P73.

15. Let A be a symmetric matrix

(a) Show that A^2 is symmetric.

(b) Show that $2A^2 - 3A + I$ is symmetric.

Proof:

we know that $A^T = A$, then

$(A^2)^T = (A \cdot A)^T = (A^T \cdot A^T) = (A \cdot A) = A^2$,

Thus A^2 is symmetric.

And

$(2A^2 - 3A + I)^T = 2(A^2)^T - 3A^T + I^T = 2A^2 - 3A + I$

Therefore $2A^2 - 3A + I$ is symmetric.

18. Prove: If $A^TA = A$, then A is symmetric and $A = A^2$.

Proof:

Since $A^T = (A^TA)^T = A^T \cdot (A^T)^T = A^TA = A$, then A is symmentric

And now we know that $A^T = A$, imply that

$A^TA = A$,

$AA = A$

$A^2 = A$.

19. Find all 3×3 diagonal matrices A that satisfy $A^2 - 3A - 4I = 0$

Solution:

Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, then

$$A^2 - 3A - 4I = \begin{bmatrix} a^2 - 3a - 4 & 0 & 0 \\ 0 & b^2 - 3b - 4 & 0 \\ 0 & 0 & c^2 - 3c - 4 \end{bmatrix} = 0,$$

i.e. $a^2 - 3a - 4 = 0$, $b^2 - 3b - 4 = 0$, $c^2 - 3c - 4 = 0$, imply that $a = b = c = 4$ or -1.

so that A is diagonal matrice, and the diagonal element is equal 4 or -1.

20. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Determine whether A is symmetric.

(a) $a_{ij} = i^2 + j^2$

(b) $a_{ij} = i^2 - j^2$

(c) $a_{ij} = 2i + 2j$

(d) $a_{ij} = 2i^2 + 3j^3$

Solution:

(a) $a_{ji} = j^2 + i^2 = i^2 + j^2 = a_{ij}$, so that A is symmetric.

(b) $a_{ji} = j^2 - i^2 \neq i^2 - j^2 = a_{ij}$, so that A is not symmetric.

(c) $a_{ji} = 2j + 2i = 2i + 2j = a_{ij}$, so that A is symmetric.

(d) $a_{ji} = 2j^2 + 3i^3 \neq 3i^2 + 2j^3 = a_{ij}$, so that A is not symmetric.

22. A squre matrix A is called skew-symmetric if $A^T = -A$, Prove:

(a) If A is an invertible skew-symmetric, then A^{-1} is skew-symmetric.

(b) if A and B are skew-symmetric, then so are A^T, $A+B$, $A-B$, and kA for any scalar k.

(c) Every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix,

[Hint. Note the identity $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$.]

Proof:

$(A^{-1})^T = (A^T)^{-1} = -A^{-1}$, so that A^{-1} is an skew-symmetric.

$(A^T)^T = A = -A^T$, so that A^T is an skew-symmetric.

$(A+B)^T = A^T + B^T = -A - B = -(A+B)$, it is true.

$(A-B)^T = A^T - B^T = -A + B = -(A-B)$, it is true.

$(kA)^T = kA^T = k(-A) = -kA$, it is true.

(c) Since $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$, and

$[\frac{1}{2}(A+A^T)]^T = \frac{1}{2}\left(A^T + (A^T)^T\right) = \frac{1}{2}(A^T + A) = \frac{1}{2}(A+A^T)$, is symmetric.

$[\frac{1}{2}(A-A^T)]^T = \frac{1}{2}\left(A^T - (A^T)^T\right) = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A-A^T)$, is skew-symmetric.

so (c) is complete proof.

P75.

14. Let A be a square matrix.

(a) Show that $(I-A)^{-1} = I + A + A^2 + A^3$ if $A^4 = 0$

(b) Show that $(I-A)^{-1} = I + A + A^2 + \cdots + A^n$ if $A^{n+1} = 0$.

Proof:

(a) Since $(I-A)(I+A+A^2+A^3) = I - A^4 = I - 0 = I$, so that $(I-A)^{-1} = I + A + A^2 + A^3$.

(b) Since $(I-A)(I+A+A^2+\cdots+A^n) = I - A^{n+1} = I$, so that $(I-A)^{-1} = I + A + A^2 + \cdots + A^n$.

19. Prove: If B is invertible, then $AB^{-1} = B^{-1}A$ if and only if $AB = BA$.

Proof:

$AB^{-1} = B^{-1}A$

$AB^{-1}B = B^{-1}AB$

$A = B^{-1}AB$

$BA = BB^{-1}AB$

$BA = AB$.

20. Prove: If A is invertible, then $A+B$ and $I+BA^{-1}$ are both invertible or both not invertible.

Proof:

if $A+B$ is invertible, then $(A+B)A^{-1} = AA^{-1} + BA^{-1} = I + BA^{-1}$ is invertible.

if $I+BA^{-1}$ is invertible, then $(I+BA^{-1})A = A + BAA^{-1} = A + B$ is inverible.

Therefore $A+B$ and $I+BA^{-1}$ are both invertible or both not invertible.