Assignment 6 (6th-week)
Page 122 1,3,5,9 Page 226. 1,5,9,13,17,29,33,37,41,45,47,51
Page 122.
Solve the following ODEs.
1. $y''' - 2y'' - 4y' + 8y = e^{-3x} + 8x^2$
Solution: *Step 1. General solution of the homoegeneous ODE.*
The characteristic equation
$\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$
$\lambda^2(\lambda - 2) - 4(\lambda - 2) = 0$
$(\lambda - 2)(\lambda^2 - 4) = 0$
$(\lambda - 2)(\lambda - 2)(\lambda + 2) = 0$
$\lambda = 2, 2, -2.$
Thus $y_h = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2} x$
Step 2. *Solution of the nonhomogeneous ODE.*
Suppose that $y_p = y_{p_1} + y_{p_2}$
Let $y_{p_1} = Ae^{-3x}$, then
$y'_{p_1} = -3Ae^{-3x}$
$y''_{p_1} = 9Ae^{-3x}$
$y'''_{p_1} = -27Ae^{-3x}$
Substitution into the given ODE
$y'''_{p_1} - 2y''_{p_1} - 4y'_{p_1} + 8y_{p_1} = -27Ae^{-3x} - 2(9Ae^{-3x}) - 4(-3Ae^{-3x}) + 8Ae^{-3x} = -25Ae^{-3x}$
imply that $-25A = 1$, $A = \frac{-1}{25}$.
Let $y_{p_2} = Bx^2 + Dx + E$, then
$y'_{p_2} = 2Bx + D$
$y''_{p_2} = 2B$
$y'''_{p_2} = 0$
Substiution into the given ODE
$y'''_{p_2} - 2y''_{p_2} - 4y'_{p_2} + 8y_{p_2} = -2(2B) - 4(2Bx + D) + 8(Bx^2 + Dx + E) =$
$8E - 4D - 4B - 8Bx + 8Dx + 8Bx^2$
imply that

$$\begin{cases} 8B = 8 \\ 8D - 8B = 0 \\ 8E - 4D - 4B = 0 \end{cases} \Rightarrow \begin{cases} B = 1 \\ D = 1 \\ E = 1 \end{cases}$$

i.e. $y_p = \frac{-1}{25} e^{-3x} + x^2 + x + 1$
Therefore the general solution is $y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2} x + \frac{-1}{25} e^{-3x} + x^2 + x + 1.$

3. $y^{iv} + 0.5y'' + 0.0625y = e^{-x} \cos 0.5x$
Solution: *Step 1. General solution of the homoegeneous ODE.*
The characteristic equation
$\lambda^4 + 0.5\lambda^2 + 0.0625 = 0$
$(\lambda^2 + 0.25)^2 = 0$
$(\lambda + 0.5i)^2(\lambda - 0.5i)^2 = 0$
$\lambda = \pm 0.5i, \pm 0.5i$
Thus $y_h = c_1 \cos \frac{1}{2}x + c_2 \sin \frac{1}{2}x + c_3 x \cos \frac{1}{2}x + c_4 x \sin \frac{1}{2}x.$
Step 2. *Solution of the nonhomogeneous ODE.*
Let $y_p = e^{-x}(A \cos \frac{1}{2}x + B \sin \frac{1}{2}x)$, then

$y_p' = (\frac{1}{2}B - A)e^{-x}\cos\frac{1}{2}x - (\frac{1}{2}A + B)e^{-x}\sin\frac{1}{2}x$
$y_p'' = (\frac{3}{4}A - B)e^{-x}\cos\frac{1}{2}x + (A + \frac{3}{4}B)e^{-x}\sin\frac{1}{2}x$
$y_p''' = (\frac{11}{8}B - \frac{1}{4}A)e^{-x}\cos\frac{1}{2}x - (\frac{11}{8}A + \frac{1}{4}B)e^{-x}\sin\frac{1}{2}x$
$y_p^{iv} = (-\frac{3}{2}B - \frac{7}{16}A)e^{-x}\cos\frac{1}{2}x + (\frac{3}{2}A - \frac{7}{16}B)e^{-x}\sin\frac{1}{2}x$

Substiution into the given ODE

$y_p^{iv} + 0.5y_p'' + 0.0625y_p$
$= (-\frac{3}{2}B - \frac{7}{16}A)e^{-x}$
$\cos\frac{1}{2}x + (\frac{3}{2}A - \frac{7}{16}B)e^{-x}\sin\frac{1}{2}x + 0.5[(\frac{3}{4}A - B)e^{-x}\cos\frac{1}{2}x + (A + \frac{3}{4}B)e^{-x}\sin\frac{1}{2}x]$
$+0.0625e^{-x}(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x)$
$= [-\frac{3}{2}B - \frac{7}{16}A + \frac{1}{2}(\frac{3}{4}A - B) + \frac{1}{16}A]e^{-x}$
$\cos\frac{1}{2}x + [\frac{3}{2}A - \frac{7}{16}B + \frac{1}{2}(A + \frac{3}{4}B) + \frac{1}{16}B]e^{-x}\sin\frac{1}{2}x$
$= -2Be^{-x}\cos\frac{1}{2}x + 2Ae^{-x}\sin\frac{1}{2}x$

imply that

$$\begin{cases} -2B = 1 \\ 2A = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = \frac{-1}{2} \end{cases}$$

i.e. $y_p = \frac{-1}{2}e^{-x}\sin\frac{1}{2}x$

Therefore the general solution is

$y =_1 \cos\frac{1}{2}x + c_2\sin\frac{1}{2}x + c_3x\cos\frac{1}{2}x + c_4x\sin\frac{1}{2}x + \frac{-1}{2}e^{-x}\sin\frac{1}{2}x.$

5. $x^3y''' + 0.75xy' - 0.75y = 9x^{5.5}$

Solution: *Step 1. General solution of the Eular-Cauchy Equation*

Let $y = x^m$, then $y' = mx^{m-1}, y'' = m(m-1)x^{m-2}, y''' = m(m-1)(m-2)x^{m-3}$,

Substiution into the ODE $x^3y''' + 0.75xy' - 0.75y = 0$

$m(m-1)(m-2)x^m + \frac{3}{4}mx^m + \frac{3}{4}x^m = 0$
$m(m-1)(m-2) + \frac{3}{4}m + \frac{3}{4} = 0$
$4m(m-1)(m-2) + 3m - 3 = 0$
$4m^3 - 12m^2 + 11m - 3 = 0$
$4m^3 - 4m^2 - 8m^2 + 8m + 3m - 3 = 0$
$4m^2(m-1) - 8m(m-1) + 3(m-1) = 0$
$(m-1)(4m^2 - 8m + 3) = 0$
$(m-1)(2m-1)(2m-3) = 0$
$m = 1, \frac{1}{2}, \frac{3}{2},$

so that $\left\{x, x^{\frac{1}{2}}, x^{\frac{3}{2}}\right\}$ is a basis, and $y_h = c_1x + c_2x^{\frac{1}{2}} + c_3x^{\frac{3}{2}}$

Step 2. *Solution of the given ODE.*

Let $y_p = Ax^{5.5}$, then

$y_p' = \frac{11}{2}Ax^{4.5}$, $y_p'' = \frac{11\cdot 9}{4}Ax^{3.5}$, $y_p''' = \frac{11\cdot 9\cdot 7}{8}Ax^{2.5}$,

Substiution into the given ODE

$x^3y_p''' + 0.75xy_p' - 0.75y_p = x^3 \cdot \frac{11\cdot 9\cdot 7}{8}Ax^{2.5} + 0.75x \cdot \frac{11}{2}Ax^{4.5} - 0.75Ax^{5.5} = 90Ax^{5.5}$

imply that $90A = 9$, $A = \frac{1}{10}$

i.e. $y_p = \frac{1}{10}x^{5.5}$

Therefore the general solution is $y = c_1x + c_2x^{\frac{1}{2}} + c_3x^{\frac{3}{2}} + \frac{1}{10}x^{5.5}$.

9. $y''' - 9y'' + 27y' - 27y = 54\sin 3x$, $y(0) = 3.5$, $y'(0) = 13.5$, $y''(0) = 38.5$

Solution: *Step 1. General solution of the homoegeneous ODE.*
The characteristic equation
$\lambda^3 - 9\lambda^2 + 27\lambda - 27 = 0$
$(\lambda - 3)^3 = 0$
$\lambda = 3, 3, 3$
Thus, $y_h = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x}$
Step 2. *Solution of the nonhomogeneous ODE.*
Let $y_p = A\sin 3x + B\cos 3x$, then
$y_p' = -3B\sin 3x + 3A\cos 3x,$
$y_p'' = -9A\sin 3x - 9B\cos 3x,$
$y_p''' = 27B\sin 3x - 27A\cos 3x,$
Substitution into the given ODE
$y_p''' - 9y_p'' + 27y_p' - 27y_p$
$= 27B\sin 3x - 27A\cos 3x - 9(-9A\sin 3x - 9B\cos 3x) + 27(-3B\sin 3x + 3A\cos 3x) - 27(A\sin 3x$
$= (54A + 54B)\cos 3x + (54A - 54B)\sin 3x$
imply that

$$\begin{cases} 54A + 54B = 0 \\ 54A - 54B = 54 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{2} \\ B = \frac{-1}{2} \end{cases}$$

i.e. $y_p = \frac{1}{2}\sin 3x + \frac{-1}{2}\cos 3x$
Therefore the general solution is $y = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x} + \frac{1}{2}\sin 3x + \frac{-1}{2}\cos 3x.$
Step 3. Solution of the initial value problem.
we have $y' = 3c_1 e^{3x} + c_2 e^{3x} + 3c_2 x e^{3x} + 2c_3 x e^{3x} + 3c_3 x^2 e^{3x} + \frac{3}{2}\cos 3x + \frac{3}{2}\sin 3x$
And $y'' = 9c_1 e^{3x} + 6c_2 e^{3x} + 9c_2 x e^{3x} + 2c_3 e^{3x} + 12c_3 x e^{3x} + 9c_3 x^2 e^{3x} + \frac{-9}{2}\sin x + \frac{9}{2}\cos 3x$

$$\begin{cases} y(0) = 3.5 \\ y'(0) = 13.5 \\ y''(0) = 38.5 \end{cases} \Rightarrow \begin{cases} \frac{7}{2} = c_1 + \frac{-1}{2} \\ \frac{27}{2} = 3c_1 + c_2 + \frac{3}{2} \\ \frac{77}{2} = 9c_1 + 6c_2 + 2c_3 + \frac{9}{2} \end{cases} \Rightarrow \begin{cases} c_1 = 4 \\ c_2 = 0 \\ c_3 = -1 \end{cases}$$

Thus we obtain answer is $y = 4e^{3x} - x^2 e^{3x} + \frac{1}{2}\sin 3x + \frac{-1}{2}\cos 3x.$

Page 226.
Find the Laplace transforms of the following functions.
1. $t^2 - 2t,$
Solution: method 1
$L\{t^2 - 2t\} = L\{t^2\} - 2L\{t\} = \frac{2!}{s^3} - \frac{2\cdot 1!}{s^2} = \frac{2}{s^3} - \frac{2}{s^2}$

5. $e^{2t}\cosh t$
Solution: Since $e^{2t}\cosh t = \frac{e^{2t}}{2}(e^t + e^{-t}) = \frac{1}{2}(e^{3t} + e^t)$, then
$L\{e^{2t}\cosh t\} = \frac{1}{2}L\{e^{3t}\} + \frac{1}{2}L\{e^t\} = \frac{1}{2}\cdot\frac{1}{s-3} + \frac{1}{2}\cdot\frac{1}{s-1}$

or let $f(t) = \cosh t$, and $F(s) = L\{f(t)\} = \frac{s}{s^2-1}.$
Then $L\{e^{2t}\cosh t\} = L\{e^{2t}f(t)\} = F(s-2) = \frac{s-2}{(s-2)^2-1}$

9. e^{3a-2bt}

Solution:
$L\{e^{3a-2bt}\} = e^{3a} \bullet L\{e^{-2bt}\} = e^{3a} \bullet \frac{1}{s+2b} = \frac{e^{3a}}{s+2b}.$

13.
Solution: we know that

$$f(t) = \begin{cases} k, & 0 < x < b \\ 0, & \text{otherwise} \end{cases}$$

then the Laplace transforms of the function $f(t)$

$$L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = \int_0^b ke^{-st}dt = \frac{-k}{s}e^{-st}|_0^b = \frac{-k(e^{-sb}-1)}{s} = \frac{k(1-e^{-sb})}{s}$$

17.
Solution: we know that

$$f(t) = \begin{cases} t, & 0 < x < b \\ 0, & \text{otherwise} \end{cases}$$

then the Laplace trnsforms of the function $f(t)$

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st}f(t)dt = \int_0^b te^{-st}dt = \frac{-1}{s}\int_0^b td(e^{-st}) \\ &= \frac{-1}{s}te^{-st}|_0^b + \frac{1}{s}\int_0^b e^{-st}dt \\ &= \frac{-1}{s}(be^{-sb}) + \frac{-1}{s^2}e^{-st}|_0^b = \\ &= \frac{-be^{-sb}}{s} + \frac{1}{s^2}(1-e^{-bs}) \end{aligned}$$

Find the inverse Laplace transforms of the following functions

29. $\frac{4s-3\pi}{s^2+\pi^2}$

Solution:
$L^{-1}\left\{\frac{4s-3\pi}{s^2+\pi^2}\right\} = 4L^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} - 3L^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\} = 4\cos\pi t - 3\sin\pi t.$

33. $\frac{n\pi L}{L^2s^2+n^2\pi^2}$

Solution:
$L^{-1}\left\{\frac{n\pi L}{L^2s^2+n^2\pi^2}\right\} = L^{-1}\left\{\frac{\frac{n\pi}{L}}{s^2+\left(\frac{n\pi}{L}\right)^2}\right\} = \sin\frac{n\pi}{L}t.$

37. $\frac{1}{(s-\sqrt{3})(s+\sqrt{5})}$

Solution:
Since $\frac{1}{(s-\sqrt{3})(s+\sqrt{5})} = \frac{1}{\sqrt{5}+\sqrt{3}}\left[\frac{1}{s-\sqrt{3}} - \frac{1}{s+\sqrt{5}}\right]$, then

$$L^{-1}\left\{\frac{1}{(s-\sqrt{3})(s+\sqrt{5})}\right\} = \frac{1}{\sqrt{5}+\sqrt{3}}\left[L^{-1}\left\{\frac{1}{s-\sqrt{3}}\right\} - L^{-1}\left\{\frac{1}{s+\sqrt{5}}\right\}\right] = \frac{e^{\sqrt{3}t} - e^{-\sqrt{5}t}}{\sqrt{5}+\sqrt{3}}$$

Applications of the s-shifting theorem.
Find the Laplace transforms.
41. $3.8te^{2.4t}$
Solution: Let $f(t) = 3.8t$, then $F(s) = L\{f(t)\} = \frac{3.8}{s^2}$,
Now

$$L\{3.8te^{2.4t}\} = L\{e^{2.4t}f(t)\} = F(s-2.4) = \frac{3.8}{(s-2.4)^2}$$

45. $e^{-kt}(a\cos t + b\sin t)$
Solution: Let $f(t) = (a\cos t + b\sin t)$, then $F(s) = L\{f(t)\} = \frac{as}{s^2+1} + \frac{b}{s^2+1}$,
Now

$$L\{e^{-kt}(a\cos t + b\sin t)\} = L\{e^{-kt}f(t)\} = F(s+k) = \frac{a(s+k)}{(s+k)^2+1} + \frac{b}{(s+k)^2+1}$$

Find the inverse Laplace transforms.
47. $\frac{7}{(s-1)^3}$
Solution: Let $F(s) = \frac{7}{s}$, and we have $f(t) = L^{-1}\left\{\frac{7}{s^3}\right\} = \frac{7}{2}L^{-1}\left\{\frac{2!}{s^3}\right\} = \frac{7}{2}t$, then

$$L^{-1}\left\{\frac{7}{(s-1)^3}\right\} = L^{-1}\{F(s-1)\} = e^t f(t) = \frac{7}{2}te^t$$

51. $\frac{15}{s^2+4s+29}$
Solution: Rewrite the from
$\frac{15}{s^2+4s+29} = \frac{15}{(s+2)^2+5^2} = \frac{3\cdot 5}{(s+2)^2+5^2}$
Let $F(s) = \frac{15}{s^2+5^2}$, and we have $f(t) = L^{-1}\left\{\frac{15}{s^2+5^2}\right\} = 3L^{-1}\left\{\frac{5}{s^2+5^2}\right\} = 3\sin 5t$, then

$$L^{-1}\left\{\frac{15}{s^2+4s+29}\right\} = L^{-1}\{F(s+2)\} = e^{-2t}f(t) = 3e^{-2t}\sin 5t$$