1. Derive the Hilbert transform $H[\cos(\omega t)] = \sin(\omega t)$

Solution:

By definition of Hilbert transform is

$$H[u(t)] = \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{u(s)}{s-t} ds$$

so that

$$H[\cos(\omega t)] = \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{\cos(\omega s)}{s-t} ds$$

change the variable $y = s - t,$ then

$$\begin{aligned} H[\cos(\omega t)] &= \frac{-1}{\pi} P \int_{-\infty}^{\infty} \frac{\cos[\omega(y+t)]}{y} dy \\ &= \frac{-1}{\pi} \left\{ \cos(\omega t) \cdot P \int_{-\infty}^{\infty} \frac{\cos(\omega y)}{y} dy - \sin(\omega t) \cdot P \int_{-\infty}^{\infty} \frac{\sin(\omega y)}{y} dy \right\} \end{aligned}$$

The integrals inside the brackets are

$$P \int_{-\infty}^{\infty} \frac{\cos(\omega y)}{y} dy = 0; \quad P \int_{-\infty}^{\infty} \frac{\sin(\omega y)}{y} dy = \pi$$

and therefore

$$H[\cos(\omega t)] = \sin(\omega t).$$

calculate is completed.

2. calculate $P \int_{-\infty}^{\infty} \frac{\cos(\omega y)}{y} dy = 0.$

Solution:

since $\frac{\cos(\omega y)}{y}$ is odd function, and integral interval is symmetric,

so then $P \int_{-\infty}^{\infty} \frac{\cos(\omega y)}{y} dy = 0.$

3. calculate $P \int_{-\infty}^{\infty} \frac{\sin(\omega y)}{y} dy = \pi.$