

(More in the next problem set.) Verify by substitution that the given functions form a basis. Solve the given initial value problem. (Show the details of your work.)

3. $y'' + 2y' + 2y = 0$, $e^{-x} \cos x, e^{-x} \sin x$, $y(0) = 1, y'(0) = -1$.

Solution: First step

Let $y_1 = e^{-x} \cos x$ and $y_2 = e^{-x} \sin x$, we want to show that y_1 and y_2 is solution of the given equation.

Since $y_1' = -e^{-x} \cos x - e^{-x} \sin x$, and

$$y_1'' = e^{-x} \cos x + e^{-x} \sin x + e^{-x} \sin x - e^{-x} \cos x = 2e^{-x} \sin x,$$

then $y_1'' + 2y_1' + 2y_1 = 2e^{-x} \sin x + 2(-e^{-x} \cos x - e^{-x} \sin x) + 2e^{-x} \cos x = 0$.

And since $y_2' = -e^{-x} \sin x + e^{-x} \cos x$, and

$$y_2'' = e^{-x} \sin x - e^{-x} \cos x - e^{-x} \cos x - e^{-x} \sin x = -2e^{-x} \cos x,$$

then $y_2'' + 2y_2' + 2y_2 = -2e^{-x} \cos x + 2(-e^{-x} \cos x - e^{-x} \sin x) + e^{-x} \sin x = 0$.

Thus $e^{-x} \cos x$ and $e^{-x} \sin x$ are solution of the given equation.

Second step, General solution,

Since $e^{-x} \cos x / e^{-x} \sin x = \cot x \neq \text{constant}$, so that $\{e^{-x} \cos, e^{-x} \sin x\}$ is linear independent, hence the general solution is

$$y = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x.$$

Third step, Particular solution,

$$y' = -c_1 e^{-x} \cos x - c_1 e^{-x} \sin x - c_2 e^{-x} \sin x + c_2 e^{-x} \cos x$$

$$\begin{cases} y(0) = 1 \\ y'(0) = -1 \end{cases} \Rightarrow \begin{cases} 1 = c_1 \\ -1 = -c_1 + c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 0 \end{cases}$$

Therefore the particular solution is $y = e^{-x} \cos x$.

Are the following functions linearly independent on the given interval?

8. $3x^2, 2x^n$ ($0 < x < 1$)

Solution:

$$3x^2 / 2x^n = \frac{3}{2} x^{2-n} \neq \text{constant}, \text{ thus } \{3x^2, 2x^n\} \text{ is linearly independent.}$$

other method in P75. calculate the Wronskian determinant,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 3x^2 & 2x^n \\ 6x & 2nx^{n-1} \end{vmatrix} = (6n - 12)x^{n+1} \neq 0.$$

Then $\{3x^2, 2x^n\}$ is linearly independent

10. $\cos^2 x, \sin^2 x$ (any interval)

Solution:

$$\cos^2 x / \sin^2 x = \cot^2 x \neq \text{constant}, \text{ thus } \{\cos^2 x, \sin^2 x\} \text{ is linearly independent.}$$

12. $x - 2, x + 2$ ($-2 < x < 2$)

Solution:

$$(x + 2) / (x - 2) \neq \text{constant}, \text{ thus } \{x - 2, x + 2\} \text{ is linearly independent.}$$

Reduce to first order and solve (showing each step in detail.)

$$18. y'' = 1 + (y')^2$$

Let $u = y'$, $u' = y''$, then given equation become

$$u' = 1 + u^2$$

$$\frac{u'}{1+u^2} = 1$$

$$\int \frac{1}{1+u^2} du = \int dx$$

$$\arctan u = x + c_1$$

$$u = \tan(x + c_1)$$

$$y' = \tan(x + c_1)$$

$$y = \int \tan(x + c_1) dx = -\ln|\cos(x + c_1)| + c_2.$$

$$22. (1 - x^2)y'' - 2xy' + 2y = 0, y_1 = x.$$

Solution: method 1. Using the formula (P.51,9),

$$\text{Rewrite the equation } y'' - \frac{2x}{(1-x^2)}y' + \frac{2}{(1-x^2)}y = 0$$

$$U = \frac{1}{y_1} e^{-\int p dx}, \text{ where } p = -\frac{2x}{(1-x^2)}.$$

$$\text{then } \int p dx = \int \frac{-2x}{(1-x^2)} dx = -\ln(x^2 - 1), \text{ and}$$

$$U = \frac{1}{x^2} e^{\ln(x^2-1)} = \frac{x^2-1}{x^2} = 1 - \frac{1}{x^2}$$

Hence the desired second solution is

$$y_2 = y_1 u = y_1 \int U dx = x \int \left(1 - \frac{1}{x^2}\right) dx = x\left(x + \frac{1}{x}\right) = x^2 + 1$$

Therefore the general solution is $y = c_1 x + c_2(x^2 + 1)$.

Find a general solution. Check your answer by substitution.

$$1. y'' - 6y - 7 = 0$$

Solution: The characteristic equation is

$$\lambda^2 - 6\lambda - 7 = 0$$

$$(\lambda - 7)(\lambda + 1) = 0$$

$$\lambda = 7 \text{ or } \lambda = -1$$

so that, we obtain the general solution

$$y = c_1 e^{7x} + c_2 e^{-x}.$$

$$4. y'' + 4\pi y' + 4\pi^2 = 0$$

Solution: The characteristic equation is

$$\lambda^2 + 4\pi\lambda + 4\pi^2 = 0$$

$$(\lambda + 2\pi)^2 = 0$$

$$\lambda = -2\pi$$

so that, we obtain the general solution

$$y = c_1 e^{-2\pi x} + c_2 x e^{-2\pi x}.$$

$$7. y'' - y' + 2.5y = 0$$

Solution : The characteristic equation is

$$\lambda^2 - \lambda + 2.5 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-10}}{2} = \frac{1 \pm 3i}{2} = \frac{1}{2} \pm \frac{3}{2}i$$

so that, we obtain the general solution

$$y = e^{\frac{1}{2}x}(A \cos \frac{3}{2}x + B \sin \frac{3}{2}x).$$

10. $y'' - 2y = 0$

Solution: The characteristic equation is

$$\lambda^2 - 2 = 0$$

$$\lambda = \pm \sqrt{2}$$

so that, we obtain the general solution

$$y = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}.$$

14. $y'' + y' - 0.96y = 0$

Solution: The characteristic equation is

$$\lambda^2 + \lambda - 0.96 = 0$$

$$\lambda^2 + \lambda - \frac{24}{25} = 0$$

$$\lambda^2 + \lambda - \frac{8}{5} \cdot \frac{3}{5} = 0$$

$$(\lambda + \frac{8}{5})(\lambda - \frac{3}{5}) = 0$$

$$\lambda = -\frac{8}{5} \text{ or } \lambda = \frac{3}{5}$$

so that, we obtain the general solution

$$y = c_1 e^{-\frac{8}{5}x} + c_2 e^{\frac{3}{5}x}.$$

Find an ODE $y'' + ay' + by = 0$ for the give basis.

15. e^{2x}, e^x

Solution: The characteristic equation is

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

so that, we obtain the ODE is $y'' - 3y' + 2y = 0$.

19. e^{4x}, e^{-4x}

Solution: The characteristic equation is

$$(\lambda - 4)(\lambda + 4) = 0$$

$$\lambda^2 - 16 = 0$$

so that, we obtain the ODE is $y'' - 16y = 0$.

Solve the initial value problem. Check that your answer satisfies the ODE as well as the initial conditions. (Show the details of your work.)

21. $y'' - 2y' - 3y = 0, y(0) = 2, y'(0) = -14$

Solution: The characteristic equation is

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = 3 \text{ or } \lambda = -1$$

so that we obtain the general solution

$$y = c_1 e^{3x} + c_2 e^{-x}$$

$$\text{And } y' = 3c_1 e^{3x} - c_2 e^{-x}$$

$$\begin{cases} y(0) = 2 \\ y'(0) = 14 \end{cases} \Rightarrow \begin{cases} 2 = c_1 + c_2 \\ 14 = 3c_1 - c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 4 \\ c_2 = -2 \end{cases}$$

Therefore the particular solution is

$$y = 4e^{3x} - 2e^{-x}.$$

24. $10y'' - 50y' + 65y = 0, \quad y(0) = 1.5, \quad y'(0) = 1.5$

Solution: Rewrite the given equation

$$2y'' - 10y' + 13 = 0$$

The characteristic equation is

$$2\lambda^2 - 10\lambda + 13 = 0$$

$$\lambda = \frac{10 \pm \sqrt{100 - 104}}{2 \cdot 2} = \frac{5}{2} \pm \frac{1}{2}i$$

so that, we obtain the general solution,

$$y = e^{\frac{5}{2}x}(A \cos \frac{1}{2}x + B \sin \frac{1}{2}x)$$

$$\text{And } y' = \frac{5}{2}e^{\frac{5}{2}x}(A \cos \frac{1}{2}x + B \sin \frac{1}{2}x) + \frac{1}{2}e^{\frac{5}{2}x}(-A \sin \frac{1}{2}x + B \cos \frac{1}{2}x)$$

$$\begin{cases} y(0) = 1.5 \\ y'(0) = 1.5 \end{cases} \Rightarrow \begin{cases} 1.5 = A \\ 1.5 = \frac{5}{2}A + \frac{1}{2}B \end{cases} \Rightarrow \begin{cases} A = 1.5 \\ B = 0.5 \end{cases}$$

Therefore the particular solution is

$$y = e^{\frac{5}{2}x}(\frac{3}{2} \cos \frac{1}{2}x + \frac{1}{2} \sin \frac{1}{2}x).$$

27. $10y'' + 5y' + 0.625y = 0, \quad y(0) = 2, \quad y'(0) = -4.5$

Solution: Rewrite the given equation

$$2y'' + y' + 0.125y = 0, \quad 16y'' + 8y' + y = 0.$$

The characteristic equation is

$$16\lambda^2 + 8\lambda + 1 = 0$$

$$(4\lambda + 1)^2 = 0$$

$$\lambda = \frac{-1}{4}$$

so that, we obtain the general solution

$$y = c_1 e^{\frac{-1}{4}x} + c_2 x e^{\frac{-1}{4}x}$$

$$\text{And } y' = \frac{-1}{4}c_1 e^{\frac{-1}{4}x} + c_2 e^{\frac{-1}{4}x} + \frac{-1}{4}c_2 x e^{\frac{-1}{4}x}.$$

$$\begin{cases} y(0) = 2 \\ y'(0) = -4.5 \end{cases} \Rightarrow \begin{cases} 2 = c_1 + 0 \cdot c_2 \\ -4.5 = \frac{-1}{4}c_1 + c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 2 \\ c_2 = -4 \end{cases}$$

Therefore the particular solution is

$$y = 2_1 e^{\frac{-1}{4}x} - 4_2 x e^{\frac{-1}{4}x}.$$

30. $y'' + 2ky' + (k^2 + w^2)y = 0, \quad y(0) = 1, \quad y'(0) = -k.$

Solution: The characteristic equation is

$$\lambda^2 + 2k\lambda + (k^2 + w^2) = 0$$

$$\lambda = \frac{-2k \pm \sqrt{4k^2 - 4(k^2 + w^2)}}{2} = -k \pm iw$$

so that, we obtain the general solution

$$y = e^{-kx}(A \cos wx + B \sin wx).$$

$$\text{And } y' = -ke^{-kx}(A \cos wx + B \sin wx) + we^{-kx}(-A \sin wx + B \cos wx)$$

$$\begin{cases} y(0) = 1 \\ y'(0) = -k \end{cases} \Rightarrow \begin{cases} 1 = A \\ -k = -k \cdot A + wB \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 0 \end{cases}$$

Therefore the particular solution is $y = e^{-kx} \cos wx$.