Assignment 11
Page 512 2,6,10,17
Show that the given integral repressts the indicated function.
Hint. Use (5),(11),or(13); the integral tells you which one, and its value tells you what function to consider.
(Show the details of your work.)

2. $\int\limits_0^\infty \frac{\sin w - w\cos w}{w^2}\sin xw dw = \{ \begin{array}{lll} \frac{\pi x}{2} & \text{if } 0 < x < 1 \\ \frac{\pi}{4} & \text{if} & x = 1 \\ 0 & \text{if} & x > 1 \end{array}$

Solution:

Since $f(x) = \{ \begin{array}{lll} \frac{\pi x}{2} & \text{if } 0 < x < 1 \\ \frac{\pi}{4} & \text{if} & x = 1 \\ 0 & \text{if} & x > 1 \end{array}$, suppose that $f(x)$ is odd function.

we have $A(w) = 0$, and

$$\begin{aligned} B(w) &= \frac{2}{\pi}\int_0^\infty f(x)\sin wx dx \\ &= \frac{2}{\pi}\int_0^1 \frac{\pi x}{2}\sin wx dx \\ &= \int_0^1 x\sin wx dx \\ &= \frac{-1}{w}\int_0^1 x d(\cos wx) \\ &= [\frac{-1}{w}x\cos wx]|_0^1 + \frac{1}{w}\int_0^1 \cos wx dx \\ &= \frac{-\cos w}{w} + \frac{1}{w^2}\sin wx|_0^1 \\ &= \frac{-\cos w}{w} + \frac{\sin w}{w^2} \\ &= \frac{\sin w - w\cos w}{w^2} \end{aligned}$$

we can obtain the Fourier sine integral,

$$\int_0^\infty B(w)\sin wx dw = \int_0^\infty \frac{\sin w - w\cos w}{w^2}\sin wx dw$$

and $f(x)$ at $x = 1$, we have

$$\frac{\lim_{x\to 1+} f(x) + \lim_{x\to 1-} f(x)}{2} = \frac{0 + \frac{\pi}{2}}{2} = \frac{\pi}{4}$$

Therefore

$$\int\limits_0^\infty \frac{\sin w - w\cos w}{w^2}\sin xw dw = \{ \begin{array}{lll} \frac{\pi x}{2} & \text{if } 0 < x < 1 \\ \frac{\pi}{4} & \text{if} & x = 1 \\ 0 & \text{if} & x > 1 \end{array}$$

6. $\int_0^\infty \frac{\sin \pi w \sin xw}{1-w^2} dw = \{ \begin{array}{ll} \frac{\pi}{2}\sin x & \text{if } 0 \le x \le \pi \\ 0 & \text{if } \quad x > \pi \end{array}$

Solution:

Since $f(x) = \{ \begin{array}{ll} \frac{\pi}{2}\sin x & \text{if } 0 \le x \le \pi \\ 0 & \text{if } \quad x > \pi \end{array}$, suppose that $f(x)$ is odd function.

we have $A(w) = 0$, and

$$\begin{aligned}
B(w) &= \frac{2}{\pi}\int_0^\infty f(x)\sin wx dx \\
&= \frac{2}{\pi}\int_0^\pi \frac{\pi}{2}\sin x \sin wx dx \\
&= \int_0^\pi \sin x \sin wx dx \\
&= \int_0^\pi \frac{1}{2}(\cos(1-w)x - \cos(1+w)x)dx \\
&= \frac{1}{2(1-w)} \cdot \sin(1-w)x|_0^\pi - \frac{1}{2(1+w)} \cdot \sin(1+w)x|_0^\pi \\
&= \frac{1}{2(1-w)} \cdot \sin(1-w)\pi - \frac{1}{2(1+w)} \cdot \sin(1+w)\pi \\
&= \frac{1}{2(1-w)}[\sin\pi\cos w\pi - \sin w\pi\cos\pi] - \frac{1}{2(1+w)}[\sin\pi\cos w\pi + \sin w\pi\cos\pi] \\
&= \frac{1}{2(1-w)}\sin w\pi - \frac{1}{2(1+w)}\sin w\pi \\
&= \left[\frac{1}{2(1-w)} - \frac{1}{2(1+w)}\right]\sin w\pi \\
&= \frac{\sin w\pi}{1-w^2}
\end{aligned}$$

we can obtain the Fourier sine integral,

$$\int_0^\infty B(w)\sin wx dw = \int_0^\infty \frac{\sin w\pi}{1-w^2}\sin wx dw$$

and $f(x)$ at $x = \pi$, we have $\lim_{x\to\pi+} f(x) = 0 = \frac{\pi}{2}\sin\pi = \lim_{x\to\pi-} f(x)$

Therefore

$$\int_0^\infty \frac{\sin \pi w \sin xw}{1-w^2} dw = \{ \begin{array}{ll} \frac{\pi}{2}\sin x & \text{if } 0 \le x \le \pi \\ 0 & \text{if } \quad x > \pi \end{array}$$

10. $f(x) = \{ \begin{array}{ll} \frac{x}{2} & \text{if } 0 < x < 1 \\ 1 - \frac{x}{2} & \text{if } 1 < x < 2 \\ 0 & \text{if } \quad x > 2 \end{array}$

Solution:

Since $f(x) = \{ \begin{array}{ll} \frac{x}{2} & \text{if } 0 < x < 1 \\ 1-\frac{x}{2} & \text{if } 1 < x < 2 \\ 0 & \text{if } \quad x > 2 \end{array}$, suppose that $f(x)$ is even function.
we have $B(w) = 0$, and

$$
\begin{aligned}
A(w) &= \frac{2}{\pi}\int_0^\infty f(x)\cos wxdx \\
&= \frac{2}{\pi}\int_0^1 \frac{x}{2}\cos wxdx + \frac{2}{\pi}\int_1^2 (1-\frac{x}{2})\cos wxdx \\
&= \frac{1}{\pi}\int_0^1 x\cos wxdx + \frac{2}{\pi}\int_1^2 (1-\frac{x}{2})\cos wxdx \\
&= \frac{1}{\pi}\cdot\frac{1}{w}\int_0^1 xd(\sin wx) + \frac{2}{\pi}\cdot\frac{1}{w}\int_1^2 (1-\frac{x}{2})d(\sin wx) \\
&= \frac{1}{\pi}\cdot\frac{1}{w}x\sin wx|_0^1 + \frac{1}{\pi}\cdot\frac{-1}{w}\int_0^1 \sin wxdx \\
&\quad +\frac{2}{\pi}\cdot\frac{1}{w}(1-\frac{x}{2})\sin wx|_1^2 + \frac{2}{\pi}\cdot\frac{-1}{w}\int_1^2 \sin wxd(1-\frac{x}{2}) \\
&= \frac{\sin w}{\pi w} + \frac{1}{\pi}\cdot\frac{1}{w^2}\cos wx|_0^1 + \frac{2}{\pi}\cdot\frac{1}{w}\cdot(\frac{-1}{2}\sin w) + \frac{1}{\pi}\cdot\frac{1}{w}\int_1^2 \sin wxdx \\
&= \frac{1}{\pi w^2}(\cos w - 1) + \frac{1}{\pi}\cdot\frac{-1}{w^2}\cos wx|_1^2 \\
&= \frac{1}{\pi w^2}(\cos w - 1) + \frac{-1}{\pi w^2}(\cos 2w - \cos w) \\
&= \frac{1}{\pi w^2}(2\cos w - \cos 2w - 1)
\end{aligned}
$$

we can obtain the Fourier cosine integral,

$$\int_0^\infty A(w)\cos wxdw = \int_0^\infty \frac{1}{\pi w^2}(2\cos w - \cos 2w - 1)\cos wxdw$$

and $f(x)$ at $x = 1$, we have $\lim_{x\to 1+} f(x) = 1 - \frac{1}{2} = \frac{1}{2} = \lim_{x\to 1-} f(x)$
and $f(x)$ at $x = 2$, we have $\lim_{x\to 2+} f(x) = 0 = 1 - \frac{2}{2} = \lim_{x\to 2-} f(x)$

17. $f(x) = \{ \begin{array}{ll} \pi - x & \text{if } 0 < x < \pi \\ 0 & \text{if } \quad x > \pi \end{array}$
Solution:
Since $f(x) = \{ \begin{array}{ll} \pi - x & \text{if } 0 < x < \pi \\ 0 & \text{if } \quad x > \pi \end{array}$, suppose that $f(x)$ is odd function.
we have $A(w) = 0$, and

$$
\begin{aligned}
B(w) &= \frac{2}{\pi}\int_0^\infty f(x)\sin wx dx \\
&= \frac{2}{\pi}\int_0^\pi (\pi - x)\sin wx dx \\
&= \frac{2}{\pi}\cdot\frac{-1}{w}\int_0^\pi (\pi - x) d(\cos wx) \\
&= \frac{2}{\pi}\cdot\frac{-1}{w}(\pi - x)\cos wx|_0^\pi + \frac{2}{\pi}\cdot\frac{1}{w}\int_0^\pi \cos wx d(\pi - x) \\
&= \frac{2}{\pi}\cdot\frac{-1}{w}[0 - \pi\cos w\pi] + \frac{2}{\pi}\cdot\frac{-1}{w}\int_0^\pi \cos wx dx \\
&= \frac{2}{w}\cos w\pi + \frac{2}{\pi}\cdot\frac{-1}{w^2}\sin wx|_0^\pi \\
&= \frac{2}{w}\cos w\pi
\end{aligned}
$$

we can obtain the Fourier sine integral,

$$\int_0^\infty B(w)\sin wx dw = \int_0^\infty \frac{2}{w}\cos w\pi \sin wx dw$$

and $f(x)$ at $x = \pi$, we have $\lim_{x\to\pi+} f(x) = 0 = \pi - \pi = \lim_{x\to\pi-} f(x)$.