Assignment2 (2nd-week)

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Test for exactness. If exact, solve. If not, use an integration factor as given or find it by inspection or from the theorems in the text. Also, if an initial condition is given, determine the corresponding particular solution.

12. $(e^{x+y} - y)dx + (xe^{x+y} + 1)dy = 0$

Test for exactness, $P = (e^{x+y} - y),\ Q = (xe^{x+y} + 1)$

Thus

$\frac{\partial P}{\partial y} = e^{x+y} - 1 \neq xe^{x+y} + e^{x+y} = \frac{\partial Q}{\partial x}.$

$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{xe^{x+y}+1}(e^{x+y} - 1 - xe^{x+y} - e^{x+y}) = -1.$

$F = \exp\int R dx = \exp\int(-1)dx = e^{-x}$

Multiplying the given equation by e^{-x}, we get the new equation.

$(e^{y} - ye^{-x})dx + (xe^{y} + e^{-x})dy = 0$

$e^{y}dx - ye^{-x}dx + xe^{y}dy + e^{-x}dy = 0$

$(e^{y}dx + xe^{y}dy) + (e^{-x}dy - ye^{-x}dx) = 0$

$(e^{y}dx + xd(e^{y})) + (e^{-x}dy + yd(e^{-x})) = 0$

$d(xe^{y}) + d(ye^{-x}) = 0$

$xe^{y} + ye^{-x} = c.$

14. $(x^4 + y^2)dx - xydy = 0,\ y(2) = 1$

Test for exactness, $P = (x^4 + y^2),\ Q = -xy$

Thus

$\frac{\partial P}{\partial y} = 2y \neq -y = \frac{\partial Q}{\partial x}.$

$R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{-xy}(2y + y) = \frac{-3}{x}.$

$F = \exp\int R dx = \exp\int(\frac{-3}{x})dx = e^{-3\ln x} = x^{-3}$

Multiplying the given equation by x^{-3}, we get the new equation

$(x + x^{-3}y^2)dx - x^{-2}ydy = 0, \quad y(2) = 1$

$xdx + x^{-3}y^2dx - x^{-2}ydy = 0$

$xdx + \frac{-1}{2}y^2d(x^{-2}) + \frac{-1}{2}x^{-2}d(y^2) = 0$

$xdx + \frac{-1}{2}d(x^{-2}y^2) = 0$

$\frac{1}{2}x^2 + \frac{-1}{2}x^{-2}y^2 = c.$

and $y(2) = 1,\ \frac{2^2}{2} + \frac{-1\cdot 1^2}{2\cdot 2^2} = c,$ imply $c = \frac{15}{8}.$

Therefore the particalar solution is $\frac{1}{2}x^2 + \frac{-1}{2}x^{-2}y^2 = \frac{15}{8}.$

16. $-\sin xy(ydx + xdy) = 0,\ y(1) = \pi$

Solution:

$-\sin xy(ydx + xdy) = 0$

$-\sin xy\ d(xy) = 0$

$\int -\sin xy d(xy) = 0$

$\cos xy + c = 0$

and $y(1) = \pi,\ \cos\pi + c = 0,$ imply $c = 1.$

Therefore the particular solution is $\cos xy + 1 = 0.$

18. $(\cos xy + \frac{x}{y})dx + (1 + \frac{x}{y}\cos xy)dy = 0$

Test for exactness, $P = (\cos xy + \frac{x}{y}),\ Q = -1 + \frac{x}{y}\cos xy$

Thus

$\frac{\partial P}{\partial y} = -x\sin xy - \frac{x}{y^2} \neq \frac{1}{y}\cos xy - x\sin xy = \frac{\partial Q}{\partial x}.$

$R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{y}{y\cos xy}\left(\frac{1}{y}\cos xy - x\sin xy + x\sin xy + \frac{x}{y^2}\right) = \frac{1}{y}.$

$F = \exp\int R^* dy = \exp\int(\frac{1}{y})dy = e^{\ln y} = y$

Multiplying the given equation by y, we get the new equation

$(y\cos xy + x)dx + (y + x\cos xy)dy = 0$

$y\cos xydx + xdx + ydy + x\cos xydy = 0$

$\cos xy(ydx + xdy) + xdx + ydy = 0$

$\cos xyd(xy) + xdx + ydy = 0$

$\sin xy + \frac{1}{2}x^2 + \frac{1}{2}y^2 = c.$

20. $(\sin y\cos y + x\cos^2 y)dx + xdy = 0$

Solution:

Test for exactness, $P = (\sin y\cos y + x\cos^2 y),\ Q = x$

Thus

$\frac{\partial P}{\partial y} = \cos^2 y - \sin^2 y - 2x\sin y\cos y \neq 1 = \frac{\partial Q}{\partial x}.$

$R^* = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{1}{\sin y\cos y + x\cos^2 y}(1 - \cos^2 y + \sin^2 y + 2x\sin y\cos y)$

$= \frac{2\sin^2 y + 2x\sin y\cos y}{\sin y\cos y + x\cos^2 y} = \frac{2\sin y(\sin y + x\cos y)}{\cos y(\sin y + x\cos y)} = 2\tan y.$

$F = \exp\int R^* dy = \exp\int(2\tan y)dy = \exp\int\frac{2\sin y}{\cos y}dy = \exp\int\frac{-2}{\cos y}d(\cos y) = \exp(-2\ln\cos y)$

$= (\cos y)^{-2}$

Multiplying the given equation by $(\cos y)^{-2}$, we get the new equation

$\left(\frac{\sin y}{\cos y}dx + x\right)dx + \frac{x}{\cos^2 y}dy = 0$

$\tan ydx + xdx + x\sec^2 ydy = 0$

$\tan ydx + xd(\tan y) + xdx = 0$

$d(x\tan y) + xdx = 0$

$x\tan y + \frac{1}{2}x^2 = c.$

Find the general solution. If an initial condition is given, find also the corresponding particular solution and graph or sketch it. (Show the details of your work.)

6. $x^2y' + 3xy = \frac{1}{x}.\ \ y(1) = -1$ (Linear equation)

Solution: method 1

Rewrite the equation $y' + \frac{3}{x}y = \frac{1}{x^3}$, we have $p(x) = \frac{3}{x}$, and integrating factor

$F(x) = \exp\left\{\int p(x)dx\right\} = \exp\left\{\int\frac{3}{x}dx\right\} = e^{3\ln x} = x^3,$

We now multiply the rewrite equation on both sides by this F.

$x^3y' + 3x^2y = 1$

$(x^3y)' = 1$

$\int d(x^3y) = \int dx$

$x^3y = x + c$

And $y(1) = -1,\ 1^3 \cdot (-1) = 1 + c,$ imply $c = -2.$

Therefore the particular solution is $x^3y = x - 2.$

method 2, Using the formula of the Linear ODE(P27, 4)

$y = e^{-h}\left(\int e^h r dx + c\right), \qquad h = \int p(x)dx.$

Rewrite the equation $y' + \frac{3}{x}y = \frac{1}{x^3}$, we have $p(x) = \frac{3}{x}, r = \frac{1}{x^3}$,

$h = \int p(x)dx = \int \frac{3}{x}dx = 3\ln x,$

And $y = e^{-3\ln x}\left(\int e^{3\ln x} \cdot \frac{1}{x^3}dx + c\right) = x^{-3}\left(\int x^3 \cdot \frac{1}{x^3}dx + c\right) = x^{-3}\left(\int dx + c\right)$

$y = x^{-3}(x + c).$

And $y(1) = -1$, $-1 = 1^{-3}(1 + c)$, imply $c = -2$.

Therefore the particular solution is $y = x^{-3}(x - 2)$.

8. $y' + 2y = 4\cos 2x, \quad y(\frac{\pi}{4}) = 2$

Solution:

we have $p(x) = 2$, and integrating factor $F(x) = \exp\left\{\int p(x)dx\right\} = \exp\left\{\int 2dx\right\} = e^{2x}$,

We now multiply the original equation on both sides by this F.

$e^{2x}y' + 2e^{2x}y = 4e^{2x}\cos 2x$

$(e^{2x}y)' = 4e^{2x}\cos 2x$

$e^{2x}y = 4\int e^{2x}\cos 2x dx$

$e^{2x}y = e^{2x}\cos 2x + e^{2x}\sin 2x + c$

$y = \cos 2x + \sin 2x + ce^{-2x}$

And $y(\frac{\pi}{4}) = 2$, $2 = \cos\frac{\pi}{2} + \sin\frac{\pi}{2} + ce^{-\frac{\pi}{2}}$, imply $c = e^{\frac{\pi}{2}}$.

Therefore the particular solution is $y = \cos 2x + \sin 2x + e^{\frac{\pi}{2} - 2x}$.

10. $y' + 4x^2 y = (4x^2 - x)e^{-\frac{x^2}{2}}$

Solution:

we have $p(x) = 4x^2$, and integrating factor

$F(x) = \exp\left\{\int p(x)dx\right\} = \exp\left\{\int 4x^2 dx\right\} = e^{\frac{4}{3}x^3}$,

Multiplying the given equation by $e^{\frac{4}{3}x^3}$, we get the new equation

$e^{\frac{4}{3}x^3}y' + 4e^{\frac{4}{3}x^3} \cdot x^2 y = (4x^2 - x)e^{\left(\frac{4}{3}x^3 - \frac{x^2}{2}\right)}$

$\left(e^{\frac{4}{3}x^3}y\right)' = (4x^2 - x)e^{\left(\frac{4}{3}x^3 - \frac{x^2}{2}\right)}$

$e^{\frac{4}{3}x^3}y = \int (4x^2 - x)e^{\left(\frac{4}{3}x^3 - \frac{x^2}{2}\right)}dx$

$e^{\frac{4}{3}x^3}y = \int e^{\left(\frac{4}{3}x^3 - \frac{x^2}{2}\right)}d\left(\frac{4}{3}x^3 - \frac{x^2}{2}\right)$

$e^{\frac{4}{3}x^3}y = e^{\left(\frac{4}{3}x^3 - \frac{x^2}{2}\right)} + c.$

12. $y'\tan x = 2y - 8, \quad y(\frac{\pi}{2}) = 0$

Solution:

Rewrite the equation $y' - 2\cot x \cdot y = -8\cot x$, we have $p(x) = -2\cot x$, and integrating factor

$F(x) = \exp\left\{\int p(x)dx\right\} = \exp\left\{\int -2\cot x dx\right\} = \exp\left\{\int \frac{-2\cos x}{\sin x}dx\right\} = \exp\left\{\int \frac{-2}{\sin x}d(\sin x)\right\} = \exp\{$

Multiplying the rewrite equation by $(\sin x)^{-2}$, we get the new equation

$(\sin x)^{-2}y' - 2\cot x(\sin x)^{-2}y = -8\cot x(\sin x)^{-2},$

$\left[(\sin x)^{-2}y\right]' = \frac{-8\cos x}{\sin^3 x}$
$(\sin x)^{-2}y = -8\int \frac{\cos x}{\sin^3 x}dx$
$(\sin x)^{-2}y = -8\int \frac{1}{\sin^3 x}d(\sin x)$
$(\sin x)^{-2}y = -8 \cdot \frac{1}{\sin^2 x} \cdot (\frac{-1}{2}) + c$
$y = 4 + c\sin^2 x$
And $y(\frac{\pi}{2}) = 0,\ 0 = 4 + c\sin^2\frac{\pi}{2}$, imply that $c = -4$,
Therefore the particular solution is $y = 4 - 4\sin^2 x$.

14. $y' + y\tan x = e^{-0.01x}\cos x, \quad y(0) = 0$
Solution:
we have $p(x) = \tan x$, and integrating factor
$F(x) = \exp\left\{\int p(x)dx\right\} = \exp\left\{\int \tan x dx\right\} = \exp\left\{\int \frac{\sin x}{\cos x}dx\right\} = \exp\left\{\int \frac{\sin x}{\cos x}dx\right\} = \exp\left\{\int \frac{-1}{\cos x}d(\cos$
Multiplying the given equation by $(\cos x)^{-1}$, we get the new equation
$(\cos x)^{-1}y + (\cos x)^{-1}\tan x \cdot y = e^{-0.01x}$
$\left[(\cos x)^{-1}y\right]^{-1} = e^{-0.01x}$
$(\cos x)^{-1}y = \int e^{-0.01x}dx = -100.0e^{-\frac{1}{100}x}$
$(\cos x)^{-1}y = -100e^{-0.01x} + c$
$y = -100e^{-0.01x}\cos x + c\cos x.$
And $y(0) = 0,\ 0 = -100 \cdot e^0\cos 0 + c\cos 0$, imply that $c = 100$.
Therefore the particular solution is $y = -100e^{-0.01x}\cos x + 100\cos x$.

16. $y'\cos^2 x + 3y = 1, \ \ y(\frac{\pi}{4}) = \frac{4}{3}$.
Solution:
Rewrite the equation $y' + 3\sec^2 x \cdot y = \sec^2 x$, we have $p(x) = 3\sec^2 x$, and integrating factor $F(x) = \exp\left\{\int p(x)dx\right\} = \exp\left\{\int 3\sec^2 x dx\right\} = \exp\{3\tan x\} = e^{3\tan x}$,
Multiplying the rewrite equation by $e^{3\tan x}$, we get the new equation
$e^{3\tan x}y' + 3e^{3\tan x}\sec^2 x \cdot y = e^{3\tan x}\sec^2 x$
$(e^{3\tan x}y)' = e^{3\tan x}\sec^2 x$
$e^{3\tan x}y = \int e^{3\tan x}\sec^2 x dx$
$e^{3\tan x}y = \int e^{3\tan x} \cdot \frac{1}{3}d(3\tan x)$
$e^{3\tan x}y = \frac{1}{3}e^{3\tan x} + c.$
And $y(\frac{\pi}{4}) = \frac{4}{3},\ e^{3\tan\frac{\pi}{4}} \cdot \frac{4}{3} = \frac{1}{3}e^{3\tan\frac{\pi}{4}} + c$, imply that $c = e^3$
Therefore the particular solution is $e^{3\tan x}y = \frac{1}{3}e^{3\tan x} + e^3$.

19. $y' = 5.7y - 6.5y^2$ (Bernoulli Equation)
Solution:
Rewrite the equation $y' - 5.7y = -6.5y^2$,
Let $u = y^{-1}$ and we know $p = -5.7,\ g = -6.5,\ (1-a) = -1$, the given function become
$u' + (-1) \cdot (-5.7)u = (-1) \cdot (-6.7)$
$u' + 5.7u = 6.5 \qquad ------(*)$
we have $p(x) = 5.7$, and integrating factor $\exp\left\{\int p(x)dx\right\} = e^{5.7x}$.

Multiplying the equation $(*)$ by $e^{5.7x}$, we get the new equation
$e^{5.7x}u' + 5.7e^{5.7x}u = 6.7e^{5.7x}$
$(e^{5.7x}u)' = 6.7e^{5.7x}$
$e^{5.7x}u = 6.7\int e^{5.7x}dx$
$e^{5.7x}u = \frac{6.7}{5.7}e^{5.7x} + c$
$u = \frac{67}{57} + ce^{-5.7x}$
the general solution is $y^{-1} = \frac{67}{57} + ce^{-5.7x}$.

21. $y' + (x+1)y = e^{x^2}y^3, \qquad y(0) = 0.5$
Solution:
Let $u = y^{-2}$, and we know $p(x) = x+1,\ g(x) = e^{x^2},\ (1-a) = -2.$
the given equation become
$u' + (-2)(x+1)u = (-2)e^{x^2} \qquad -------(*)$
we have $p(x) = -2(x+1)$, and integrating factor $\exp\left\{\int -2(x+1)dx\right\} = e^{-(x^2+2x)}$.
Multiplying the equation $(*)$ by $e^{-(x^2+2x)}$, we get the new equation
$e^{-(x^2+2x)}u' + (-2)(x+1)e^{-(x^2+2x)} \cdot u = (-2)e^{-2x}$
$[e^{-(x^2+2x)}u]' = -2e^{-2x}$
$e^{-(x^2+2x)}u = \int(-2e^{-2x})dx$
$e^{-(x^2+2x)}u = e^{-2x} + c$
$u = e^{x^2} + ce^{(x^2+2x)}$
i.e. $y^{-2} = e^{x^2} + ce^{(x^2+2x)}$,
And $y(0) = 0.5,\ (0.5)^{-2} = e^0 + ce^0$, imply that $c = 3$,
Therefore the particular solution is $y^{-2} = e^{x^2} + 3e^{(x^2+2x)}$.

23. $2yy' + y^2\sin x = \sin x, \quad y(0) = \sqrt{2}$
Solution:
Rewrite the equation $y' + \frac{\sin x}{2}y = \frac{\sin x}{2y}$,
Let $u = y^2$, and $p(x) = \frac{\sin x}{2},\ g(x) = \frac{\sin x}{2},\ (1-a) = 2.$
the given equation become
$u' + 2 \cdot \frac{\sin x}{2} \cdot u = 2 \cdot \frac{\sin x}{2}$
$u' + \sin x \cdot u = \sin x$
$u' = (1-u)\sin x$
$\frac{u'}{1-u} = \sin x$
$\int \frac{1}{1-u}du = \int \sin x dx$
$-\ln(1-u) = -\cos x + c$
$1 - u = ce^{\cos x}$
$u = 1 - ce^{\cos x}$
i.e. $y^2 = 1 - ce^{\cos x}$
And $y(0) = \sqrt{2},\ 2 = 1 - ce^{\cos 0}$, imply that $c = -e^{-1}$
Therefore the particular solution is $y^2 = 1 + e^{\cos x - 1}$.