Assignment I (1st-week)

Page 8, 5,7,9,12,14; Page 18, 2,4,6,8,10,12,14,16; Page 25, 2,4,6,8,10

State the order the ODE. Verify that the given function is a solution (a, b, c are arbitrary contants.)

5. $y' = 1 + y^2$. $y = \tan(x + c)$

Solution:

$y' = \sec^2(x + c) = 1 + \tan^2(x + c) = 1 + y^2$

7. $y'' + 2y' + 10y = 0$. $y = 4e^{-x}\sin 3x$

Solution:

Since

$y' = -4e^{-x}\sin 3x + 12e^{-x}\cos 3x$

$y'' = 4e^{-x}\sin 3x - 12e^{-x}\cos 3x - 12e^{-x}\cos 3x - 36e^{-x}\sin 3x = -32e^{-x}\sin 3x - 24e^{-x}\cos 3x$

then

$y'' + 2y' + 10y = (-32e^{-x}\sin 3x - 24e^{-x}\cos 3x) + 2(-4e^{-x}\sin 3x + 12e^{-x}\cos 3x) + 40e^{-x}\sin 3x =$

9. $y''' = \cos x$, $y = -\sin x + ax^2 + bx + c$

Solution:

Since

$y' = -\cos x + 2ax + b$,

$y'' = \sin x + 2a$

$y''' = \cos x$

Verify that y is a solution of the ODE. Determine from y the particular solution satisfying the given initial condition. Sketch of graph this soluton.

12. $y' = y - x$, $y = ce^x + x + 1$, $y(0) = 3$

Solution:

$y' = ce^x + 1 = (ce^x + x + 1) - x = y - x$

so that y is a general solution of the equation. And $y(0) = 3$

$3 = ce^0 + 0 + 1, \quad 2 = c$

Therefore the particular solution is $y = 2e^x + x + 1$.

14. $y' = y\tan x$, $y = c\sec x$. $y(0) = \frac{1}{2}\pi$

Solution:

$y' = c\sec x\tan x = y\tan x$

so that y is a general solution of the equation. And $y(0) = \frac{1}{2}\pi$

$\frac{1}{2}\pi = c\sec\frac{1}{2}\pi, \ c = \frac{1}{2}\pi.$

Therefore the particular solution is $y = \frac{1}{2}\pi\sec x$.

Find a general solution. Show the steps of derivation. Check your answer by substitution.

2. $y' + (x + 2)y^2 = 0$

Solution:

$y' = -(x+2)y^2$
$\frac{1}{y^2}y' = -(x+2)$
$\int \frac{1}{y^2}dy = \int -(x+2)dx$
$\frac{-1}{y} = -\frac{x^2}{2} - 2x + c$

4. $y' = (y+9x)^2 \quad (y+9x) = v$
Solution:
Let $y + 9x = v$, then $v' = y' + 9$, then equation become
$v' - 9 = v^2$
$v' = v^2 + 9$
$\frac{1}{v^2+9}v' = 1$
$\int \frac{1}{v^2+9}dv = \int dx$
$\frac{1}{3}\arctan\frac{v}{3} = x + c$
$\arctan\frac{v}{3} = 3(x+c)$
$\frac{v}{3} = \tan 3(x+c)$
$v = 3\tan 3(x+c)$
$y + 9x = 3\tan 3(x+c)$.

6. $y' = (4x^2 + y^2)/(xy)$
Solution:
$y' = \frac{4x}{y} + \frac{y}{x}$.
Let $u = \frac{y}{x}$, $y = ux$ then $y' = u'x + u$, then equation become
$u'x + u = \frac{4}{u} + u$
$u'x = \frac{4}{u}$
$\frac{1}{u}u' = \frac{4}{x}$
$\int \frac{1}{u}du = \int \frac{4}{x}dx$
$\ln u = 4\ln x + c$
$u = cx^4$

8. $y'e^{\pi x} = y^2 + 1$
Solution:
$\frac{1}{y^2+1}y' = e^{-\pi x}$
$\int \frac{1}{y^2+1}dy = \int e^{-\pi x}dx$
$\arctan y = \frac{-1}{\pi}e^{-\pi x} + c$
$y = \tan(\frac{-e^{-\pi x}}{\pi} + c)$.

Find the particular solution. Show the steps of derivation, beginnig with the general solution. (L, R, b are constants.)
10. $yy' + 4x = 0,\ y(0) = 3$
Solution:
$yy' = -4x$
$\int ydy = \int(-4x)dx$
$\frac{y^2}{2} = -2x^2 + c$

$y^2 = -4x^2 + c$
And $y(0) = 3$, imply that
$3^2 = c,\ 9 = c$
Therefore the particular solution is $y^2 = 4x^2 + 9$.

12. $2xyy' = 3y^2 + x^2,\ y(1) = 2$
Solution:
$y' = \frac{3}{2} \bullet \frac{y}{x} + \frac{1}{2} \bullet \frac{x}{y}$
Let $u = \frac{y}{x}$, $y = ux$, then $y' = u'x + u$, the equation become
$u'x + u = \frac{3}{2}u + \frac{1}{2u}$
$u'x = \frac{1}{2}(u + \frac{1}{u})$
$u'x = \frac{1}{2}(\frac{u^2+1}{u})$
$\frac{2u}{u^2+1}u' = \frac{1}{x}$
$\int \frac{2u}{u^2+1}du = \int \frac{1}{x}dx$
$\int \frac{1}{u^2+1}d(u^2 + 1) = \ln|x| + c$
$\ln(u^2 + 1) = \ln|x| + c$
$u^2 + 1 = ce^{|x|}$
$\frac{y^2}{x^2} + 1 = ce^{|x|}$
And $y(1) = 2$, imply that
$\frac{2^2}{1^2} + 1 = ce^1$
$5 = ce^1$
$c = 5e^{-1}$
Therefore the particular solution is $\frac{y^2}{x^2} + 1 = e^{|x|-1}$.

14. $y' = \frac{y}{x} + \frac{2x^3}{y}\cos(x^2),\ y(\sqrt{\frac{\pi}{2}}) = \sqrt{\pi}$
Solution:
Let $u = \frac{y}{x}$, $y = ux$, then $y' = u'x + u$, the equation become
$u'x + u = u + \frac{x^2\cos(x^2)}{u}$
$u'x = \frac{x^2\cos(x^2)}{u}$
$\frac{u'}{u} = x\cos(x^2)$
$\int \frac{1}{u}du = \int x\cos(x^2)dx$
$\ln|u| = \sin(x^2) + c$
$u = ce^{\sin(x^2)}$
$\frac{y}{x} = ce^{\sin(x^2)}$
And $y(\sqrt{\frac{\pi}{2}}) = \sqrt{\pi}$, imply that
$\frac{\sqrt{\pi}}{\sqrt{\frac{\pi}{2}}} = ce^{\sin(\frac{\pi}{2})}$
$\sqrt{2} = ce^1$
$c = \sqrt{2}e^{-1}$
Therefore the particular solution is $\frac{y}{x} = \sqrt{2}e^{\sin(x^2)-1}$.

16. $xy' = y + 4x^5\cos^2(\frac{y}{x}),\ y(2) = 0.$
Solution:

$y' = \frac{y}{x} + 4x^4 \cos^2(\frac{y}{x})$

Let $u = \frac{y}{x},\ y = ux,$ then $y' = u'x + x,$ the equation become

$u'x + u = u + 4x^4 \cos^2 u$

$u'x = 4x^4 \cos^2 u$

$\frac{u'}{\cos^2 u} = 4x^3$

$\sec^2 u \cdot u' = 4x^3$

$\int \sec^2 u du = \int 4x^3 dx$

$\tan u = x^4 + c$

$\tan \frac{y}{x} = x^4 + c$

And $y(2) = 0,$ imply that

$\tan \frac{0}{2} = 2^4 + c$

$c = -16.$

Therefore the particular solution is $\tan \frac{y}{x} = x^4 - 16.$

2. $(x - y)(dx - dy) = 0$

Solution:

$(x - y)dx - (x - y)dy = 0.$

Test for exactness, $M = (x - y),\ N = (y - x).$

Thus

$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x}.$

Now, $u(x, y) = \int M dx + k(y) = \int (x - y)dx + k(y) = \frac{x^2}{2} - xy + k(y)$

And $N = \frac{\partial u}{\partial y} = -x + k'(y),$ imply that $k'(y) = y,\ k(y) = \frac{y^2}{2} + c$

Therefore the general solution is $\frac{x^2}{2} - xy + \frac{y^2}{2} = c.$

4. $(e^y - ye^x)dx + (xe^y - e^x)dy = 0$

Solution:

Test for exactness, $M = e^y - ye^x,\ N = xe^y - e^x$

Thus

$\frac{\partial M}{\partial y} = e^y - e^x = \frac{\partial N}{\partial x}.$

Now, $u(x, y) = \int M dx + k(y) = \int (e^y - ye^x)dx + k(y) = xe^y - ye^x + k(y)$

And $N = \frac{\partial u}{\partial y} = xe^y - e^x + k'(y),$ imply that $k'(y) = 0,\ k(y) = c$

Therefore the general solution is $xe^y - ye^x = c.$

6. $e^x(\cos y dx - \sin y dy) = 0$

Solution:

Test for exactness, $M = e^x(\cos y),\ N = -e^x \sin y$

Thus

$\frac{\partial M}{\partial y} = -e^x \sin y = \frac{\partial N}{\partial x}.$

Now $u(x, y) = \int M dx + k(y) = \int (e^x \cos y)dx + k(y) = e^x \cos y + k(y)$

And $N = \frac{\partial u}{\partial y} = -e^x \sin y + k'(y),$ imply that $k'(y) = 0,\ k(y) = c$

Therefore the general solution is $e^x \cos y = c.$

8. $(2x + \frac{1}{y} - \frac{y}{x^2})dx + (2y + \frac{1}{x} - \frac{x}{y^2})dy = 0$
Solution:
Test for exactness, $M = (2x + \frac{1}{y} - \frac{y}{x^2}),\ N = (2y + \frac{1}{x} - \frac{x}{y^2})$
Thus
$\frac{\partial M}{\partial y} = \frac{-1}{y^2} + \frac{-1}{x^2} = \frac{\partial N}{\partial x}.$
Now $u(x, y) = \int M dx + k(y) = \int(2x + \frac{1}{y} - \frac{y}{x^2})dx + k(y) = x^2 + \frac{x}{y} + \frac{y}{x} + k(y)$
And $N = \frac{\partial u}{\partial y} = \frac{-x}{y^2} + \frac{1}{x} + k'(y),$ imply that $k'(y) = 2y,\ k(y) = y^2 + c$
Therefore the general solution is $x^2 + y^2 + \frac{x}{y} + \frac{y}{x} = c.$

10. $-2xy\sin(x^2)dx + \cos(x^2)dy = 0$
Solution:
Test for exactness, $M = -2xy\sin(x^2),\ N = \cos(x^2)$
Thus
$\frac{\partial M}{\partial y} = -2x\sin(x^2) = \frac{\partial N}{\partial x}.$
Now $u(x, y) = \int M dx + k(y) = \int -2xy\sin(x^2)dx + k(y) = y\cos(x^2) + k(y)$
And $N = \frac{\partial u}{\partial y} = \cos(x^2) + k'(y),$ imply that $k'(y) = 0,\ k(y) = c$
Therefore the general solution is $y\cos(x^2) + c.$