Fourier series of function $f(x)$

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where fourier coefficients

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$$

Page 486 13,16,19,22,24

13. $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1dx = \frac{1}{\pi}\int_{0}^{\frac{\pi}{2}} 1dx = \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos nxdx = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} \cos nxdx$

$= \frac{2}{\pi}\cdot\frac{1}{n}\sin nx|_{0}^{\frac{\pi}{2}} = \frac{2}{\pi}\cdot\frac{1}{n}\sin\frac{n\pi}{2}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin nxdx = 0$

hence, $f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \frac{1}{2} + \sum_{n=1}^{\infty}(\frac{2}{\pi}\cdot\frac{1}{n}\sin\frac{n\pi}{2})\cos nx$

$= \frac{1}{2} + \frac{2}{\pi}(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \frac{1}{7}\cos 7x + \cdots\cdots)$

16. $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} xdx = 0,$ since x is even function.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$ since $x\cos nx$ is even function.

$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos nxdx = 0$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$ since $x\sin nx$ is even function.

$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin nxdx = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} x\sin nxdx = \frac{2}{\pi}\cdot\frac{-1}{n}\int_{0}^{\frac{\pi}{2}} xd(\cos nx)$

$= \frac{2}{\pi}\cdot\frac{-1}{n}x\cos nx|_{0}^{\frac{\pi}{2}} + \frac{2}{\pi}\cdot\frac{1}{n}\int_{0}^{\frac{\pi}{2}}\cos nxdx$

$= \frac{-2}{n\pi}(\frac{\pi}{2}\cos\frac{n\pi}{2}) + \frac{2}{\pi}\cdot\frac{1}{n^2}\sin nx|_{0}^{\frac{\pi}{2}}$

$= \frac{-1}{n}\cos\frac{n\pi}{2} + \frac{2}{n^2\pi}\sin\frac{n\pi}{2}$

hence, $f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=1}^{\infty}(\frac{-1}{n}\cos\frac{n\pi}{2} + \frac{2}{n^2\pi}\sin\frac{n\pi}{2})\sin nx$

19. $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{0}^{\pi} xdx + \frac{1}{2\pi}\int_{-\pi}^{0}[-(x+\pi)]dx = 0$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_{0}^{\pi} x\cos nxdx + \frac{1}{\pi}\int_{-\pi}^{0}[-(x+\pi)]\cos nxdx$

$= \frac{1}{\pi}\int_{0}^{\pi} x\cos nxdx + \frac{1}{\pi}\int_{-\pi}^{0} -x\cos nxdx + \frac{1}{\pi}\int_{-\pi}^{0} -\pi\cos nxdx$

$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nxdx + (-1)\int_{-\pi}^{0}\cos nxdx$

$= \frac{2}{\pi}\cdot\frac{1}{n}\int_{0}^{\pi} xd(\sin nx) + (\frac{-1}{n})\sin nx|_{-\pi}^{0}$

$$= \frac{2}{n\pi}x\sin nx|_0^\pi + \frac{-2}{n\pi}\int_0^\pi \sin nxdx + 0$$
$$= \frac{-2}{n\pi} \cdot \frac{-1}{n}\cos nx|_0^\pi = \frac{2}{n^2\pi}(\cos n\pi - 1)$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_0^\pi x\sin nxdx + \frac{1}{\pi}\int_{-\pi}^0 [-(x+\pi)]\sin nxdx$$
$$= \frac{1}{\pi}\int_0^\pi x\sin nxdx + \frac{1}{\pi}\int_{-\pi}^0 -x\sin nxdx + \frac{1}{\pi}\int_{-\pi}^0 -\pi\sin nxdx$$
$$= \frac{1}{\pi}\int_{-\pi}^\pi |x|\sin nxdx + (-1)\int_{-\pi}^0 \sin nxdx$$
$$= \frac{1}{n}\cos nx|_{-\pi}^0 = \frac{1}{n}(1-\cos n\pi)$$

hence, $f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$
$$= \sum_{n=1}^{\infty}[\frac{2}{n^2\pi}(\cos n\pi - 1)\cos nx + \frac{1}{n}(1-\cos n\pi)\sin nx]$$

22. $f(x) = x^2 \qquad (0 < x < 2\pi)$
$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{2\pi} x^2dx = \frac{1}{2\pi} \cdot \frac{1}{3}x^3|_0^{2\pi} = \frac{1}{2\pi} \cdot \frac{1}{3} \cdot (2\pi)^3$$
$$= \frac{4\pi^2}{3}$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nxdx = \frac{1}{\pi} \cdot \frac{1}{n}\int_0^{2\pi} x^2d(\sin nx)$$
$$= \frac{1}{n\pi}x^2\sin nx|_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi}\sin nxd(x^2)$$
$$= -\frac{2}{n\pi}\int_0^{2\pi} x\sin nxdx = \frac{2}{n^2\pi}\int_0^{2\pi} xd(\cos nx)$$
$$= \frac{2}{n^2\pi}x\cos nx|_0^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi}\cos nxdx$$
$$= \frac{2}{n^2\pi}(2\pi - 0) - \frac{2}{n^3\pi}\sin nx|_0^{2\pi}$$
$$= \frac{4}{n^2}$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nxdx = \frac{1}{\pi} \cdot \frac{-1}{n}\int_0^{2\pi} x^2d(\cos nx)$$
$$= \frac{-1}{n\pi}x^2\cos nx|_0^{2\pi} + \frac{1}{n\pi}\int_0^{2\pi}\cos nxd(x^2)$$
$$= \frac{-1}{n\pi}(2\pi)^2 + \frac{2}{n\pi}\int_0^{2\pi} x\cos nxdx$$
$$= \frac{-4\pi}{n} + \frac{2}{n^2\pi}\int_0^{2\pi} xd(\sin nx)$$
$$= \frac{-4\pi}{n} + \frac{2}{n^2\pi}x\sin nx|_0^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi}\sin nxdx$$
$$= \frac{-4\pi}{n} + \frac{2}{n^3\pi}\cos nx|_0^{2\pi} = \frac{-4\pi}{n}$$

hence, $f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$
$$= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}(\frac{4}{n^2}\cos nx + \frac{-4\pi}{n}\sin nx)$$
$$= \frac{4\pi^2}{3} + [4(\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\cdots)$$
$$- 4\pi(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\cdots)]$$

24. $f(x) = \{ \begin{matrix} -4x, & \text{if } -\pi < x < 0 \\ 4x, & \text{if } 0 < x < \pi \end{matrix}$
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^\pi 4xdx = \frac{4}{\pi}\int_0^\pi xdx = \frac{4}{\pi} \cdot \frac{1}{2}x^2|_0^\pi = 2\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{2}{\pi}\int_0^\pi 4x\cos nxdx = \frac{8}{\pi} \cdot \frac{1}{n}\int_0^\pi xd(\sin nx)$$

$$= \frac{8}{n\pi} x \sin nx|_0^\pi - \frac{8}{n\pi}\int_0^\pi \sin nx dx$$
$$= \frac{8}{n^2\pi}\cos nx|_0^\pi = \frac{8}{n^2\pi}(\cos n\pi - 1)$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$$

hence, $f(x) \sim a_0 + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx)$
$= 2\pi + \sum_{n=1}^\infty \frac{8}{n^2\pi}(\cos n\pi - 1)\cos nx$

Fourier series of function $f(x)$

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x$$

where fourier coefficients

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$$

Page 490 1,3,7,10,17

1. $f(x) = -1\ (-2 < x < 0),\ f(x) = 1\ (0 < x < 2),\quad p = 4$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{4}\int_{-2}^{2} f(x)dx = 0$ since $f(x)$ is odd function.

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}xdx = 0$

since $f(x)\cos\frac{n\pi}{2}x$ is odd function.

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}xdx$
$= \int_0^2 \sin\frac{n\pi}{2}xdx = \frac{-2}{n\pi}\cos\frac{n\pi}{2}x|_0^2 = \frac{-2}{n\pi}(\cos n\pi - 1)$
$= \frac{2}{n\pi}(1 - \cos n\pi)$

hence, $f(x) \sim a_0 + \sum_{n=1}^\infty a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x$
$= \sum_{n=1}^\infty \frac{2}{n\pi}(1 - \cos n\pi)\sin\frac{n\pi}{2}x$

3. $f(x) = x^2\ (-1 < x < 1),\quad p = 2$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_{-1}^{1} x^2 dx$ since x^2 is even function.
$= \int_0^1 x^2 dx = \frac{1}{3}x^3|_0^1 = \frac{1}{3}$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx$ since $x^2\cos n\pi x$ is even function.
$= \int_{-1}^{1} x^2\cos n\pi x dx = 2\int_0^1 x^2 \cos n\pi x dx$
$= \frac{2}{n\pi}\int_0^1 x^2 d(\sin n\pi x) = \frac{2}{n\pi}x^2\sin n\pi x|_0^1 + \frac{-2}{n\pi}\int_0^1 \sin n\pi x d(x^2)$
$= \frac{-4}{n\pi}\int_0^1 x\sin n\pi x dx = \frac{4}{n^2\pi^2}\int_0^1 x d(\cos n\pi x)$
$= \frac{4}{n^2\pi^2}x\cos n\pi x|_0^1 + \frac{-4}{n^2\pi^2}\int_0^1 \cos n\pi x dx$

$= \frac{4}{n^2\pi^2}(\cos n\pi) + \frac{4}{n^3\pi^3}\sin n\pi x|_0^1$

$= \frac{4}{n^2\pi^2}(\cos n\pi)$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$ since $x^2\sin n\pi x$ is odd function.

$= \int_{-1}^{1} x^2\sin n\pi xdx = 0$

hence, $f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x$

$= \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos n\pi)\cos n\pi x$

7. $f(x) = |x| \quad (-1 < x < 1), \quad p = 2$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_{-1}^{1}|x|dx$ since $|x|$ is even function.

$= \int_{0}^{1} xdx = \frac{1}{2}x^2|_0^1 = \frac{1}{2}$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx$ since $|x|\cos n\pi x$ is even function.

$= \int_{-1}^{1}|x|\cos n\pi xdx = 2\int_{0}^{1} x\cos n\pi xdx$

$= \frac{2}{n\pi}\int_{0}^{1} xd(\sin n\pi x) = \frac{2}{n\pi}x\sin n\pi x|_0^1 + \frac{-2}{n\pi}\int_{0}^{1}\sin n\pi xdx$

$= \frac{2}{n^2\pi^2}\cos n\pi x|_0^1 = \frac{2}{n^2\pi^2}(\cos n\pi - 1)$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx$ since $|x|\sin n\pi x$ is odd function.

$= \int_{-1}^{1}|x|\sin n\pi xdx = 0$

hence, $f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x$

$= \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}(\cos n\pi - 1)\cos n\pi x$

10. $f(x) = 0 \quad (-2 < x < 0), \quad f(x) = x \quad (0 < x < 2), \quad p = 4$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{4}\int_{-2}^{2} f(x)dx = \frac{1}{4}\int_{0}^{2} xdx + \frac{1}{4}\int_{-2}^{0} 0dx$

$= \frac{1}{4}\cdot\frac{1}{2}x^2|_0^2 = \frac{1}{4}\cdot\frac{1}{2}\cdot 2^2 = \frac{1}{2}$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}xdx = \frac{1}{4}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}xdx$

$= \frac{1}{4}\int_{0}^{2} x\cos\frac{n\pi}{2}xdx = \frac{1}{4}\cdot\frac{2}{n\pi}\int_{0}^{2} xd(\sin\frac{n\pi}{2}x)$

$= \frac{1}{2n\pi}x\sin\frac{n\pi}{2}x|_0^2 + \frac{-1}{2n\pi}\int_{0}^{2}\sin\frac{n\pi}{2}xdx$

$= \frac{-1}{2n\pi}\cdot\frac{-2}{n\pi}\cos\frac{n\pi}{2}x|_0^2$

$= \frac{1}{n^2\pi^2}(\cos n\pi - 1)$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}xdx = \frac{1}{4}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}xdx$

$= \frac{1}{4}\int_{0}^{2} x\sin\frac{n\pi}{2}xdx = \frac{1}{4}\cdot\frac{-2}{n\pi}\int_{0}^{2} xd(\cos\frac{n\pi}{2}x)$

$= \frac{-1}{2n\pi}x\cos\frac{n\pi}{2}x|_0^2 + \frac{1}{2n\pi}\int_{0}^{2}\cos\frac{n\pi}{2}xdx$

$= \frac{-1}{2n\pi}(2\cos n\pi) + \frac{1}{2n\pi}\cdot\frac{2}{n\pi}\sin\frac{n\pi}{2}x|_0^2$

$= \frac{-1}{n\pi}(\cos n\pi)$

hence, $f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x$

$= \frac{1}{2} + \sum_{n=1}^{\infty} [\frac{1}{n^2\pi^2}(\cos n\pi - 1)\cos \frac{n\pi}{2} x + \frac{-1}{n\pi}(\cos n\pi)\sin \frac{n\pi}{2} x]$

17. Using Prob.3. show that

$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots\cdots = \frac{1}{12}\pi^2$

Proof:by Prob.3.

$x^2 \sim \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(\cos n\pi)\cos n\pi x$

setting $x = 0$, we have

$0^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$0 = \frac{1}{3} + \frac{4}{\pi^2}(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\cdots)$

$\frac{1}{3} = \frac{4}{\pi^2}(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots)$

$\frac{\pi^2}{12} = (1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots)$