Assignment 7 (7-8th-week)
Page 232 1,3,5,7,10,14,18,22,27,31
Find the Laplace transform by differentiation
1. te^{kt}
Solution:
let $y = te^{kt}$,
then $y' = e^{kt} + kte^{kt}$, and $y'' = ke^{kt} + ke^{kt} + k^2te^{kt} = 2ke^{kt} + k^2te^{kt}$
and then $y(0) = 0$, $y'(0) = 1$
Now,
$L\{y''\} = L\{2ke^{kt} + k^2te^{kt}\}$
$s^2Y - sy(0) - y'(0) = 2k \cdot L\{e^{kt}\} + k^2 \cdot L\{te^{kt}\}$
$s^2Y - 1 = 2k \cdot \frac{1}{s-k} + k^2Y$
$(s^2 - k^2)Y = \frac{2k}{s-k} - 1$
$(s^2 - k^2)Y = \frac{s+k}{s-k}$
$Y = \frac{1}{(s-k)^2}$
i.e. $L\{te^{kt}\} = \frac{1}{(s-k)^2}$

3. $\sin^2 \omega t$
Solution:
let $y = \sin^2 \omega t$,
then $y' = 2\sin\omega t \cos\omega t \cdot \omega = \omega \sin 2\omega t$, and $y'' = \omega\cos 2\omega t \cdot 2\omega = 2\omega^2 \cos 2\omega t$
and then $y(0) = 0$, $y'(0) = 0$
Now,
$L\{y''\} = 2\omega^2 L\{\cos 2\omega t\}$
$s^2Y - sy(0) - y'(0) = 2\omega^2 \cdot \frac{s}{s^2-4\omega^2}$
$s^2Y = \frac{2\omega^2 s}{s^2-4\omega^2}$
$Y = \frac{2\omega^2}{s(s^2-4\omega^2)}$
i.e. $L\{\sin^2 \omega t\} = \frac{2\omega^2}{s(s^2-4\omega^2)}$

5. $\sinh^2 at$
Solution:
let $y = \sinh^2 at$
then $y' = 2\sinh at \cosh at \cdot a = 2a \sinh at \cosh at$
and $y'' = 2a^2(\sinh^2 at + \cosh^2 at) = 2a^2(1 + 2\sinh^2 at) = 2a^2 + 4a^2 \sinh^2 at$
and then $y(0) = 0$, $y'(0) = 0$
Now,
$L\{y''\} = L\{2a^2 + 4a^2 \sinh^2 at\}$
$s^2Y - sy(0) - y'(0) = \frac{2a^2}{s} + 4a^2 L\{\sinh^2 at\}$
$s^2Y = \frac{2a^2}{s} + 4a^2Y$
$(s^2 - 4a^2)Y = \frac{2a^2}{s}$
$Y = \frac{2a^2}{s(s^2-4a^2)}$
i.e. $L\{\sinh^2 at\} = \frac{2a^2}{s(s^2-4a^2)}$

7. $t\sin\frac{\pi t}{2}$
Solution:
let $y = t\sin\frac{\pi t}{2}$
then $y' = \sin\frac{\pi t}{2} + \frac{\pi}{2}t\cos\frac{\pi t}{2}$,
and $y'' = \frac{\pi}{2}\cos\frac{\pi t}{2} + \frac{\pi}{2}\cos\frac{\pi t}{2} - \frac{\pi^2}{4}t\sin\frac{\pi t}{2} = \pi\cos\frac{\pi t}{2} - \frac{\pi^2}{4}t\sin\frac{\pi t}{2}$
and then, $y(0) = 0$, $y'(0) = 0$
Now,
$L\{y''\} = L\{\pi\cos\frac{\pi t}{2} - \frac{\pi^2}{4}t\sin\frac{\pi t}{2}\}$
$s^2Y - sy(0) - y'(0) = \pi\cdot\frac{s}{s^2+(\frac{\pi}{2})^2} - \frac{\pi^2}{4}Y$
$(s^2 + \frac{\pi^2}{4})Y = \frac{4\pi s}{4s^2+\pi^2}$
$(\frac{4s^2+\pi^2}{4})Y = \frac{4\pi s}{4s^2+\pi^2}$
$Y = \frac{16\pi s}{(4s^2+\pi^2)^2}$
i.e. $L\{t\sin\frac{\pi t}{2}\} = \frac{16\pi s}{(4s^2+\pi^2)^2}$

Intitial value problems
10. $y' + 4y = 0$, $y(0) = 2.8$
Solution:
$sY - y(0) + 4Y = 0$
$(s+4)Y = 2.8$
$Y = \frac{2.8}{s+4}$
hence, $y = 2.8e^{-4t}$

14. $y'' - 4y + 4y = 0$, $y(0) = 2.1$, $y'(0) = 3.9$
Solution:
$s^2Y - sy(0) - y'(0) - 4(sY - y(0)) + 4Y = 0$
$s^2Y - 2.1s - 3.9 - 4(sY - 2.1) + 4Y = 0$
$(s^2 - 4s + 4)Y = 2.1s - 4.5$
$Y = \frac{2.1s-4.5}{(s-2)^2} = \frac{2.1}{(s-2)} - \frac{0.3}{(s-2)^2}$
Since $L\{\frac{1}{s^2}\} = t$, then $L\{\frac{1}{(s-2)}\} = te^{2t}$
hence, $y = 2.1e^{2t} - 0.3te^{2t}$

18. $y'' + 9y = 10e^{-t}$, $y(0) = y'(0) = 0$
Solution:
$L\{y'' + 9y\} = L\{10e^{-t}\}$
$s^2Y - sy(0) - y'(0) + 9Y = \frac{10}{s+1}$
$(s^2+9)Y = \frac{10}{s+1}$
$Y = \frac{10}{(s+1)(s^2+9)} = \frac{1}{s+1} - \frac{s}{s^2+9} + \frac{1}{s^2+9}$

hence, $y = e^{-t} - \cos 3t + \frac{1}{3}\sin 3t$

22. $y'' - 2y' - 3y = 0$, $y(1) = 4$, $y'(1) = 5$
Solution:
let $t = \tilde{t} + 1$, and $\tilde{y}(\tilde{t}) = y(t)$, then we have

$\tilde{y}'' - 2\tilde{y}' - 3\tilde{y} = 0$, $\tilde{y}(0) = 4$, $\tilde{y}'(0) = 5$
Now,
$s^2\tilde{Y} - s\tilde{y}(0) - \tilde{y}'(0) - 2s\tilde{Y} + 2\tilde{y}(0) - 3\tilde{Y} = 0$
$(s^2 - 2s - 3)\tilde{Y} = 4s + 5 - 8$
$\tilde{Y} = \frac{4s-3}{(s-3)(s+1)} = \frac{15}{4} \cdot \frac{1}{s-3} + \frac{1}{4} \cdot \frac{1}{s+1}$
hence, $\tilde{y} = \frac{15}{4}e^{3\tilde{t}} + \frac{1}{4}e^{-\tilde{t}}$
Therefore, $y = \frac{15}{4}e^{3t-3} + \frac{1}{4}e^{-t+1}$

Inverse Laplace transform
27. $\frac{1}{s^2+s/2}$
Solution:
Since $L^{-1}\{\frac{1}{s+\frac{1}{2}}\} = e^{-\frac{1}{2}t}$, then
$L^{-1}\{\frac{1}{s^2+s/2}\} = L^{-1}\{\frac{1}{s} \cdot \frac{1}{s+\frac{1}{2}}\} = \int_0^t e^{-\frac{1}{2}\tau} d\tau = 2 - 2e^{-\frac{1}{2}t}$

31. $\frac{5}{s^3-5s^2}$
Solution:
Since $L^{-1}\{\frac{5}{s-5}\} = 5e^{5t}$, then
$L^{-1}\{\frac{5}{s^2-5s}\} = L^{-1}\{\frac{1}{s} \cdot \frac{5}{s-5}\} = \int_0^t 5e^{5\tau} d\tau = e^{5t} - 1$
and then
$L^{-1}\{\frac{5}{s^3-5s^2}\} = L^{-1}\{\frac{1}{s} \cdot \frac{5}{s^2-5s}\} = \int_0^t (e^{5\tau} - 1) d\tau = \frac{1}{5}e^{5t} - t - \frac{1}{5}$

Page 240 2,6,10,14,16,18,22,24,28,32,34
Find the Laplace transform
2. t $(0 < t < 1)$
Solution:
$$L\{t \cdot (1 - u(t-1))\} = L\{t + t \cdot u(t-1)\}$$
$$= L\{t + (t-1) \cdot u(t-1) + u(t-1)\}$$
$$= \frac{1}{s^2} + \frac{e^{-s}}{s^2} + e^{-s}$$

6. t^2 $(t > 3)$
Solution:
$$L\{t^2 \cdot u(t-3)\} = L\{(t-3)^2 \cdot u(t-3) + (6t-9) \cdot u(t-3)\}$$
$$= L\{(t-3)^2 \cdot u(t-3) + 6(t-3) \cdot u(t-3) + 9 \cdot u(t-3)\}$$
$$= e^{-3s} \cdot \frac{2}{s^3} + 6e^{-3s} \cdot \frac{1}{s^2} + 9e^{-3s}$$

10. $\sin \omega t$ $(0 < t < 2)$
Solution:
Using formula $L\{f(t) \cdot u(t-a)\} = e^{-as}L\{f(t+a)\}$,
$L\{\sin \omega t \cdot (1 - u(t-2))\}$
$= L\{\sin \omega t - \sin \omega t \cdot u(t-2)\}$
$= L\{\sin \omega t\} - L\{\sin \omega t \cdot u(t-2)\}$
$= \frac{\omega}{s^2+\omega^2} - e^{-2s} \cdot L\{\sin \omega(t+2)\}$
$= \frac{\omega}{s^2+\omega^2} - e^{-2s} \cdot L\{\sin \omega t \cos 2\omega + \cos \omega t \sin 2\omega\}$
$= \frac{\omega}{s^2+\omega^2} - e^{-2s} \cdot (\frac{\omega \cdot \cos 2\omega}{s^2+\omega^2} + \frac{s \cdot \sin 2\omega}{s^2+\omega^2})$

$= \frac{\omega - \omega \cdot e^{-2s} \cos 2\omega - s \cdot e^{-2s} \sin 2\omega}{s^2 + \omega^2}$

Inverse transform
14. $\frac{se^{-s}}{s^2+\omega^2}$
Solution:
since $L^{-1}\{\frac{s}{s^2+\omega^2}\} = \cos \omega t$, then
$L^{-1}\{\frac{se^{-s}}{s^2+\omega^2}\} = u(t-1) \cdot \cos \omega(t-1)$

16. $s^{-2} - (s^{-2} + s^{-1})e^{-s}$
Solution:
since $s^{-2} - (s^{-2} + s^{-1})e^{-s} = \frac{1}{s^2} - (\frac{1}{s^2} + \frac{1}{s})e^{-s} = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$,
and we have $L^{-1}\{\frac{1}{s}\} = 1$, and $L^{-1}\{\frac{1}{s^2}\} = t$, then
$L^{-1}\{s^{-2} - (s^{-2} + s^{-1})e^{-s}\} = t - (t-1)u(t-1) - u(t-1) = t - t \cdot u(t-1)$
$= t[1 - u(t-1)]$

18. $e^{-\pi s}/(s^2 + 2s + 2)$
Solution:
since $L^{-1}\{\frac{1}{s^2+2s+2}\} = L\{\frac{1}{(s+1)^2+1}\} = e^{-t} \sin t$, then
$L^{-1}\{e^{-\pi s}/(s^2 + 2s + 2)\} = e^{-t+\pi} \sin(t + \pi) = -e^{-t+\pi} \sin t$

22. $2.5(e^{-3.8s} - e^{-2.6s})/s$
Solution:
since $2.5(e^{-3.8s} - e^{-2.6s})/s = 2.5 \cdot \frac{e^{-3.8s}}{s} - 2.5 \cdot \frac{e^{-2.6s}}{s}$, then
$L^{-1}\{2.5(e^{-3.8s} - e^{-2.6s})/s\} = 2.5u(t-3.8) - 2.5u(t-2.6)$
$= 2.5[u(t-3.8) - u(t-2.6)]$

Intitial value problems
24. $9y'' - 6y' + y = 0$, $y(0) = 3$, $y'(0) = 1$,
Solution:

$9[s^2Y - sy(0) - y'(0)] - 6[sY - y(0)] + Y = 0$
$9s^2Y - 27s - 9 - 6(sY - 3) + Y = 0$
$(9s^2 - 6s + 1)Y = 27s - 9$
$Y = \frac{27s-9}{(9s^2-6s+1)} = \frac{9(3s-1)}{(3s-1)^2} = \frac{9}{(3s-1)}$
$Y = \frac{3}{s-\frac{1}{3}}$
hence , $y = 3e^{\frac{1}{3}t}$

28. $y'' + 3y' + 2y = r(t)$, $r(t) = 1$ if $0 < t < 1$ and 0 if $t > 1$;
$y(0) = 0$, $y'(0) = 0$
Solution:
$y'' + 3y' + 2y = 1 - u(t-1)$
$s^2Y - sy(0) - y'(0) + 3[sY - y(0)] + 2Y = \frac{1}{s} - \frac{e^{-s}}{s}$
$(s^2 + 3s + 2)Y = \frac{1}{s} - \frac{e^{-s}}{s}$
$Y = \frac{1}{s(s+2)(s+1)} - \frac{e^{-s}}{s(s+2)(s+1)}$

since $\frac{1}{s(s+2)(s+1)} = \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{(s+1)} + \frac{1}{2} \cdot \frac{1}{(s+2)}$, then
$L^{-1}\{\frac{1}{s(s+2)(s+1)}\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$, and then
$L^{-1}\{\frac{e^{-s}}{s(s+2)(s+1)}\} = [\frac{1}{2} - e^{-t+1} + \frac{1}{2}e^{-2t+2}] \cdot u(t-1)$

hence, $y = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - [\frac{1}{2} - e^{-t+1} + \frac{1}{2}e^{-2t+2}] \cdot u(t-1)$

32. $y'' + 8y' + 15y = r(t) \quad r(t) = 35e^{2t}$ if $0 < t < 2$ and 0 if $t > 2$;
$y(0) = 3$, $y'(0) = -8$
Solution:
$y'' + 8y' + 15y = 35e^{2t} \cdot [1 - u(t-2)]$
$y'' + 8y' + 15y = 35e^{2t} - 35e^{2t} \cdot u(t-2)$
$y'' + 8y' + 15y = 35e^{2t} - 35e^{4}e^{2(t-2)} \cdot u(t-2)$
$s^2Y - sy(0) - y'(0) + 8[sY - y(0)] + 15Y = \frac{35}{s-2} - \frac{35e^4e^{-2s}}{s-2}$
$s^2Y - 3s + 8 + 8[sY - 3] + 15Y = \frac{35}{s-2} - \frac{35e^4e^{-2s}}{s-2}$
$(s^2 + 8s + 15)Y = \frac{35}{s-2} - \frac{35e^4e^{-2s}}{s-2} + 3s + 16$
$Y = \frac{35}{(s-2)(s+3)(s+5)} - \frac{35e^4e^{-2s}}{(s-2)(s+3)(s+5)} + \frac{3s+16}{(s+3)(s+5)}$

since $\frac{3s+16}{(s+3)(s+5)} = \frac{7}{2} \cdot \frac{1}{(s+3)} - \frac{1}{2} \cdot \frac{1}{(s+5)}$, then
$L^{-1}\{\frac{3s+16}{(s+3)(s+5)}\} = \frac{7}{2}e^{-3t} - \frac{1}{2}e^{-5t}$

since $\frac{35}{(s-2)(s+3)(s+5)} = \frac{1}{(s-2)} - \frac{7}{2} \cdot \frac{1}{(s+3)} + \frac{5}{2} \cdot \frac{1}{(s+5)}$, then
$L^{-1}\{\frac{35}{(s-2)(s+3)(s+5)}\} = e^{2t} - \frac{7}{2}e^{-3t} + \frac{5}{2}e^{-5t}$
Now,
$$\begin{aligned} L^{-1}\{\frac{35e^4e^{-2s}}{(s-2)(s+3)(s+5)}\} &= e^4 \cdot L^{-1}\{\frac{35e^{-2s}}{(s-2)(s+3)(s+5)}\} \\ &= e^4(e^{2t-4} - \frac{7}{2}e^{-3t+6} + \frac{5}{2}e^{-5t+10}) \cdot u(t-2) \\ &= (e^{2t} - \frac{7}{2}e^{-3t+10} + \frac{5}{2}e^{-5t+14}) \cdot u(t-2) \end{aligned}$$

hence, $y = \frac{7}{2}e^{-3t} - \frac{1}{2}e^{-5t} + e^{2t} - \frac{7}{2}e^{-3t} + \frac{5}{2}e^{-5t} - (e^{2t} - \frac{7}{2}e^{-3t+10} + \frac{5}{2}e^{-5t+14}) \cdot u(t-2)$
i.e. $y = e^{2t} + 2e^{-5t} - (e^{2t} - \frac{7}{2}e^{-3t+10} + \frac{5}{2}e^{-5t+14}) \cdot u(t-2)$

34. $y'' + 2y' + 5y = 10\sin t$ if $0 < t < 2\pi$ and 0 if $t > 2\pi$; $y(\pi) = 1, y'(\pi) = 2e^{-\pi} - 2$
Solution:
$y'' + 2y' + 5y = 10\sin t[1 - u(t-\pi)]$
let $t = \tilde{t} + \pi$, and $\tilde{y}(\tilde{t}) = y(t)$, then we have $\tilde{y}(0) = 1, \tilde{y}'(0) = 2e^{-\pi} - 2$
$\tilde{y}'' + 2\tilde{y}' + 5\tilde{y} = 10\sin(\tilde{t} + \pi)[1 - u(\tilde{t})]$
$\tilde{y}'' + 2\tilde{y}' + 5\tilde{y} = 10\sin(\tilde{t} + \pi)[1 - 1]$
$\tilde{y}'' + 2\tilde{y}' + 5\tilde{y} = 0$
$s^2\tilde{Y} - s\tilde{y}(0) - \tilde{y}'(0) + 2[s\tilde{Y} - \tilde{y}(0)] + 5\tilde{Y} = 0$
$s^2\tilde{Y} - s - 2e^{-\pi} + 2 + 2[s\tilde{Y} - 1] + 5\tilde{Y} = 0$
$(s^2 + 2s + 5)\tilde{Y} = s + 2e^{-\pi}$

$\tilde{Y} = \frac{s}{(s^2+2s+5)} + \frac{2e^{-\pi}}{(s^2+2s+5)}$

$L^{-1}\{\frac{s}{(s^2+2s+5)}\} = L^{-1}\{\frac{s}{(s+1)^2+2^2}\} = e^{-\tilde{t}}\cos 2t$
since, $L^{-1}\{\frac{2}{(s^2+2s+5)}\} = L^{-1}\{\frac{2}{(s+1)^2+2^2}\} = e^{-\tilde{t}}\sin 2\tilde{t}$, then
$L^{-1}\{\frac{2e^{-\pi}}{(s^2+2s+5)}\} = [e^{-\tilde{t}+\pi}\sin 2(\tilde{t}-\pi)]\cdot u(\tilde{t}-\pi)$
$= [e^{-\tilde{t}+\pi}\sin 2\tilde{t}]\cdot u(\tilde{t}-\pi)$

hence, $\tilde{y} = e^{-\tilde{t}}\cos 2t + [e^{-\tilde{t}+\pi}\sin 2\tilde{t}]\cdot u(\tilde{t}-\pi)$,
i.e. $y = e^{-t+\pi}\cos 2(t-\pi) + [e^{-t+2\pi}\sin 2(t-\pi)]\cdot u(t-2\pi)$.