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2. Let  $f(x) = x$  if  $0 < x < k$ ,  $f(x) = 0$  if  $x > k$ . Find  $\hat{f}_c(w)$ .

Solution:

$$\begin{aligned}\hat{f}_c(w) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx dx = \sqrt{\frac{2}{\pi}} \int_0^k x \cos wx dx = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w} \int_0^k x d(\sin wx) \\ &= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w} x \sin wx \Big|_0^k - \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w} \int_0^k \sin wx dx \\ &= \sqrt{\frac{2}{\pi}} \cdot \frac{k}{w} \sin kw + \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w^2} \cos wx \Big|_0^k \\ &= \sqrt{\frac{2}{\pi}} \cdot \frac{k}{w} \sin kw + \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w^2} (\cos kw - 1)\end{aligned}$$

11. Find  $F_s(e^{-\pi x})$  by integration.

Solution:

let  $f(x) = e^{-\pi x}$ , then  $f'(x) = -\pi e^{-\pi x}$ ,  $f''(x) = \pi^2 e^{-\pi x}$ ,  
and  $f(0) = 1$ ,  $f'(0) = -\pi$

$$\begin{aligned}F_s(f''(x)) &= -w^2 F_s(f(x)) + \sqrt{\frac{2}{\pi}} w f(0) \\ F_s(\pi^2 e^{-\pi x}) &= -w^2 F_s(e^{-\pi x}) + \sqrt{\frac{2}{\pi}} w \cdot 1 \\ \pi^2 F_s(e^{-\pi x}) &= -w^2 F_s(e^{-\pi x}) + \sqrt{\frac{2}{\pi}} w \\ [\pi^2 + w^2] F_s(e^{-\pi x}) &= \sqrt{\frac{2}{\pi}} w \\ \text{hence, } F_s(e^{-\pi x}) &= \sqrt{\frac{2}{\pi}} \cdot \frac{w}{\pi^2 + w^2}\end{aligned}$$

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Find the Fourier transform of  $f(x)$  (without using Table III in Sec. 11.10)

Show the details,

$$2. f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \quad (k > 0) \end{cases}$$

Solution:

$$\begin{aligned}\hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{kx} \cdot e^{-iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{(1-2i)x} dx = \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{k-iw} e^{(k-iw)x} \Big|_0^{\infty} \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{k-iw} (\lim_{x \rightarrow \infty} e^{(k-iw)x} - 1)\end{aligned}$$

$$5. f(x) = \begin{cases} k & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$\begin{aligned} \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 ke^{-iwx} dx \\ &= \frac{k}{\sqrt{2\pi}} \int_{-1}^1 e^{-iwx} dx \\ &= \frac{k}{\sqrt{2\pi}} \cdot \frac{-1}{iw} e^{-iwx} \Big|_{-1}^1 \\ &= \frac{k}{\sqrt{2\pi}} \cdot \frac{-1}{iw} (e^{-iw} - e^{iw}) \\ &= \frac{k}{\sqrt{2\pi}} \cdot \frac{-1}{iw} [(\cos w - i \sin w) - (\cos w + i \sin w)] \\ &= \frac{k}{\sqrt{2\pi}} \cdot \frac{-1}{iw} (-2i \sin w) = \frac{k\sqrt{2}}{w\sqrt{\pi}} \cdot \sin w \end{aligned}$$

$$8. f(x) = \begin{cases} xe^{-x} & \text{if } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$\begin{aligned} \hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^0 xe^{-x} \cdot e^{-iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^0 xe^{-(1+iw)x} dx \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{(1+iw)} \int_{-1}^0 x d[e^{-(1+iw)x}] \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{(1+iw)} xe^{-(1+iw)x} \Big|_{-1}^0 + \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(1+iw)} \int_{-1}^0 e^{-(1+iw)x} dx \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{(1+iw)} (0 + e^{(1+iw)}) + \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(1+iw)} \int_{-1}^0 e^{-(1+iw)x} dx \\ &= \frac{-1}{\sqrt{2\pi}} \cdot \frac{e^{(1+iw)}}{(1+iw)} + \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{(1+iw)^2} e^{-(1+iw)x} \Big|_{-1}^0 \\ &= \frac{-1}{\sqrt{2\pi}} \cdot \frac{e^{(1+iw)}}{(1+iw)} + \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{(1+iw)^2} (0 + e^{(1+iw)}) \\ &= \frac{-1}{\sqrt{2\pi}} \cdot \frac{e^{(1+iw)}}{(1+iw)} + \frac{-1}{\sqrt{2\pi}} \cdot \frac{e^{(1+iw)}}{(1+iw)^2} \end{aligned}$$

2.  $u_{xx} = u$

Solution:

$$u_{xx} - u = 0$$

Since no  $y$ -derivatives occur, we can solve this like  $u'' - u = 0$ .

the characteristic equation is  $\lambda^2 - 1 = 0$ , then  $\lambda = 1, -1$

hence, the solution is

$$u(x, y) = A(y)e^x + B(y)e^{-x}$$

6.  $u_{xx} = 4y^2u$

Solution:

$$u_{xx} - u = 0$$

Since no  $y$ -derivatives occur, we can solve this like  $u'' - 4y^2u = 0$ .

the characteristic equation is  $\lambda^2 - 4y^2 = 0$ , then  $\lambda = 2y, -2y$

hence, the solution is

$$u(x, y) = A(y)e^{2yx} + B(y)e^{-2yx}$$

Table

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	one-dimensional wave equation
$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	one-dimensional heat equation
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	two-dimensional Laplace equation
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$	two-dimensional Poisson equation
$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$	two-dimensional wave equation
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$	three-dimensional Laplace equation

15.  $u = \sin 8x \cos 2t$  satisfy one-dimensional wave equation

Solution:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\sin 8x \cos 2t) = -2 \sin 8x \sin 2t$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}(-2 \sin 8x \sin 2t) = -4 \cos 2t \sin 8x$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\sin 8x \cos 2t) = 8 \cos 2t \cos 8x$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(8 \cos 2t \cos 8x) = -64 \cos 2t \sin 8x$$

then

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \\ -4 \cos 2t \sin 8x &= -64c^2 \cos 2t \sin 8x \\ c^2 &= \frac{1}{16} \\ c &= \frac{1}{4}, \frac{-1}{4} \end{aligned}$$

19.  $u = e^{-\pi^2 t} \sin 4x$  satisfy one-dimensional heat equation

Solution:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(e^{-\pi^2 t} \sin 4x) = -\pi^2 e^{-\pi^2 t} \sin 4x$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^{-\pi^2 t} \sin 4x) = 4e^{-\pi^2 t} \cos 4x$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(4e^{-\pi^2 t} \cos 4x) = -16e^{-\pi^2 t} \sin 4x$$

then

$$\begin{aligned}\frac{\partial u}{\partial t} &= c^2 \frac{\partial^2 u}{\partial x^2} \\ -\pi^2 e^{-\pi^2 t} \sin 4x &= -16c^2 e^{-\pi^2 t} \sin 4x \\ c^2 &= \frac{\pi^2}{16} \\ c &= \frac{\pi}{4}, \frac{-\pi}{4}\end{aligned}$$

23.  $u = \cos 2y \sinh 2x$  satisfy two-dimensional Laplace equation

Solution:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\cos 2y \sinh 2x) = 2 \cosh 2x \cos 2y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(2 \cosh 2x \cos 2y) = 4 \sinh 2x \cos 2y$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\cos 2y \sinh 2x) = -2 \sinh 2x \sin 2y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(-2 \sinh 2x \sin 2y) = -4 \sinh 2x \cos 2y$$

then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4 \sinh 2x \cos 2y - 4 \sinh 2x \cos 2y = 0$$

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