{(2)}(-1/2)}{2!} = \frac{\frac{6}{8}\cdot\left(\frac{-1}{2}\right)}{2} = \frac{-3}{16}$

(c) $\tanh z$;

Solution:

let $\tanh z = f(z) = \frac{p(z)}{q(z)} = \frac{\sinh z}{\cosh z}$,

and we know

$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$, $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

$\cosh z = 0 \Leftrightarrow z = 0$

Thus residue $B = \frac{p(0)}{q'(0)} = \frac{\sinh 0}{\sinh 0} = 1$

(d) $\frac{\exp z}{z^2+\pi^2}$

Solution:

let $f(z) = \frac{\phi(z)}{z+i\pi}$, and $\phi(z) = \frac{\exp z}{z-i\pi}$

Thus

order $m = 1$, $\mathrm{Res}_{z=-i\pi} f(z) = \phi(-i\pi) = \frac{-1}{-2i\pi} = \frac{-i}{2\pi}$

let $f(z) = \frac{\psi(z)}{z-i\pi}$, and $\psi(z) = \frac{\exp z}{z+i\pi}$

Thus

order $m = 1$, $\mathrm{Res}_{z=i\pi} f(z) = \psi(i\pi) = \frac{-1}{2i\pi} = \frac{i}{2\pi}$

2. Show that te point $z = 0$ is a simple pole of the function $f(z) = \csc z$ and that the residue there is 1 by appealing to

(a) the Laurent series for $\csc z$ in Exercise 11(a), Sec. 51;

(b) the corollary in Sec. 57

Solution:

(a) $\csc z = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right] z^3 + \cdots, \quad (0 < |z| < \pi)$

Thus $\mathrm{Res}_{z=0} \csc z = 1$

(b) $\csc z = \frac{1}{\sin z}$

$\mathrm{Res}_{z=0} \csc z = \frac{1}{(\sin z)'}|_{z=0} = \frac{1}{\cos z}|_{z=0} = 1$

Show that
(a) $\text{Res}_{z=-1}\frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}}$ $(|z|>0, 0<\arg z<2\pi)$
Solution:
$\text{Res}_{z=-1}\frac{z^{1/4}}{z+1} = z^{1/4}|_{z=-1} = (-1)^{1/4} = \frac{1+i}{\sqrt{2}}$

(b) $\text{Res}_{z=i}\frac{\log z}{(z^2+1)^2}$
order $m=2$, and $\frac{d}{dz}(\frac{\log z}{(z+i)^2}) = \frac{\frac{1}{z}(z+i)^2-2(z+i)\log z}{(z+i)^4} = \frac{\frac{1}{z}(z+i)-2\log z}{(z+i)^3}$
$\frac{d}{dz}(\frac{\log z}{(z+i)^2})|_{z=i} = \frac{2+2\log i}{-8i} = \frac{2+2(\log 1+i\frac{\pi}{2})}{-8i} = \frac{i\pi+2}{-8i} = \frac{\pi+2i}{8}$
Thus
$\text{Res}_{z=i}\frac{\log z}{(z^2+1)^2} = \frac{d}{dz}(\frac{\log z}{(z+i)^2})|_{z=i} = \frac{\pi+2i}{8}$

(c) $\text{Res}_{z=z_n}(z\sec z) = (-1)^{n+1}z_n$, where $z_n = \frac{\pi}{2}+n\pi(n=0,\pm1,\pm2,\cdots)$
Solution:
$\text{Res}_{z=z_n}(z\sec z)$ $=\text{Res}_{z=z_n}(\frac{z}{\cos z}) = \frac{z}{(\cos z)'}|_{z=z_n} = \frac{z}{\sin z}|_{z=z_n} = \frac{z_n}{(-1)^{n+1}} =$ $(-1)^{n+1}z_n$

(d) $\text{Res}_{z=\pi i}\frac{\exp(zt)}{\sinh z} + \text{Res}_{z=-\pi i}\frac{\exp(zt)}{\sinh z} = -2\cos\pi t$

4. Find the value of the integral

$$\int_c \frac{3z^3+2}{(z-1)(z^2+9)}dz,$$

taken counterclockwise around the circle (a) $|z-2|=2$ (b) $|z|=4$.
Solution:
(a) since C is curve $|z-2|=2$, point $z_1=1$ inside the curve
$B_1 = \text{Res}_{z=1}\frac{3z^3+2}{(z-1)(z^2+9)} = \frac{3z^3+2}{(z^2+9)}|_{z=1} = \frac{5}{10} = \frac{1}{2}$
$\int_c \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i\cdot\frac{1}{2} = \pi i$

(b) since C is curve $|z|=4$, point $z_1=1, z_2=3i$, $z_3=-3i$
$B_2 = \text{Res}_{z=3i}\frac{3z^3+2}{(z-1)(z^2+9)} = \frac{3z^3+2}{(z-1)(z+3i)} = \frac{3\cdot(3i)^3+2}{(3i-1)\cdot 6i} = \frac{5}{4}+\frac{49}{12}i$
$B_3 = \text{Res}_{z=-3i}\frac{3z^3+2}{(z-1)(z^2+9)} = \frac{3z^3+2}{(z-1)(z-3i)} = \frac{3\cdot(-3i)^3+2}{-(-3i-1)\cdot 6i} = \frac{5}{4}-\frac{49}{12}i$
$\int_c \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i\cdot(B_1+B_2+B_3) = 2\pi i\cdot(3) = 6\pi i$

5. Find the value of the integral

$$\int_c \frac{dz}{z^3(z+4)},$$

taken counterclockwise around the circle (a) $|z|=2$; (b) $|z+2|=3$.
Solution:
(a) since C is curve $|z=2|$, point $z_1=0$ inside the curve,
$B_1 = \text{Res}_{z=0}\frac{1}{z^3(z+4)} = \frac{1}{2!}\left[\frac{1}{(z+4)}\right]''|_{z=0} = \frac{1}{2!}\cdot\frac{2}{(z+4)^3}|_{z=0} = \frac{1}{64}$

$\int_c \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i \cdot \frac{1}{64} = \frac{\pi i}{32}$

(b) since C is curve $|z+2| = 3$,point $z_1 = 0$, $z_2 = -4$ inside the curve,
$B_2 = \text{Res}_{z=-4} \frac{1}{z^3(z+4)} = \frac{1}{z^3}|_{z=-4} = \frac{-1}{64}$
$\int_c \frac{3z^3+2}{(z-1)(z^2+9)}dz = 2\pi i \cdot (B_1 + B_2) = 2\pi i \cdot 0 = 0$

6. Let C be the circle $|z| = 2$, described in the positive sense, and evaluate the integral
(a) $\int_c \tan z dz$
Solution:
isolated point $z_n = \frac{\pi}{2} + n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$
$B_n = \text{Res}_{z=z_n} \tan z = \text{Res}_{z=z_n} \frac{\sin z}{\cos z} = \frac{\sin z}{(\cos z)'}|_{z=z_n} = -1$
C is curve $|z| = 2$, point $\frac{\pi}{2}, -\frac{\pi}{2}$ inside the curve,
$\int_c \tan z dz = 2\pi i \cdot (B_0 + B_{-1}) = 2\pi i \cdot (-2) = -4\pi i$

(b) $\int_c \frac{dz}{\sinh 2z}$
Solution:
isolate point $z_n = n\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$
$B_n = \text{Res}_{z=z_n} \frac{1}{\sinh 2z} = \frac{1}{(\sinh 2z)'}|_{z=z_n} = \frac{1}{2\cosh 2z}|_{z=z_n} = \frac{-1}{2}$
C is curve $|z| = 2$, point 0 inside the curve,
$\int_c \frac{dz}{\sinh 2z} = 2\pi i \cdot (B_1) = 2\pi i \cdot (\frac{-1}{2}) = -\pi i$

(c) $\int_c \frac{\cosh \pi z}{z(z^2+1)} dz$
Solution:
since C is curve $|z| = 2$, point $z_0 = 0$, $z_1 = i$, $z_2 = -i$ inside the curve.
$B_1 = \text{Res}_{z=0} \frac{\cosh \pi z}{z(z^2+1)} = \frac{\cosh \pi z}{(z^2+1)}|_{z=0} = 1$
$B_2 = \text{Res}_{z=i} \frac{\cosh \pi z}{z(z^2+1)} = \frac{\cosh \pi z}{z(z+i)}|_{z=i} = \frac{\cosh i\pi}{-2} = \frac{1}{2}$
$B_3 = \text{Res}_{z=-i} \frac{\cosh \pi z}{z(z^2+1)} = \frac{\cosh \pi z}{z(z-i)}|_{z=-i} = \frac{\cosh -i\pi}{-2} = \frac{1}{2}$
$\int_c \frac{\cosh \pi z}{z(z^2+1)} dz = 2\pi i \cdot (B_1 + B_2 + B_3) = 2\pi i \cdot 2 = 4\pi i$

7. Use Theorem 2 in Sec. 54, involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when
(a) $f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}$;
Solution:
since C is curve $|z| = 3$, point $z_0 = 0$, $z_1 = 1$, $z_2 = -\frac{5}{2}$ inside the curve.
$B_1 = \text{Res}_{z=0} \frac{(3z+2)^2}{z(z-1)(2z+5)} = \frac{(3z+2)^2}{(z-1)(2z+5)}|_{z=0} = -\frac{4}{5}$
$B_2 = \text{Res}_{z=1} \frac{(3z+2)^2}{z(z-1)(2z+5)} = \frac{(3z+2)^2}{z(2z+5)}|_{z=1} = \frac{25}{7}$
$B_3 = \text{Res}_{z=1} \frac{(3z+2)^2}{z(z-1)(2z+5)} = \frac{(3z+2)^2}{2z(z-1)}|_{z=-\frac{5}{2}} = \frac{121}{70}$
$\int_c f(z)dz = 2\pi i \cdot (B_1 + B_2 + B_3) = 2\pi i \cdot (4.5) = 9\pi i$

(a) $f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}$;
Solution:

since C is curve $|z| = 3$, point $z_0 = 0$, $z_1 = 1$, $z_2 = -\frac{5}{2}$ inside the curve.
$\text{Res}_{z=0}\frac{1}{z^2}f(\frac{1}{z})$ $=\text{Res}_{z=0}\frac{1}{z^2}\frac{z(3+2z)^2}{(1-z)(2+5z)} = \frac{(3+2z)^2}{(1-z)(2+5z)}|_{z=0} = \frac{3^2}{2} = 4.5$
$\int_c f(z)dz = 2\pi i\cdot\text{Res}_{z=0}\frac{1}{z^2}f(\frac{1}{z}) = 9\pi i$

(b) $f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}$
Solution:
since C is curve $|z| = 3$,
$\text{Res}_{z=0}\frac{1}{z^2}f(\frac{1}{z})$ $=\text{Res}_{z=0}\frac{1}{z^2}\cdot\frac{z(z-3)}{(z+1)(z^4+2)} = \frac{(z-3)}{(z+1)(z^4+2)}|_{z=0} = \frac{-3}{2} = -1.5$
$\int_c f(z)dz = 2\pi i\cdot\text{Res}_{z=0}\frac{1}{z^2}f(\frac{1}{z}) = -3\pi i$

(c) $f(z) = \frac{z^3 e^{1/z}}{1+z^3}$
Solution:
since C is curve $|z| = 3$,
$\text{Res}_{z=0}\frac{1}{z^2}f(\frac{1}{z})$ $=\text{Res}_{z=0}\frac{1}{z^2}\cdot\frac{e^z}{z^3+1} = \left(\frac{e^z}{z^3+1}\right)'|_{z=0} = \frac{e^z(z^3+1)-e^z(2z^2)}{(z^3+1)^2}|_{z=0} = \frac{1}{1} = 1$
$\int_c f(z)dz = 2\pi i\cdot\text{Res}_{z=0}\frac{1}{z^2}f(\frac{1}{z}) = 2\pi i$

8. Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \qquad \text{and} \qquad y = \pm\left(N + \frac{1}{2}\right)\pi$$

where N is a positive integer. Show that

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i\left[\frac{1}{6} + 2\sum_{n=1}^{N}\frac{(-1)^n}{n^2\pi^2}\right].$$

Then, using the fact that the value of this integral tends to zero as N tends to infinity (Exercise 17, Sec. 33), point out how it follows that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

Solution:
Singular points $z_p = 0$, $z_n = \left(n + \frac{1}{2}\right)\pi$, $(n = 0, \pm 1, \pm 2, \ldots)$

9. Show that

$$\int_c \frac{dz}{(z^2-1)^2+3} = \frac{\pi}{2\sqrt{2}}$$

where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2, y = 0$ and $y = 1$.

Suggestion: By observing that the four zeros of the polynomial $q(z) = (z^2 - 1)^2 + 3$ are the square roots of the numbers $1 \pm \sqrt{3}i$; show that the reciprocal $1/q(z)$ is analytic inside and on C except at the points

$$z_0 = \frac{\sqrt{3}+i}{\sqrt{2}} \qquad \text{and} \qquad -\overline{z}_0 = \frac{-\sqrt{3}+i}{\sqrt{2}}$$

Then apply the corollary in Sec. 57.

Solution:

Singular points $z_0 = \frac{\sqrt{3}+i}{\sqrt{2}}$ and $-\overline{z}_0 = \frac{-\sqrt{3}+i}{\sqrt{2}}$ inside the curve C.

$B_1 = \text{Res}_{z=z_0} \frac{1}{(z^2-1)^2+3} = \frac{1}{((z^2-1)^2+3)'}|_{z=z_0} = \frac{1}{2(z^2-1)\cdot 2z}|_{z=z_0} = \frac{1}{4z(z^2-1)}|_{z=z_0} = \frac{1}{4\cdot\frac{\sqrt{3}+i}{\sqrt{2}}\cdot((\frac{\sqrt{3}+i}{\sqrt{2}})^2-1)} = -\frac{1}{16}i\sqrt{2} - \frac{1}{48}\sqrt{6}$

$B_2 = \text{Res}_{z=-\overline{z}_0} \frac{1}{(z^2-1)^2+3} = \frac{1}{((z^2-1)^2+3)'}|_{z=-\overline{z}_0} = \frac{1}{2(z^2-1)\cdot 2z}|_{z=-\overline{z}_0} = \frac{1}{4z(z^2-1)}|_{z=-\overline{z}_0} = \frac{1}{4\cdot\frac{-\sqrt{3}+i}{\sqrt{2}}\cdot((\frac{-\sqrt{3}+i}{\sqrt{2}})^2-1)} = \frac{1}{48}\sqrt{6} - \frac{1}{16}i\sqrt{2}$

Thus

$\int_c \frac{dz}{(z^2-1)^2+3} = 2\pi i \cdot (B_1 + B_2) = 2\pi i \cdot \left(-\frac{\sqrt{2}}{8}i\right) = \frac{\pi}{2\sqrt{2}}$

10. Consider the function

$$f(z) = \frac{1}{[q(z)]^2}$$

where q is analytic at $z_0, q(z_0) = 0$, and $q'(z_0) \neq 0$. Show that z_0 is a pole of order $m = 2$ of the function f, with residue

$$B_0 = -\frac{q''(z_0)}{[q'(z_0)]^3}$$

Suggestion: Note that z_0 is a zero of order $m = 1$ of the function q, so that equation (5), Sec. 57, holds. Then write

$$f(z) = \frac{\Phi(z)}{(z-z_0)^2} \qquad \text{where} \qquad \Phi(z) = \frac{1}{[g(z)]^2}.$$

The desired form of the residue $B_0 = \Phi'(z_0)$ can be obtained by showing that $q'(z_0) = g(z_0)$ and $q''(z_0) = 2g'(z_0)$.

Solution:

let $f(z) = \frac{\Phi(z)}{(z-z_0)^2}$, and $\Phi(z) = \frac{1}{[g(z)]^2}$ and $q(z) = (z - z_0)g(z)$

then

$q'(z) = g(z) + (z - z_0)g'(z)$

$q''(z) = g'(z) + g'(z) + (z - z_0)g''(z)$

imply that $g(z_0) = q'(z_0), 2g(z_0) = q''(z_0)$

Thus

$\text{Res}_{z=z_0} f(z) = \Phi'(z) = \left[\frac{1}{[g(z)]^2}\right]' |_{z=z_0} = \frac{2g(z)\cdot g'(z)}{[g(z)]^4}|_{z=z_0} = \frac{2g'(z)}{[g(z)]^3}|_{z=z_0} = \frac{2g'(z_0)}{[g(z_0)]^3} = -\frac{q''(z_0)}{[q'(z_0)]^3}$

11. Apply the result in Exercise 10 to find the residue at $z = 0$ of function
(a) $f(z) = \csc^2 z$;
Solution:
$\mathrm{Res}_{z=0} \csc^2 z = \mathrm{Res}_{z=0} \frac{1}{\sin^2 z} = -\frac{[\sin z]''}{([\sin z]')^3}|_{z=0} = -\frac{\sin z}{\cos^3 z}|_{z=0} = 0$

(b) $f(z) = \frac{1}{(z+z^2)^2}$
Solution:
$\mathrm{Res}_{z=0} \frac{1}{(z+z^2)^2} = \frac{-[z+z^2]''}{([z+z^2]')^3}|_{z=0} = \frac{-2}{(1+2z)^3}|_{z=0} = -2$

12. Let p and q denote functions that are analytic at a point z_0, where $p(z_0) \neq 0$ and $q(z_0) = 0$. Show that if the quotient $p(z)/q(z)$ has a pole of order m at z_0, then z_0 is a zero to order m of q.(Compare the therem in Sec. 57).
Proof:
To Show this

$$\mathrm{Res}_{z=z_0} \frac{p(z)}{q(z)} =$$

we assume that p and q are as stated and observe that, because of the conditions on q, the point z_0 is a zero of order $m = 1$ of that function. According to the lemma, then,

$$q(z) = (z - z_0)^m g(z)$$

where $g(z)$ is analytic and nonzero at z_0. Furthermore, the theorem tells us that z_0 is a simple pole of $p(z)/q(z)$; and equation (3) in the prrof of the theorem becomes

$$\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{(z - z_0)^m}$$

Now $p(z)/g(z)$ is analytic and nonzero at z_0, and it follows from the theorem in Sec. 56 that

$$\mathrm{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{\frac{d^{(m-1)}}{dz^{m-1}}[p(z)/g(z)]}{(m-1)!}|_{z=z_0}$$

$\frac{d}{dz}(\frac{p}{g}) = \frac{p'g - pg'}{g^2} = \frac{p'}{g} - \frac{pg'}{g^2} = \frac{p'g - pg'}{g^2}$
$\frac{d}{dz^2}(\frac{p}{g}) = \frac{p''g + p'g'}{g^2} - \frac{(p'g' + pg'')g^2 - (pg')(2gg')}{g^4} = \frac{p''}{g^2} + \frac{p'g'}{g^2} - \frac{p'g'}{g^2} - \frac{pg''}{g^2} - \frac{2pg'g'}{g^3} = \frac{p''}{g^2} - \frac{pg''}{g^2} + \frac{2pg'g'}{g^3}$
$= \frac{p''g-}{g^3}$
...
$\frac{d^{m-1}}{dz^{m-1}}(\frac{p}{g}) =$
thus

$$\mathrm{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{1}{(m-1)!}$$