Assignment4 (4th-week)

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Find a real general solution, showing the details of your work.

2. $4x^2y'' + 4xy' - y = 0$

Solution: Rewrite the equation

$x^2y'' + xy' - \frac{1}{4}y = 0$

it is a Cauchy-Euler equation, the auxiliary equation is

$m^2 - \frac{1}{4} = 0$

$m = \pm\frac{1}{2}$

Therefore the general solution is $y = c_1x^{\frac{1}{2}} + c_2x^{\frac{-1}{2}}$.

4. $x^2y'' + 3xy' + y = 0$

Solution: It is a Cauchy-Euler equation, the auxiliary equation is

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1$

Therefore the general solution is $y = c_1x^{-1} + c_2 \ln x \cdot x^{-1}$.

6. $2x^2y'' + 4xy' + 5y = 0$

Solution: Rewrite the equation

$x^2y'' + 2xy' + \frac{5}{2}y = 0$

it is a Cauchy-Euler equation, the auxiliary equation is

$m^2 + m + \frac{5}{2} = 0$

$m = \frac{-1\pm\sqrt{1-10}}{2} = \frac{-1}{2} \pm \frac{3}{2}i$

Therefore the general solution is $y = x^{\frac{-1}{2}}[A\cos(\frac{3}{2}\ln x) + B\sin(\frac{3}{2}\ln x)]$.

Solve and graph the solution, showing the details of your work.

11. $x^2y'' - 4xy' + 6y = 0,\ y(1) = 1,\ y'(1) = 0$

Solution: It is a Cauchy-Euler equation, the auxiliary equation is

$m^2 - 5m + 6 = 0$

$(m-3)(m-2) = 0$

$m = 3$ or $m = 2$

Therefore the genaeral solution is $y = c_1x^3 + c_2x^2$

And $y' = 3c_1x^2 + 2c_2x$

$$\begin{cases} y(1) = 1 \\ y'(1) = 0 \end{cases} \begin{cases} 1 = c_1 + c_2 \\ 0 = 3c_1 + 2c_2 \end{cases} \Rightarrow \begin{cases} c_1 = -2 \\ c_2 = 3 \end{cases}$$

Thus we obtain the answer is $y = -2x^3 + 3x^2$.

12. $x^2y'' + 3xy' + y = 0, \quad y(1) = 4, y'(1) = -2$

Solution: It is a Cauchy-Euler equation, the auxiliary equation is

$m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$m = -1$

Therefore the general solution is $y = c_1 x^{-1} + c_2 \ln x \cdot x^{-1}$

And $y' = -c_1 x^{-2} + c_2 x^{-1} \cdot x^{-1} - c_2 \ln x \cdot x^{-2}$

$$\left\{ \begin{array}{c} y(1) = 4 \\ y'(1) = -2 \end{array} \right. \Rightarrow \left\{ \begin{array}{c} 4 = c_1 \\ -2 = -c_1 + c_2 \end{array} \right. \Rightarrow \left\{ \begin{array}{c} c_1 = 4 \\ c_2 = 2 \end{array} \right.$$

Thus we obtain the answer is $y = 4x^{-1} + 2 \ln x \cdot x^{-1}$.

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Find an ODE (1) for which the given functions are solutions. Show linear independence (a) by considering quotients, (b) by Theorem 2.

1. $e^{0.5x}, e^{-0.5x}$

Solution: The Wronskian determinat is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{0.5x} & e^{-0.5x} \\ \frac{1}{2} e^{0.5x} & \frac{-1}{2} e^{-0.5x} \end{vmatrix} = -\frac{1}{2} - \frac{1}{2} = -1 \neq 0$$

then $\{e^{0.5x}, e^{-0.5x}\}$ is linear independence,

The characteristic equation is

$(\lambda - 0.5)(\lambda + 0.5) = 0$

$\lambda^2 - 0.25 = 0.$

Therefore the ODE is $y'' - 0.25y = 0.$

7. $\cos(2 \ln x), \sin(2 \ln x)$

Solution: The Wronskian determinat is

$$W = \begin{vmatrix} \cos(2 \ln x) & \sin(2 \ln x) \\ \frac{-2}{x} \sin(2 \ln x) & \frac{2}{x} \cos(2 \ln x) \end{vmatrix} = \frac{1}{x}(2 \cos^2(2 \ln x) + 2 \sin^2(2 \ln x)) = \frac{2}{x} \neq 0$$

then $\{\cos(2 \ln x), \sin(2 \ln x)\}$ is linear independence,

The auxiliary equation is

$[m - 2i][m + 2i] = 0$

$m^2 + 4 = 0$

Therefore the ODE is $x^2 y'' + xy' + 4y = 0.$

12. $e^{-2x} \cos wx, e^{-2x} \sin wx$

Solution: The Wronskian determinat is

$$W = \begin{vmatrix} e^{-2x} \cos wx & e^{-2x} \sin wx \\ -2e^{-2x} \cos wx - we^{-2x} \sin wx & -2e^{-2x} \sin wx + we^{-2x} \cos wx \end{vmatrix} =$$

$$w(\cos^2 wx) e^{2(-2x)} + w(\sin^2 wx) e^{2(-2x)} = we^{-4x}$$

then $\{e^{-2x} \cos wx, e^{-2x} \sin wx\}$ is linear independence,

The characteristic equation is

$[\lambda - (-2 + iw)][\lambda - (-2 - iw)] = 0$

$\lambda^2 + 4\lambda + (4 + w^2) = 0$

Therefore the ODE is $y'' + 4y' + (4 + w^2)y = 0.$

16. $e^{-kx} \cos \pi x, e^{-kx} \sin \pi x$

Solution: The Wronskian determinat is

$$W = \begin{vmatrix} e^{-kx}\cos\pi x & e^{-kx}\sin\pi x \\ -ke^{-kx}\cos\pi x - \pi e^{-kx}\sin\pi x & -ke^{-kx}\sin\pi x + \pi e^{-kx}\cos\pi x \end{vmatrix} =$$

$$\pi(\cos^2\pi x)e^{2(-kx)} + \pi(\sin^2\pi x)e^{2(-kx)} = \pi e^{-2kx}$$

then $\{e^{-kx}\cos\pi x, e^{-kx}\sin\pi x\}$ is linear independence,

The characteristic equation is

$[\lambda - (-k + i\pi)][\lambda - (-k - i\pi)] = 0$

$\lambda^2 + 2k\lambda + k^2 + \pi^2 = 0$

Therefore the ODE is $y'' + 2ky' + (k^2 + \pi^2)y = 0$

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Find a (real) general solution. Which rule are you using?

(Show each step of your calculation.)

2. $y'' + 4y' + 3.75y = 109\cos 5x$

Soluiton *Step 1. General solution of the homogeneous ODE.*

The characteristic equation

$$\lambda^2 + 4\lambda + 3.75 = \lambda^2 + 4\lambda + \frac{15}{4} = \left(\lambda + \frac{5}{2}\right)\left(\lambda + \frac{3}{2}\right) = 0$$

shows that a real general solution of the homogeneous ODE is

$$y_h = c_1 e^{\frac{-5}{2}x} + c_2 e^{\frac{-3}{2}x}$$

Step 2. *Solution of the nonhomogeneous ODE.*

Let $y_p = A\cos 5x + B\sin 5x$, then

$y_p' = 5B\cos 5x - 5A\sin 5x$

$y_p'' = -25A\cos 5x - 25B\sin 5x$

Substitution into the given ODE

$y_p'' + 4y_p' + 3.75y_p$

$= (-25A\cos 5x - 25B\sin 5x) + 4(5B\cos 5x - 5A\sin 5x) + \frac{15}{4}(A\cos 5x + B\sin 5x)$

$= (20B - \frac{85}{4}A)\cos 5x + (-20A - \frac{85}{4}B)\sin 5x$

imply that

$$\begin{cases} 20B - \frac{85}{4}A = 109 \\ -20A - \frac{85}{4}B = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{68}{25} \\ B = \frac{64}{25} \end{cases}$$

i.e. $y_p = -\frac{68}{25}\cos 5x + \frac{64}{25}\sin 5x$

Therefore the general solution is $y = c_1 e^{\frac{-5}{2}x} + c_2 e^{\frac{-3}{2}x} - \frac{68}{25}\cos 5x + \frac{64}{25}\sin 5x.$

5. $y'' + y' - 6y = 6x^3 - 3x^2 + 12x$

Soluiton: *Step 1. General solution of the homogeneous ODE.*

The characteristic equation

$$\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2) = 0$$

Thus, $y_h = c_1 e^{-3x} + c_2 e^{2x}$.

Step 2. *Solution of the nonhomogeneous ODE.*

Let $y_p = Ax^3 + Bx^2 + Dx + E$, then

$y_p' = 3Ax^2 + 2Bx + D$

$y_p'' = 6Ax + 2B$

Substitution into the given ODE
$y_p'' + y_p' - 6y_p = (6Ax + 2B) + (3Ax^2 + 2Bx + D) - 6(Ax^3 + Bx^2 + Dx + E)$
$= (2B + D - 6E) + (6A + 2B - 6D)x + (3A - 6B)x^2 - 6Ax^3$
imply that

$$\begin{cases} -6A = 6 \\ 3A - 6B = -3 \\ 6A + 2B - 6D = 12 \\ 2B + D - 6E = 0 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 0 \\ D = -3 \\ E = -\frac{1}{2} \end{cases}$$

i.e. $y_p = -x^3 - 3x - \frac{1}{2}$.
Therefore the general solution is $y = c_1 e^{-3x} + c_2 e^{2x} - x^3 - 3x - \frac{1}{2}$.

8. $y'' + 10y' + 25y = 100\sinh 5x$
Solution: Rewrite the equation
$y'' + 10y' + 25y = 50e^{5x} - 50e^{-5x}$
Step 1. General solution of the homogeneous ODE.
The characteristic equation

$$\lambda^2 + 10\lambda + 25 = (\lambda + 5)^2 = 0$$

Thus, $y_h = c_1 e^{-5x} + c_2 x e^{-5x}$.
Step 2. *Solution of the nonhomogeneous ODE.*
Suppose that $y_p = y_{p_1} + y_{p_2}$
Let $y_{p_1} = Ae^{5x}$, then
$y_{p_1}' = 5Ae^{5x}$
$y_{p_1}'' = 25Ae^{5x}$
Substitution into the Rewrite ODE
$y_{p_1}'' + 10y_{p_1}' + 25y_{p_1} = 25Ae^{5x} + 10 \cdot 5Ae^{5x} + 25Ae^{5x} = 100Ae^{5x}$
imply that $100A = 50,\ A = \frac{1}{2}$
Let $y_{p_2} = Bx^2 e^{-5x}$, then
$y_{p_2}' = 2Bxe^{-5x} - 5Bx^2 e^{-5x}$
$y_{p_2}'' = 2Be^{-5x} - 20Bxe^{-5x} + 25Bx^2 e^{-5x}$
Substitution into the Rewrite ODE
$y_{p_2}'' + 10y_{p_2}' + 25y_{p_2} = (2Be^{-5x} - 20Bxe^{-5x} + 25Bx^2e^{-5x}) + 10(2Bxe^{-5x} - 5Bx^2e^{-5x}) + 25Bx^2e^{-5x}$
$= 2Be^{-5x}$
imply that $2B = -50,\ B = -25$
i.e. $y_p = \frac{1}{2}e^{5x} - 25x^2 e^{-5x}$
Therefore the general solution is $y = c_1 e^{-5x} + c_2 x e^{-5x} + \frac{1}{2}e^{5x} - 25x^2 e^{-5x}$.

11. $y'' + 1.44y = 24\cos 1.2x$
Soluiton: *Step 1. General solution of the homogeneous ODE.*
The characteristic equation

$$\lambda^2 + 1.44 = (\lambda + 1.2i)(\lambda - 1.2i) = 0$$

Thus, $y_h = c_1 \cos 1.2x + c_2 \sin 1.2x$.
Step 2. *Solution of the nonhomogeneous ODE.*
Let $y_p = Ax\cos 1.2x + Bx\sin 1.2x$, then

$y_p' = A\cos 1.2x + B\sin 1.2x + 1.2Bx\cos 1.2x - 1.2Ax\sin 1.2x$
$y_p'' = 2.4B\cos 1.2x - 2.4A\sin 1.2x - 1.44Ax\cos 1.2x - 1.44Bx\sin 1.2x$

Substitution into the given ODE
$y_p'' + 1.44y_p$
$= [2.4B\cos 1.2x - 2.4A\sin 1.2x - 1.44Ax\cos 1.2x - 1.44Bx\sin 1.2x]$
$+1.44[Ax\cos 1.2x + Bx\sin 1.2x]$
$= 2.4B\cos 1.2x - 2.4A\sin 1.2x$
imply that

$$\begin{cases} -2.4A = 0 \\ 2.4B = 24 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 10 \end{cases}$$

i.e. $y_p = 10x\sin 1.2x$
Therefore the general solution is $y = c_1\cos 1.2x + c_2\sin 1.2x + 10x\sin 1.2x$

14. $y'' + 2y' + y = 2x\sin x$
Soluiton: *Step 1. General solution of the homogeneous ODE.*
The characteristic equation

$$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$$

Thus, $y_h = c_1e^{-x} + c_2xe^{-x}$.
Step 2. *Solution of the nonhomogeneous ODE.*
Let $y_p = (Ax + B)\cos x + (Dx + E)\sin x$, then
$y_p' = A\cos x + D\sin x - (Ax + B)\sin x + (Dx + E)\cos x$
$y_p'' = 2D\cos x - 2A\sin x - (Ax + B)\cos x - (Dx + E)\sin x$
Substitution into the given ODE
$2x\sin x = y_p'' + 2y_p' + y_p$
$= [2D\cos x - 2A\sin x - (Ax + B)\cos x - (Dx + E)\sin x]$
$+ 2[A\cos x + D\sin x - (Ax + B)\sin x + (Dx + E)\cos x] + [(Ax + B)\cos x + (Dx + E)\sin x]$
$= (2D - Ax - B + 2A + 2Dx + 2E + Ax + B)\cos x + (-2A - Dx - E + 2D - 2Ax - 2B + Dx + E$
$= (2D - 2B - 2A - 2Ax)\sin x + (2A + 2D + 2E + 2xD)\cos x$
imply that

$$\begin{cases} -2A = 2 \\ 2D - 2B - 2A = 0 \\ 2D = 0 \\ 2A + 2D + 2E = 0 \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 1 \\ D = 0 \\ E = 1 \end{cases}$$

i.e. $y_p = (-x + 1)\cos x + \sin x$
Therefore the general solution is $y = c_1e^{-x} + c_2xe^{-x} + (-x + 1)\cos x + \sin x$.

Solve the initial value problem. State which rules you are using. Show each step of your calculation in detail.
16. $y'' + 4y = 16\cos 2x, \quad y(0) = 0, y'(0) = 0$
Soluiton: *Step 1. General solution of the homogeneous ODE.*
The characteristic equation

$$\lambda^2 + 4 = (\lambda + 2i)(\lambda - 2i) = 0$$

Thus, $y_h = c_1\cos 2x + c_2\sin 2x$

Step 2. *Solution of the nonhomogeneous ODE.*
Let $y_p = Ax\cos 2x + Bx\sin 2x$, then
$y_p' = A\cos 2x + B\sin 2x + 2Bx\cos 2x - 2Ax\sin 2x$
$y_p'' = 4B\cos 2x - 4A\sin 2x - 4Ax\cos 2x - 4Bx\sin 2x$
Substitution into the given ODE
$y_p'' + 4y_p = (4B\cos 2x - 4A\sin 2x - 4Ax\cos 2x - 4Bx\sin 2x) + 4(Ax\cos 2x + Bx\sin 2x)$
$= 4B\cos 2x - 4A\sin 2x$
imply that
$$\begin{cases} 4B = 16 \\ -4A = 0 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 4 \end{cases}$$
i.e. $y_p = 4x\sin 2x$
Therefore the general solution is $y = c_1\cos 2x + c_2\sin 2x + 4x\sin 2x$.
Step 3. Solution of the initial value problem.
Since $y' = -2c_1\sin 2x + 2c_2\cos 2x + 4\sin 2x + 4x\cos 2x$
$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases} \Rightarrow \begin{cases} 0 = c_1 \\ 0 = 2c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$
Thus we obtain the answer is $y = 4x\sin 2x$.

18. $y'' - 2y' = 12e^{2x} - 8e^{-2x}, \quad y(0) = -2, y'0) = 12$
Soluiton: *Step 1. General solution of the homogeneous ODE.*
The characteristic equation
$$\lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0$$
Thus, $y_h = c_1e^{2x} + c_2$
Step 2. *Solution of the nonhomogeneous ODE.*
Suppose that $y_p = y_{p_1} + y_{p_2}$
Let $y_{p_1} = Axe^{2x}$, then
$y_{p_1}' = Ae^{2x} + 2Axe^{2x}$
$y_{p_1}'' = 4Ae^{2x} + 4Axe^{2x}$
Substitution into the given ODE
$y_{p_1}'' - 2y_{p_1}' = (4Ae^{2x} + 4Axe^{2x}) - 2(Ae^{2x} + 2Axe^{2x}) = 2Ae^{2x}$
imply that $2A = 12,\ A = 6$.
Let $y_{p_2} = Be^{-2x}$, then $y_{p_2}' = -2Be^{-2x},\ y_{p_2}'' = 4Be^{-2x}$, Substitution into the given ODE
$y_{p_2}'' - 2y_{p_2}' = 4Be^{-2x} + 4Be^{-2x} = 8Be^{-2x}$
imply that $8B = -8,\ B = -1$.
i.e. $y_p = 6xe^{2x} - e^{-2x}$
Therefore the general solution is $y = c_1e^{2x} + c_2 + 6xe^{2x} - e^{-2x}$.
Step 3. Solution of the initial value problem.
And $y' = 2c_1e^{2x} + 12xe^{2x} + 6e^{2x} + 2e^{-2x}$
$$\begin{cases} y(0) = -2 \\ y'0) = 12 \end{cases} \Rightarrow \begin{cases} -2 = c_1 + c_2 - 1 \\ 12 = 2c_1 + 6 + 2 \end{cases} \Rightarrow \begin{cases} c_1 = 2 \\ c_2 = -3 \end{cases}$$
Thus we obtain the answer is $y = 2e^{2x} - 3 + 6xe^{2x} - e^{-2x}$.

20. $y'' + 2y' + 10y = 17\sin x - 37\sin 3x, \quad y(0) = 6.6, y'(0) = -2.2$

Soluiton: *Step 1. General solution of the homogeneous ODE.*
The characteristic equation

$$\lambda^2 + 2\lambda + 10 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i$$

Thus, $y_h = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$
Step 2. *Solution of the nonhomogeneous ODE.*
Suppose that $y_p = y_{p_1} + y_{p_2}$
Let $y_{p_1} = A\sin x + B\cos x$, then
$y'_{p_1} = A\cos x - B\sin x$
$y''_{p_1} = -B\cos x - A\sin x$
Substitution into the given ODE
$y''_{p_1} + 2y'_{p_1} + 10y_{p_1} = (-B\cos x - A\sin x) + 2(A\cos x - B\sin x) + 10(A\sin x + B\cos x)$
$= (2A + 9B)\cos x + (9A - 2B)\sin x$
imply that

$$\begin{cases} 9A - 2B = 17 \\ 2A + 9B = 0 \end{cases} \Rightarrow \begin{cases} A = 1.8 \\ B = -0.4 \end{cases}$$

Let $y_{p_2} = D\sin 3x + E\cos 3x$, then
$y'_{p_2} = 3D\cos 3x - 3E\sin 3x$
$y''_{p_2} = -9E\cos 3x - 9D\sin 3x$
Substitution into the given ODE
$y''_{p_2} + 2y'_{p_2} + 10y_{p_2}$
$= (-9E\cos 3x - 9D\sin 3x) + 2(3D\cos 3x - 3E\sin 3x) + 10(D\sin 3x + E\cos 3x)$
$= (6D + E)\cos 3x + (D - 6E)\sin 3x$
imply that

$$\begin{cases} 6D + E = 0 \\ D - 6E = -37 \end{cases} \Rightarrow \begin{cases} D = -1 \\ E = 6 \end{cases}$$

i.e. $y_p = 1.8\sin x - 0.4\cos x - \sin 3x + 6\cos 3x$
Therefore the general solution is
$y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) + 1.8\sin x - 0.4\cos x - \sin 3x + 6\cos 3x.$
Step 3. Solution of the initial value problem.
And $y' = 1.$
$8\cos x + 0.4\sin x - 3\cos 3x - 18\sin 3x - e^{-x}(c_1\cos 3x + c_2 \sin 3x) + e^{-x}(3c_2\cos 3x - 3c_1 \sin 3x)$

$$\begin{cases} y(0) = 6.6 \\ y'(0) = -2.2 \end{cases} \Rightarrow \begin{cases} 6.6 = c_1 - 0.4 + 6 \\ -2.2 = 1.8 - 3 - c_1 + 3c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 0 \end{cases}$$

Thus we obtain the answer is $y = e^{-x}\cos 3x + 1.8\sin x - 0.4\cos x - \sin 3x + 6\cos 3x.$