Page 101 1,3,5,7. Page 111. 2,4,8,13,18. Page 115. 4,7,9,11.

Page 101

Solve the given nonhomogeneous ODE by variation of parameters or undermined coefficients. Give a general solution. (Show the details of your work.)

1. $y'' + y = \csc x$

Soluiton *Step 1. General solution of the homogeneous ODE.*

The characteristic equation

$\lambda^2 + 1 = 0$

$\lambda = \pm i$

Thus $y_h = c_1 \cos x + +c_2 \sin x$,

hence we have the basis of solutions $y_1 = \cos x,\ y_2 = \sin x$

Using the following formula to obtain the particular solution

$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$

we also need the Wronskian deteminat

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

and

$\int \frac{y_2 r}{W} dx = \int \sin x \cdot \csc x dx = \int dx = x,$

$\int \frac{y_1 r}{W} dx = \int \cos x \cdot \csc x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{1}{\sin x} d(\sin x) = \ln|\sin x|,$

i.e. $y_p = -x \cos x + \sin x \cdot \ln|\sin x|$.

Therefore the general solution is $y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \cdot \ln|\sin x|$.

3. $x^2 y'' - 2xy' + 2y = x^3 \cos x$

Solution: Rewrite the equation

$y'' - \frac{1}{x} y' + \frac{2}{x^2} y = x \cos x$

It is Cauchy-Euler equation, the auxiliary equation is

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0$

Thus $y_h = c_1 x + c_2 x^2$, and the basis of solutions $y_1 = x,\ y_2 = x^2$.

Using the following formula to obtain the particular solution

$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$

we also need the Wronskian deteminat

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$$

and

$\int \frac{y_2 r}{W} dx = \int \frac{x^2 \cdot x \cos x}{x^2} dx = \int x \cos x dx = \cos x + x \sin x$

$\int \frac{y_1 r}{W} dx = \int \frac{x \cdot x \cos x}{x^2} dx = \int \cos x dx = \sin x,$

i.e. $y_p = -x(\cos x + x \sin x) + x^2 \sin x$.

Therefore the general solution is $y = c_1 x + c_2 x^2 + x(\cos x + x \sin x) + x^2 \sin x$.

5. $y'' + y = \tan x$

Solution:The characteristic equation

$\lambda^2 + 1 = 0$

$\lambda = \pm i.$

Thus $y_h = c_1 \cos x + c_2 \sin x$, hence we have the basis of solutions $y_1 = \cos x,\ y_2 = \sin x$

Using the following formula to obtain the particular solution

$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$

we also need the Wronskian deteminat

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

and

$\int \frac{y_2 r}{W} dx = \int \sin x \cdot \tan x dx = \int \frac{\sin^2 x}{\cos x} dx = \int \frac{1-\cos^2 x}{\cos x} dx$

$= \int (\sec x - \cos x) dx = \ln|\sec x + \tan x| - \sin x$

$\int \frac{y_1 r}{W} dx = \int \cos x \cdot \tan x dx = \int \sin x dx = -\cos x,$

$y_p = -\cos x(\ln|\sec x + \tan x| - \sin x) - \sin x \cos x = -\cos x \cdot \ln|\sec x + \tan x|$

Therefore the general solution is $y = c_1 \cos x + c_2 \sin x + -\cos x \cdot \ln|\sec x + \tan x|$.

7. $y'' + y = \cos x + \sec x$

Solution:

Solution:The characteristic equation

$\lambda^2 + 1 = 0$

$\lambda = \pm i.$

Thus $y_h = c_1 \cos x + c_2 \sin x$, hence we have the basis of solutions $y_1 = \cos x,\ y_2 = \sin x$

Using the following formula to obtain the particular solution

$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$

we also need the Wronskian deteminat

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

and

$\int \frac{y_2 r}{W} dx = \int \sin x(\cos x + \sec x) dx = -\int (\cos x + \frac{1}{\cos x}) d(\cos x) = -(\frac{1}{2}\cos^2 x + \ln|\cos x|)$

$\int \frac{y_1 r}{W} = \int \cos x(\cos x + \sec x) dx = \int (\cos^2 x + 1) dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + x = \frac{3}{2}x + \frac{1}{4}\sin 2x$

i.e. $y_p = \cos x(\frac{1}{2}\cos^2 x + \ln|\cos x|) + \sin x(\frac{3}{2}x + \frac{1}{4}\sin 2x)$

$= \frac{1}{2}\cos^3 x + \cos x \cdot \ln|\cos x| + \frac{3}{2}x \sin x + \frac{1}{4}\sin x \sin 2x$

where $\frac{1}{2}\cos^3 x + \frac{1}{4}\sin x \sin 2x = \frac{1}{2}\cos^3 x + \frac{1}{2}\sin^2 \cos x = \frac{1}{2}(\cos^2 x + \sin^2 x)\cos x = \frac{1}{2}\cos x$

Therefore the general solution is $y = c_1 \cos x + c_2 \sin x + \cos x \cdot \ln|\cos x| + \frac{3}{2}x \sin x.$

P111. To get a feel for higher order ODEs, show that the given functions are solutions and form a basis on any interval. Use Wronskian. (In Prob. 2, $x > 0$)

2. $1, x^2, x^4, \quad x^2 y''' - 3xy'' + 3y' = 0$

Solution: Step 1. Show that $\{1, x^2, x^4\}$ are solution of the function $x^2 y''' - 2xy'' + 3y' = 0$

Since $x^2(1)''' - 3x(1)'' + 3(1)' = 0$, and

$x^2(x^2)''' - 3x(x^2)'' + 3(x^2)' = 0 - 3x \cdot 2 + 3 \cdot 2x = 0$, and

$x^2(x^4)''' - 3x(x^4)'' + 3(x^4)' = x^2 \cdot 24x - 3x \cdot 12x^2 + 3 \cdot 4x^3 = 0$

$\{1, x^2, x^4\}$ are solution of the given function

Step 2. Use Wronskian determinat

$$W = \begin{vmatrix} 1 & x^2 & x^4 \\ 0 & 2x & 4x^3 \\ 0 & 2 & 12x^2 \end{vmatrix} = 16x^3 \neq 0, \text{ for } x > 0$$

Thus $\{1, x^2, x^4\}$ is linear independent. $\{1, x^2, x^4\}$ is a basis

4. $e^{2x}\cos x, e^{2x}\sin x, e^{-2x}\cos x, e^{-2x}\sin x,$ $\quad y^{\text{iv}} - 6y'' + 25y = 0.$

Solution: Step 1. Show that $\{e^{2x}\cos x, e^{2x}\sin x, e^{-2x}\cos x, e^{-2x}\sin x\}$ are solution of the function $y^{\text{iv}} - 6y'' + 25y = 0$

Since

$(e^{2x}\cos x)^{\text{iv}} - 6(e^{2x}\cos x)'' + 25(e^{2x}\cos x)$

$= [-7e^{2x}\cos x - 24e^{2x}\sin x] - 6[3e^{2x}\cos x - 4e^{2x}\sin x] + 25e^{2x}\cos x = 0$

and

$(e^{2x}\sin x)^{\text{iv}} - 6(e^{2x}\sin x)'' + 25(e^{2x}\sin x)$

$= [24e^{2x}\cos x - 7e^{2x}\sin x] - 6[4e^{2x}\cos x + 3e^{2x}\sin x] + 25e^{2x}\sin x = 0$

and

$(e^{-2x}\cos x)^{\text{iv}} - 6(e^{-2x}\cos x)'' + 25(e^{-2x}\cos x)$

$= [24e^{-2x}\sin x - 7e^{-2x}\cos x] - 6[3e^{-2x}\cos x + 4e^{-2x}\sin x] + 25e^{-2x}\cos x = 0$

and

$(e^{-2x}\sin x)^{\text{iv}} - 6(e^{-2x}\sin x)'' + 25(e^{-2x}\sin x)$

$= [-24e^{-2x}\cos x - 7e^{-2x}\sin x] - 6[3e^{-2x}\sin x - 4e^{-2x}\cos x] + 25e^{-2x}\sin x = 0$

So that $\{e^{2x}\cos x, e^{2x}\sin x, e^{-2x}\cos x, e^{-2x}\sin x\}$ are solution of the given function.

Step 2. Use Wronskian determinat

$$W = \begin{vmatrix} e^{2x}\sin x & e^{2x}\cos x & e^{-2x}\sin x & e^{-2x}\cos x \\ e^{2x}\cos x + 2e^{2x}\sin x & 2e^{2x}\cos x - e^{2x}\sin x & e^{-2x}\cos x - 2e^{-2x}\sin x & -2e^{-2x}\cos x - e^{-2x}\sin x \\ 4e^{2x}\cos x + 3e^{2x}\sin x & 3e^{2x}\cos x - 4e^{2x}\sin x & 3e^{-2x}\sin x - 4e^{-2x}\cos x & 3e^{-2x}\cos x + 4e^{-2x}\sin x \\ 11e^{2x}\cos x + 2e^{2x}\sin x & 2e^{2x}\cos x - 11e^{2x}\sin x & 11e^{-2x}\cos x - 2e^{-2x}\sin x & -2e^{-2x}\cos x - 11e^{-2x}\sin x \end{vmatrix}$$

$$= 320(\cos^4 x) + 320(\sin^4 x) + 640(\cos^2 x\sin^2 x) = 320(\sin^2 x + \cos^2 x)^2 = 320 \neq 0$$

Thus $\{e^{2x}\cos x, e^{2x}\sin x, e^{-2x}\cos x, e^{-2x}\sin x\}$ is linear independent. it is a basis.

Are the given functions linearly independent of dependent on the positive x –axis? (Give a reason.)

8. $x + 1, x + 2, x$

Solution: Use Wronskian determinat

$$W = \begin{vmatrix} x+1 & x+2 & x \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Thus $\{x + 1, x + 2, x\}$ is linearly dependent.

13. $\sin 2x, \sin x, \cos x$

Solution:Use Wronskian determinat

$$W = \begin{vmatrix} \sin 2x & \sin x & \cos x \\ \cos 2x & \cos x & -\sin x \\ -\sin 2x & -\sin x & -\cos x \end{vmatrix} = 0$$

Thus $\{\sin 2x, \sin x, \cos x\}$ is linearly dependent.

18. $\cosh x, \sinh x, \cosh^2 x$

Solution:Use Wronskian determinat

$$W = \begin{vmatrix} \cosh x & \sinh x & \cosh^2 x \\ \sinh x & \cosh x & 2\cosh x \sinh x \\ \cosh x & \sinh x & 2\cosh^2 x + 2\sinh^2 x \end{vmatrix} = \cosh^4 x - 2\sinh^4 x + \cosh^2 x \sinh^2 x \neq 0$$

Thus $\left\{\cosh x, \sinh x, \cosh^2 x\right\}$ is linearly independent.

P115.

Find an ODE(1) for which the given functions form a basis of solutions.

4. $\cos x, \sin x, x\cos x, x\sin x$

Solution:The characteristic equation

$(\lambda - i)^2(\lambda + i)^2 = 0$

$(\lambda^2 + 1)^2 = 0$

$\lambda^4 + 2\lambda^2 + 1 = 0$

Therefore the ODE is $y^{\text{iv}} + 2y'' + y = 0$

Solve the given ODE. (Show the details of your work.)

7. $y''' + y' = 0$

Solution: The characteristic equation

$\lambda^3 + \lambda = 0$

$\lambda(\lambda + i)(\lambda - i) = 0$

$\lambda = 0,$ or $\pm i$

Therefore the general solution is $y = c_1 + c_2 \cos x + c_3 \sin x.$

9. $y''' + y'' - y' - y = 0$

Solution: The characteristic equation

$\lambda^3 + \lambda^2 - \lambda - 1 = 0$

$\lambda^2(\lambda + 1) - (\lambda + 1) = 0$

$(\lambda + 1)(\lambda^2 - 1) = 0$

$\lambda = -1, -1, 1$

Therefore the general solution is $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{x}.$

11. $y''' - 3y'' - 4y' + 6y = 0$

Solution: The characteristic equation

$\lambda^3 - 3\lambda^2 - 4\lambda + 6 = 0$

$\lambda^3 - \lambda^2 - 2\lambda^2 + 2\lambda - 6\lambda + 6 = 0$

$\lambda^2(\lambda - 1) - 2\lambda(\lambda - 1) - 6(\lambda - 1) = 0$

$(\lambda - 1)(\lambda^2 - 2\lambda - 6) = 0$

$\lambda - 1 = 0$ or $\lambda^2 - 2\lambda - 6 = 0$
$\lambda = 1,$ or $\lambda = \frac{2 \pm \sqrt{4+24}}{2} = 1 \pm \sqrt{7}.$
Therefore the general solution is $y = c_1 e^x + c_2 e^{(1+\sqrt{7})x} + c_3 e^{(1-\sqrt{7})x}$.