Assignment 8 (9-10th-week)
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Find the solutions of following initial value problems.
1. $y'' + y = \delta(t-2\pi),\ y(0) = 10,\ y'(0) = 0$
Solution:
$s^2Y - sy(0) - y'(0) + Y = e^{-2\pi s}$
$(s^2+1)Y - 10s = e^{-2\pi s}$
$Y = \frac{e^{-2\pi s}}{(s^2+1)} + \frac{10s}{(s^2+1)}$

since $L^{-1}\{\frac{10s}{(s^2+1)}\} = 10\cos t$, and
since $L^{-1}\{\frac{1}{(s^2+1)}\} = \sin t$,
then $L^{-1}\{\frac{e^{-2\pi s}}{(s^2+1)}\} = \sin(t-2\pi)\cdot u(t-2\pi) = \sin t \cdot u(t-2\pi)$
Hence, $y = \sin t \cdot u(t-2\pi) + 10\cos t$

3. $y'' - y = 10\delta(t-\frac{1}{2}) - 100\delta(t-1),\ y(0) = 10,\ y'(0) = 1$
Solution:
$s^2Y - sy(0) - y'(0) - Y = 10e^{-\frac{1}{2}s} - 100e^{-s}$
$(s^2-1)Y = 10e^{-\frac{1}{2}s} - 100e^{-s} + 10s + 1$
$Y = \frac{10e^{-\frac{1}{2}s}}{(s+1)(s-1)} - \frac{100e^{-s}}{(s+1)(s-1)} + \frac{10s+1}{(s+1)(s-1)}$

since $\frac{10s+1}{(s+1)(s-1)} = \frac{9}{2}\cdot\frac{1}{s+1} + \frac{11}{2}\cdot\frac{4}{s-1}$, then $L^{-1}\{\frac{10s+1}{(s+1)(s-1)}\} = \frac{9}{2}e^{-t} + \frac{11}{2}e^{t}$
since $\frac{1}{(s+1)(s-1)} = \frac{1}{2}\cdot\frac{1}{(s-1)} - \frac{1}{2}\cdot\frac{1}{(s+1)}$, then $L^{-1}\{\frac{1}{(s+1)(s-1)}\} = \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$
Now, $L^{-1}\{\frac{10e^{-\frac{1}{2}s}}{(s+1)(s-1)}\} = (5e^{t-\frac{1}{2}} - 5e^{-t+\frac{1}{2}})\cdot u(t-\frac{1}{2})$
and, $L^{-1}\{\frac{100e^{-s}}{(s+1)(s-1)}\} = (50e^{t-1} - 50e^{-t+1})\cdot u(t-1)$

Hence, $y = 5e^{t-\frac{1}{2}} - 5e^{-t+\frac{1}{2}})\cdot u(t-\frac{1}{2}) - (50e^{t-1} - 50e^{-t+1})\cdot u(t-1) + \frac{9}{2}e^{-t} + \frac{11}{2}e^{t}$

5. $y'' + 4y' + 5y = [1-u(t-10)]e^{t} - e^{10}\delta(t-10),\ y(0) = 0,\ y'(0) = 1$
Solution:
$s^2Y - sy(0) - y'(0) + 4sY - 4y(0) + 5Y = L\{e^{t}\} - L\{e^{10}u(t-10)e^{t-10}\} - e^{10}\cdot e^{-10s}$
$(s^2+4s+5)Y = \frac{1}{s-1} - \frac{e^{10}\cdot e^{-10s}}{s-1} - e^{10}\cdot e^{-10s} + 1$
$Y = \frac{1}{s-1}\cdot\frac{1}{s^2+4s+5} - \frac{e^{10}\cdot e^{-10s}}{(s-1)}\cdot\frac{1}{s^2+4s+5} - \frac{e^{10}\cdot e^{-10s}}{s^2+4s+5} + \frac{1}{s^2+4s+5}$

since $\frac{1}{s^2+4s+5} = \frac{1}{(s+2)^2+1^2}$, then $L^{-1}\{\frac{1}{s^2+4s+5}\} = e^{-2t}\sin t$
And since $\frac{1}{s-1}\cdot\frac{1}{s^2+4s+5} = \frac{1}{10}\cdot\frac{1}{s-1} + \frac{-\frac{1}{10}s-\frac{1}{2}}{s^2+4s+5}$
$= \frac{1}{10}\cdot\frac{1}{s-1} - \frac{1}{10}\cdot\frac{s+2}{s^2+4s+5} - \frac{3}{10}\frac{1}{s^2+4s+5}$
then $L^{-1}\{\frac{1}{s-1}\cdot\frac{1}{s^2+4s+5}\} = \frac{1}{10}e^{t} - \frac{1}{10}e^{-2t}\cos t - \frac{3}{10}e^{-2t}\sin t$
Now, $L^{-1}\{\frac{e^{10}\cdot e^{-10s}}{s^2+4s+5}\} = e^{10}\cdot e^{-2t+20}\sin(t-10)\cdot u(t-10)$

$= e^{-2t+30}\sin(t-10)\cdot u(t-10)$

and $L^{-1}\{\frac{e^{10}\cdot e^{-10s}}{s-1}\cdot\frac{1}{s^2+4s+5}\}$

$= e^{10}[\frac{1}{10}e^{t-10}-\frac{1}{10}e^{-2t+20}\cos(t-10)-\frac{3}{10}e^{-2t+20}\sin(t-10)]\ \cdot u(t-10)$

$= [\frac{1}{10}e^{t}-\frac{1}{10}e^{-2t+30}\cos(t-10)-\frac{3}{10}e^{-2t+30}\sin(t-10)]\ \cdot u(t-10)$

Hence,

$$y = \frac{1}{10}e^{t}-\frac{1}{10}e^{-2t}\cos t-\frac{3}{10}e^{-2t}\sin t$$
$$-[\frac{1}{10}e^{t}-\frac{1}{10}e^{-2t+30}\cos(t-10)-\frac{3}{10}e^{-2t+30}\sin(t-10)]\ \cdot u(t-10)$$
$$-e^{-2t+30}\sin(t-10)\cdot u(t-10)+e^{-2t}\sin t$$

i.e.

$$y = \frac{1}{10}e^{t}-\frac{1}{10}e^{-2t}\cos t+\frac{7}{10}e^{-2t}\sin t$$
$$-[\frac{1}{10}e^{t}-\frac{1}{10}e^{-2t+30}\cos(t-10)+\frac{7}{10}e^{-2t+30}\sin(t-10)]\ \cdot u(t-10)$$

11. $y''+3y'-4y = 2e^{t}-8e^{2}\delta(t-2),\ y(0) = 2,\ y'(0) = 0$

Solution:

$s^2Y-sy(0)-y'(0)+3sY-3y(0)-4Y = \frac{2}{s-1}-8e^{2}\cdot e^{-2s}$

$(s^2+3s-4)Y = \frac{2}{s-1}-8e^{2}\cdot e^{-2s}+2s+6$

$Y = \frac{2}{(s-1)(s+4)(s-1)}-\frac{8e^{2}\cdot e^{-2s}}{(s+4)(s-1)}+\frac{2s+3}{(s+4)(s-1)}$

since $\frac{2s+6}{(s+4)(s-1)} = \frac{2}{5}\cdot\frac{1}{s+4}+\frac{8}{5}\cdot\frac{1}{s-1}$, then $L^{-1}\{\frac{2s+3}{(s+4)(s-1)}\} = \frac{2}{5}e^{-4t}+\frac{8}{5}e^{t}$

since $\frac{2}{(s-1)(s+4)(s-1)} = \frac{2}{25}\cdot\frac{1}{s+4}+\frac{-\frac{2}{25}s+\frac{12}{25}}{(s-1)^2} = \frac{2}{25}\cdot\frac{1}{s+4}-\frac{2}{25}\cdot\frac{(s-1)}{(s-1)^2}+\frac{2}{5}\cdot\frac{1}{(s-1)^2}$

and so $L^{-1}\{\frac{1}{s^2}\} = t$, imply that $L^{-1}\{\frac{1}{(s-1)^2}\} = te^{t}$

then $L^{-1}\{\frac{2}{(s-1)(s+4)(s-1)}\} = \frac{2}{25}e^{-4t}-\frac{2}{25}e^{t}+\frac{2}{5}te^{t}$

Now, $\frac{1}{(s+4)(s-1)} = \frac{1}{5}(\frac{1}{s-1}-\frac{1}{s+4})$, then $L^{-1}\{\frac{1}{(s+4)(s-1)}\} = \frac{1}{5}(e^{t}-e^{-4t})$

and then $L^{-1}\{\frac{8e^{2}\cdot e^{-2s}}{(s+4)(s-1)}\} = 8e^{2}\cdot\frac{1}{5}(e^{t-2}-e^{-4t+8})\cdot u(t-2)$

$= \frac{8}{5}(e^{t}-e^{-4t+10})\cdot u(t-2)$

Hence, $y = \frac{2}{25}e^{-4t}-\frac{2}{25}e^{t}+\frac{2}{5}te^{t}-\frac{8}{5}(e^{t}-e^{-4t+10})\cdot u(t-2)+\frac{2}{5}e^{-4t}+\frac{8}{5}e^{t}$

i.e. $y = \frac{12}{25}e^{-4t}+\frac{38}{25}e^{t}+\frac{2}{5}te^{t}-\frac{8}{5}(e^{t}-e^{-4t+10})\cdot u(t-2)$

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Find by integration

2. $1*1$

Solution:

$1*1 = \int_0^t 1\cdot 1d\tau = \tau|_0^t = t$

8. $\sin t * \cos t$

Solution:

$$\sin t * \cos t = \int_0^t \sin\tau\cos(t-\tau)d\tau = \frac{1}{2}\int_0^t \sin t + \sin(2\tau - t)d\tau$$
$$= \frac{1}{2}\sin t\int_0^t d\tau + \frac{1}{2}\int_0^t \sin(2\tau - t)d\tau$$
$$= \frac{1}{2}t\sin t + \frac{1}{4}\int_0^t \sin(2\tau - t)d(2\tau - t)$$
$$= \frac{1}{2}t\sin t + \frac{-1}{4}\cos(2\tau - t)|_0^t$$
$$= \frac{1}{2}t\sin t + \frac{-1}{4}(\cos t - \cos t) = \frac{1}{2}t\sin t$$

12,16 Inverse Transforms

12. $\frac{1}{s^2(s-2)}$

Solution:

since $\frac{1}{s^2(s-2)} = \frac{-\frac{1}{4}s-\frac{1}{2}}{s^2} + \frac{1}{4}\cdot\frac{1}{s-2} = \frac{-1}{4}\cdot\frac{1}{s} + \frac{-1}{2}\cdot\frac{1}{s^2} + \frac{1}{4}\cdot\frac{1}{s-2}$

then $L^{-1}\{\frac{1}{s^2(s-2)}\} = \frac{-1}{4} + \frac{-1}{2}t + \frac{1}{4}e^{2t}$

16. $\frac{5}{(s^2+1)(s^2+25)}$

Solution:

since $\frac{5}{(s^2+1)(s^2+25)} = \frac{5}{24}\cdot\frac{1}{(s^2+1)} - \frac{1}{24}\frac{5}{(s^2+25)}$

then $L^{-1}\{\frac{5}{(s^2+1)(s^2+25)}\} = \frac{5}{24}\sin t - \frac{1}{24}\sin 5t$

Using the convolution theorem, solve:

18. $y'' + y = \sin t,\ y(0) = 0,\ y'(0) = 0$

Solution:

$s^2Y - sy(0) - y'(0) + Y = \frac{1}{s^2+1}$

$(s^2+1)Y = \frac{1}{s^2+1}$

$Y = \frac{1}{(s^2+1)^2}$

Hence, $y = L^{-1}\{\frac{1}{s^2+1}\} * L^{-1}\{\frac{1}{s^2+1}\} = \sin t * \sin t$

$$\sin t * \sin t = \int_0^t \sin\tau\sin(t-\tau)d\tau$$
$$= \int_0^t \frac{1}{2}[\cos(2\tau - t) - \cos t]d\tau$$
$$= \frac{1}{2}\int_0^t \cos(2\tau - t)d\tau - \frac{1}{2}\int_0^t \cos t d\tau$$
$$= \frac{1}{2}\sin t - \frac{1}{2}t\cos t$$

i.e. $y = \frac{1}{2}\sin t - \frac{1}{2}t\cos t$

20. $y'' + 4y' + 5y = 2e^{-2t},\ y(0) = 0,\ y'(0) = 0$

Solution:

$s^2Y - sy(0) - y'(0) + 4sY - 4y(0) + 5Y = \frac{2}{s+2}$

$(s^2 + 4s + 5)Y = \frac{2}{s+2}$

$Y = 2\cdot\frac{1}{s+2}\cdot\frac{1}{s^2+4s+5}$

Hence, $y = 2L^{-1}\{\frac{1}{s+2}\} * L^{-1}\{\frac{1}{s^2+4s+5}\} = 2e^{-2t} * L^{-1}\{\frac{1}{(s+2)^2+1}\}$

$$= 2e^{-2t} * e^{-2t}\sin t$$

$2e^{-2t} * e^{-2t}\sin t = \int_0^t e^{-2(t-\tau)}e^{-2\tau}\sin\tau d\tau$
$= \int_0^t e^{-2t}\sin\tau d\tau$
$= e^{-2t}(1-\cos t)$
i.e. $y = e^{-2t}(1-\cos t)$

24. $y'' + 5y' + 6y = \delta(t-3),\ y(0) = 1,\ y'(0) = 0$
Solution:
$s^2Y - sy(0) - y'(0) + 5sY - 5y(0) + 6Y = e^{-3s}$
$(s^2 + 5s + 6)Y = e^{-3s} + s + 5$
$Y = \frac{e^{-3s}+s+5}{s^2+5s+6} = \frac{e^{-3s}}{(s+2)(s+3)} + \frac{s+5}{(s+2)(s+3)}$

since $\frac{s+5}{(s+2)(s+3)} = \frac{3}{s+2} - \frac{2}{(s+3)}$, then
$L^{-1}\{\frac{s+5}{(s+2)(s+3)}\} = 3e^{-2t} - 2e^{-3t}$
since $\frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{(s+3)}$, then
$L^{-1}\{\frac{1}{(s+2)(s+3)}\} = e^{-2t} - e^{-3t}$, and then
$L^{-1}\{\frac{e^{-3s}}{(s+2)(s+3)}\} = [e^{-2t+6} - e^{-3t+9}] \cdot u(t-3)$

hence, $y = 3e^{-2t} - 2e^{-3t} + [e^{-2t+6} - e^{-3t+9}] \cdot u(t-3)$

Integral Equation
27. $y(t) - \int_0^t y(\tau)d\tau = 1$
Solution:
$y(t) - y(t) * 1 = 1$
$Y - Y \cdot L\{1\} = \frac{1}{s}$
$Y - \frac{1}{s}Y = \frac{1}{s}$
$(\frac{s-1}{s})Y = \frac{1}{s}$
$Y = \frac{1}{s-1}$

Hence, $y = e^t$

32. $y(t) - \int_0^t y(\tau)(t-\tau)d\tau = 2 - \frac{t^2}{2}$
Solution:
$y(t) - y(t) * t = 2 - \frac{t^2}{2}$
$Y - Y \cdot \frac{1}{s^2} = \frac{2}{s} - \frac{2}{2s^3}$
$(\frac{s^2-1}{s^2})Y = (\frac{2s^2-1}{s^3})$
$Y = \frac{2s^2-1}{s(s^2-1)} = \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{s+1} + \frac{1}{2} \cdot \frac{1}{s-1}$

Hence, $y = 1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^t$

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Find the Laplace transform

2. $-t\cosh 2t$

Solution:

since $L\{\cosh 2t\} = \frac{s}{s^2-4}$,

then $L\{-t\cosh 2t\} = \frac{d}{ds}(\frac{s}{s^2-4}) = \frac{1}{s^2-4} - \frac{2s^2}{(s^2-4)^2}$

6. $t^2 \sin 3t$

Solution:

since $L\{\sin 3t\} = \frac{3}{s^2+9}$

then $L\{t\sin 3t\} = -\frac{d}{ds}(\frac{3}{s^2+9}) = \frac{6s}{(s^2+9)^2}$

and then $L\{t^2 \sin 3t\} = -\frac{d}{ds}(\frac{6s}{(s^2+9)^2}) = \frac{24s^2}{(s^2+9)^3} - \frac{6}{(s^2+9)^2}$

10. $t\cos \omega t$

Solution:

since $L\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$

then $L\{t\cos \omega t\} = -\frac{d}{ds}(\frac{s}{s^2+\omega^2}) = \frac{2s^2}{(s^2+\omega^2)^2} - \frac{1}{s^2+\omega^2}$

Inverse Laplace transform

14. $\frac{s}{(s^2+16)^2}$

Solution:

Let $F'(s) = \frac{s}{(s^2+16)^2}$, then

$F(s) = \int \frac{s}{(s^2+16)^2} ds = \frac{-1}{2} \cdot \frac{1}{s^2+16} = \frac{-1}{8} \cdot \frac{4}{s^2+16}$

and then $f(t) = L^{-1}\{F(s)\} = \frac{-1}{8} \sin 4t$

Hence, $L^{-1}\{\frac{s}{(s^2+16)^2}\} = L^{-1}\{F'(s)\} = -tf(t) = -t \cdot \frac{-1}{8} \sin 4t = \frac{1}{8} t \sin 4t$

18. $\ln \frac{s+a}{s+b}$

Solution:

Let $F(s) = \ln \frac{s+a}{s+b} = \ln(s+a) - \ln(s+b)$, then$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$,

and then $-tf(t) = L^{-1}\{F'(s)\} = e^{-at} - e^{-bt}$

Hence, $f(t) = \frac{e^{-bt}-e^{-at}}{t}$

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1. $y_1' = -y_1 - y_2,\ y_2' = y_1 - y_2,\ y_1(0) = 0,\ y_2(0) = 1,$

Solution:

$sY_1 - y_1(0) = -Y_1 - Y_2$

$(s+1)Y_1 + Y_2 = 0,$ (1)

$sY_2 - y_2(0) = Y_1 - Y_2$

$-Y_1 + (s+1)Y_2 = 1$ (2)

$(1) - (2) \times (s+1)$

$$((s+1)^2 + 1)Y_2 = s + 1$$
$$Y_2 = \frac{s+1}{(s+1)^2 + 1}$$

hence, $y_2 = e^{-t}\cos t$

$(1) \times [-(s+1)] + (2)$

$$-((s+1)^2 + 1)Y_1 = 1$$
$$Y_1 = \frac{-1}{(s+1)^2 + 1}$$

hence, $y_1 = -e^{-t}\sin t$

9. $y_1' = 5y_1 + 5y_2 - 15\cos t + 27\sin t,\ y_2' = -10y_1 - 5y_2 - 150\sin t,\ y_1(0) = 2,\ y_2(0) = 2,$

Solution:

$sY_1 - y_1(0) = 5Y_1 + 5Y_2 - \frac{15s}{s^2+1} + \frac{27}{s^2+1}$

$(s-5)Y_1 - 5Y_2 = -\frac{15s}{s^2+1} + \frac{27}{s^2+1} + 2$ (1)

$sY_2 - y_2(0) = -10Y_1 - 5Y_2 - \frac{150}{s^2+1}$

$10Y_1 + (s+5)Y_2 = -\frac{150}{s^2+1} + 2$ (2)

$(1) \times 10 - (2) \times (s-5)$

$y_1 = 4\cos 5t + 6\sin 5t - 2\cos t - 25\sin t$

$y_2 = 2\cos 5t - 10\sin 5t + 20\sin t$

15. $y_1' = -3y_1 + y_2 + u(t-1)e^t,\ y_2' = -4y_1 + 2y_2 + u(t-1)e^t,$

$y_1(0) = 0, y_2(0) = 3$

Solution:

$sY_1 - y_1(0) = -3Y_1 + Y_2 + e \cdot L\{u(t-1)e^{t-1}\}$

$(s+3)Y_1 - Y_2 = \frac{e \cdot e^{-s}}{s-1}$ (1)

$sY_2 - y_2(0) = -4Y_1 + 2Y_2 + e \cdot L\{u(t-1)e^{t-1}\}$

$4Y_1 + (s-2)Y_2 = \frac{e \cdot e^{-s}}{s-1} + 3$ (2)

$(1) \times (s-2) + 2$

$$(s^2 + s - 6 + 4)Y_1 = \frac{e \cdot e^{-s}}{s-1} \cdot (s-1) + 3$$
$$Y_1 = \frac{e \cdot e^{-s}}{(s+2)(s-1)} + \frac{3}{(s+2)(s-1)}$$

since $\frac{3}{(s+2)(s-1)} = \frac{1}{s-1} - \frac{1}{s+2}$, then $L^{-1}\{\frac{3}{(s+2)(s-1)}\} = e^t - e^{-2t}$

and then $L^{-1}\{\frac{e \cdot e^{-s}}{(s+2)(s-1)}\} = \frac{e}{3}L^{-1}\{\frac{3e^{-s}}{(s+2)(s-1)}\} = \frac{e}{3}(e^{t-1} - e^{-2t+2})u(t-1)$

$= \frac{1}{3}(e^t - e^{-2t+3})u(t-1)$

hence, $y_1 = e^t - e^{-2t} + \frac{1}{3}(e^t - e^{-2t+3})u(t-1)$

$(1) \times 4 - (3) \times (s+3)$

$$
\begin{aligned}
(-4 - s^2 - s + 6)Y_2 &= \frac{e \bullet e^{-s}}{s-1} \bullet [4 - s - 3] - 3(s+3) \\
(s+2)(s-1)Y_2 &= e \bullet e^{-s} + 3(s+3) \\
Y_2 &= \frac{e \bullet e^{-s}}{(s+2)(s-1)} + \frac{3(s+3)}{(s+2)(s-1)}
\end{aligned}
$$

since $\frac{3(s+3)}{(s+2)(s-1)} = \frac{4}{s-1} - \frac{1}{s+2}$, then $L^{-1}\{\frac{3(s+3)}{(s+2)(s-1)}\} = 4e^t - e^{-2t}$

since $\frac{1}{(s+2)(s-1)} = \frac{1}{3}(\frac{1}{s-1} - \frac{1}{s+2})$, then $L^{-1}\{\frac{1}{(s+2)(s-1)}\} = \frac{1}{3}(e^t - e^{-2t})$

and then $L^{-1}\{\frac{e \bullet e^{-s}}{(s+2)(s-1)}\} = e \bullet \frac{1}{3}(e^{t-1} - e^{-2t+2})u(t-1)$

$= \frac{1}{3}(e^t - e^{-2t+3})u(t-1)$

hence, $y_2 = 4e^t - e^{-2t} + \frac{1}{3}(e^t - e^{-2t+3})u(t-1)$