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1. Find the residue at $z = 0$ of the function

(a) $\frac{1}{z+z^2} = \frac{1}{z} \cdot \frac{1}{1+z} = \frac{1}{z} \cdot (1 - z + z^2 - z^3 + \cdots + (-1)^n z^n + \cdots)$

$= \frac{1}{z} - 1 + z - z^2 + z^3 + \cdots + (-1)^{n+1} z^n + \cdots$

Ans:1

(b) $z\cos(\frac{1}{z}) = z \cdot (1 - \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{4!} \cdot \frac{1}{z^4} - \cdots + \frac{(-1)^n}{(2n)!} \cdot \frac{1}{z^{2n}} + \cdots)$

$= z - \frac{1}{2!} \cdot \frac{1}{z} + \frac{1}{4!} \cdot \frac{1}{z^3} - \cdots + \frac{(-1)^n}{(2n)!} \cdot \frac{1}{z^{2n-1}} + \cdots$

Ans:$-\frac{1}{2}$

(c) $\frac{z-\sin z}{z} = \frac{1}{z}\left[z - (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots + \frac{(-1)^n z^{2n+1}}{(2n+1)} + \cdots)\right]$

$= \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \cdots + \frac{(-1)^{n+1} z^{2n}}{(2n+1)} + \cdots$

Ans:0

(d) $\frac{\cot z}{z^4} = \frac{1}{z^4}[\quad]$

(e) $\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^4} \cdot \frac{1}{1-z^2} \cdot \sinh z$

$= \frac{1}{z^4}[1 + z^2 + z^4 + z^6 + \cdots]\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]$

Ans:$\frac{1}{3!} + 1 = \frac{7}{6}$

2. Use Theorem 1, Sec. 54, to evaluate the integral of each ot these functions aroud the circle $|z| = 3$ in the positive sense:

(a) $\frac{\exp(-z)}{z^2} = \frac{1}{z^2}\left[1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \cdots + \frac{(-1)^n z^n}{n!} + \cdots\right] = \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \cdots$

Solution:

$\int_c \frac{\exp(-z)}{z^2} dz = 2\pi i \cdot (-1) = -2\pi i$

(b) $\frac{\exp(-z)}{(z-1)^2} = \frac{\exp(1-z)}{e(z-1)^2} = \frac{1}{e(z-1)^2}\left[1 - (z-1) + \frac{(1-z)^2}{2!} - \frac{(1-z)3}{3!} + \cdots\right]$

Solution:

$\int_c \frac{\exp(-z)}{(z-1)^2} dz = 2\pi i \cdot (\frac{-1}{e}) = \frac{-2\pi i}{e}$

(c) $z^2 \exp(\frac{1}{z}) = z^2(1 + \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{3!} \cdot \frac{1}{z^3} + \cdots)$

Solution:

$\int_c z^2 \exp(\frac{1}{z}) dz = 2\pi i \cdot (\frac{1}{3!}) = \frac{\pi i}{3}$

(d) $\frac{z+1}{z^2-2z}$

Solution:

If $f(z) = \frac{z+1}{z^2-2z}$, then $\frac{1}{z^2}f(\frac{1}{z}) = \frac{1}{z^2}\left[\frac{\frac{1}{z}+1}{\frac{1}{z^2}-\frac{2}{z}}\right] = \frac{1}{z^2}\left[\frac{z+z^2}{1-2z}\right] = \frac{1}{z}\left[\frac{1+z}{1-2z}\right]$

$\frac{1}{z^2}f(\frac{1}{z}) = \frac{1}{z}(1+z)(1+2z+4z^2+\cdots)$

$\int_c f(z)dz = 2\pi i \text{Res}_{z=0}\left[\frac{1}{z^2}f(\frac{1}{z})\right] = 2\pi i \cdot 1 = 2\pi i$

(d) $\frac{z+1}{z^2-2z}$

Solution:

$f(z) = \frac{z+1}{z(z-2)} = (-\frac{z+1}{2z})\left(\frac{1}{1-\frac{1}{2}z}\right) = (-\frac{1}{2} - \frac{1}{2z})(1 + \frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{8}z^3 + \cdots)$

$\text{Res}_{z=0}f(z) = \frac{-1}{2}$

$f(z) = \frac{z+1}{z(z-2)} = (\frac{z+1}{z-2})(\frac{1}{2+z-2}) = \left(\frac{z+1}{2(z-2)}\right)\left(\frac{1}{1+\frac{1}{2}(z-2)}\right)$

$= \left(\frac{1}{2} + \frac{3}{2(z-2)}\right)\left(1 + \frac{(z-2)}{2} + \frac{(z-2)^2}{4} + \frac{(z-2)^3}{8} + \cdots\right)$

$\text{Res}_{z=2}f(z) = \frac{3}{2}$

$\int_c f(z)dz = 2\pi i \cdot (\frac{-1}{2} + \frac{3}{2}) = 2\pi i$

3. Use Theorem 2, Sec. 54, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

(a) $\frac{z^5}{1-z^3}$

Solution:

If $f(z) = \frac{z^5}{1-z^3}$, then $\frac{1}{z^2}f(\frac{1}{z}) = \frac{1}{z^2} \cdot \frac{\frac{1}{z^5}}{1-\frac{1}{z^3}} = \frac{1}{z^2} \cdot \frac{1}{z^5-z^2} = \frac{-1}{z^4} \cdot \frac{1}{1-z^3}$.

$\frac{1}{z^2}f(\frac{1}{z}) = \frac{-1}{z^4}(1+z^3+z^6+z^9+\cdots)$

$\text{Res}_{z=0}\frac{1}{z^2}f(\frac{1}{z}) = -1$

$\int_c f(z)dz = 2\pi i \text{Res}_{z=0}\frac{1}{z^2}f(\frac{1}{z}) = -2\pi i$

(b) $\frac{1}{1+z^2}$

Solution:

If $f(z) = \frac{1}{1+z^2}$, then $\frac{1}{z^2}f(\frac{1}{z}) = \frac{1}{z^2} \cdot \frac{1}{1+\frac{1}{z^2}} = \frac{1}{1+z^2} = 1 + z^2 + z^4 + z^6 + \cdots$

$\text{Res}_{z=0}\frac{1}{z^2}f(\frac{1}{z}) = 0$

$\int_c f(z)dz = 0$

(c) $\frac{1}{z}$

Soluiton:

If $f(z) = \frac{1}{z}$, then $\frac{1}{z^2}f(\frac{1}{z}) = \frac{1}{z^2} \cdot z = \frac{1}{z}$

$\text{Res}_{z=0}\frac{1}{z^2}f(\frac{1}{z}) = 1$

$\int_c f(z)dz = 2\pi i$

4. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential

singular point:

5. Show that the singular point of each of the following functions is a pole. determine the order m of that pole and the corresponding residue B.

(a) $\frac{1-\cosh z}{z^3}$

Solution:

$\frac{1-\cosh z}{z^3} = \frac{1}{z^3}(1-(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\cdots)) = -\frac{1}{2!}\bullet\frac{1}{z}-\frac{z}{4!}-\frac{z^3}{6!}+\cdots$

Ans: $m = 1, B = -\frac{1}{2}$

(b) $\frac{1-\exp(2z)}{z^4}$

Solution:

$\frac{1-\exp(2z)}{z^4} = \frac{1}{z^4}(1-(1+2z+\frac{2^2z^2}{2!}+\frac{2^3z^3}{3!}+\frac{2^4z^4}{4!}+\cdots))$

Ans: $m = 3, B = \frac{-2^3}{3!} = \frac{-4}{3}$

(c) $\frac{\exp(2z)}{(z-1)^2}$

Solution:

$\frac{\exp(2z)}{(z-1)^2} = \frac{e^2\exp(2z-2)}{(z-1)^2} = \frac{e^2}{(z-1)^2}\left[1+2(z-1)+\frac{2^2(z-1)^2}{2!}+\frac{2^3(z-1)^3}{3!}+\cdots\right]$

Ans: $m = 2, B = 2e^2$

6. Suppose that a function f is analytic at z_0, and consider the quotient

$$g(z) = \frac{f(z)}{z-z_0}$$

Show that

(a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with resodie $f(z_0)$;

(b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

Suggestion: As pointed out in Sec. 44, there is a Taylar series for $f(z)$ about z_0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

Solution:

The Expansion fo $f(z)$ into a Taylor series about the point z_0.

$f(z) = f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots$

$\frac{f(z)}{z-z_0} = \frac{f(z_0)}{z-z_0}+\frac{f'(z_0)}{1!}+\frac{f''(z_0)}{2!}(z-z_0)+\cdots$

Thus

$\text{Res}_{z=z_0}g(z) = f(z_0)$

If $f(z_0) = 0$, then $\text{Res}_{z=z_0}g(z) = 0$, i.e. z_0 is a removable singular point of g.

7. Let the degrees of the polynomials

$$P(z) = a_0+a_1z+a_2z^2+a_3z^3+\cdots+a_nz^n$$

and

$$Q(z) = b_0+b_1z+b_2z^2+b_3z^3+\cdots+b_mz^m$$

be such that $m \geq n+2$.

(a) write $(z \neq 0)$

$$\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)}$$

as the quotient of two polynomials, and point out why $z = 0$ is a removable singular point of that quotient.

(b) Use the final result in part (a) and Theorem 2 in Sec. 54 to show that if all of the zeros of $Q(z)$ are interior to a given simple closed contour C, then

$$\int_c \frac{P(z)}{Q(z)} dz = 0$$

[Compare Exercise $3(b)$].

Solution:

(a) $\frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} = \frac{1}{z^2} \cdot \frac{a_0+a_1z^{-1}+a_2z^{-2}+a_3z^{-3}+\cdots+a_nz^{-n}}{b_0+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+\cdots+b_mz^{-m}}$

Res$_{z=0} \frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} = 0$

(b) $\int_c \frac{P(z)}{Q(z)} dz = 2\pi i$ •Res$_{z=0} \frac{1}{z^2} \cdot \frac{P(1/z)}{Q(1/z)} = 0$