Theorem 10.7

Let $f(z)$ is an analytic function, and $f(e^{it})$ is convergence, for $0 \le t \le 2\pi$, then for any given $\varepsilon_n \downarrow 0$, exist starlike functions $S_0(z), S_1(z), \widetilde{S}_1(z), S_2(z), \widetilde{S}_2(z), \ldots, S_n(z), \widetilde{S}_n(z), \ldots$, such that $\left| f(z) - \sum_{k=0}^{n} \left(S_k + \widetilde{S}_k \right) \right| \le \varepsilon_n$, for any $|z| \le 1$, it is hold. And where $\widetilde{S}_0(z) = 0$.

Prove: Since $f(z) = \sum_{k=0}^{\infty} a_k z^k$,

First Step, for $\varepsilon_1 > 0$, $\exists N_1$, when $n \ge N_1$, we have

$$\left| f(z) - \sum_{k=0}^{N_1} a_k z^k \right| \le \varepsilon_1$$

and

$$\sum_{k=0}^{N_1} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots + a_{N_1} z^{N_1} = S_0(z) + S_1(z) + \widetilde{S}_1(z)$$

denote $S_0(z) = a_0 + a_1 z$,

$$S_1(z) = \frac{A_1}{2}\left(z + \sum_{k=2}^{N_1} \frac{a_k}{A_1} z^k \right)$$

$$S_1(z) = \frac{A_1}{2}\left(-z + \sum_{k=2}^{N_1} \frac{a_k}{A_1} z^k \right)$$

where $|A_1| \ge \sum_{k=2}^{N_1} k|a_k|$, $S_0(z), S_1(z), \widetilde{S}_1(z)$ are 1-starlike functions.

Second Step, for $0 < \varepsilon_2 < \varepsilon_1$, $\exists N_2 > N_1$, when $n \ge N_2$, we have

$$\left| f(z) - \sum_{k=0}^{N_2} a_k z^k \right| \le \varepsilon_2$$

and

$$\sum_{k=0}^{N_2} a_k z^k = \sum_{k=0}^{N_1} a_k z^k + (a_{N_1+1} z^{N_1+1} + a_{N_1+2} z^{N_1+2} + \cdots + a_{N_2} z^{N_2})$$
$$= S_0(z) + S_1(z) + \widetilde{S}_1(z) + S_2(z), + \widetilde{S}_2(z)$$

denote

$$S_2(z) = z^{N_1-1} \cdot \frac{A_2}{2}\left(z + \sum_{k=2}^{N_2-N_1+1} \frac{a_{(N_1+k-1)}}{A_2} z^k \right)$$

$$S_2(z) = z^{N_1-1} \cdot \frac{A_2}{2}\left(-z + \sum_{k=2}^{N_2-N_1+1} \frac{a_{(N_1+k-1)}}{A_2} z^k \right)$$

where $|A_2| \ge \sum_{k=2}^{N_2-N_1+1} k \left| a_{(N_1+k-1)} \right|$, $S_2(z), \widetilde{S}_2(z)$ are N_1-starlike functions.

Third step, by induction, let with ε_n, we can find that
$S_0(z), S_1(z), \widetilde{S}_1(z), S_2(z), \widetilde{S}_2(z), \ldots, S_n(z), \widetilde{S}_n(z)$,

then for ε_{n+1}, $\exists N_{n+1} > N_n$, when $n \ge N_{n+1}$, we have

$$\left| f(z) - \sum_{k=0}^{N_{n+1}} a_k z^k \right| \le \varepsilon_{n+1}$$

and

$$\begin{aligned}\sum_{k=0}^{N_{n+1}} a_k z^k &= \sum_{k=0}^{N_n} a_k z^k + \sum_{k=N_n}^{N_{n+1}} a_k z^k \\ &= S_0(z) + S_1(z) + \widetilde{S}_1(z) + S_2(z), + \widetilde{S}_2(z) + \ldots + S_n(z) + \widetilde{S}_n(z) + S_{n+1}(z) + \widetilde{S}_{n+1}(z)\end{aligned}$$

denote

$$\begin{aligned}S_{N_{n+1}}(z) &= z^{N_n - 1} \cdot \frac{A_{n+1}}{2}\left(z + \sum_{k=2}^{N_{n+1}-N_n+1} \frac{a_{(N_n+k-1)}}{A_2} z^k\right) \\ \widetilde{S}_{N_{n+1}}(z) &= z^{N_n - 1} \cdot \frac{A_{n+1}}{2}\left(-z + \sum_{k=2}^{N_{n+1}-N_n+1} \frac{a_{(N_n+k-1)}}{A_2} z^k\right)\end{aligned}$$

where $|A_{n+1}| \geq \sum_{k=2}^{N_{n+1}-N_n+1} k\left|a_{(N_n+k-1)}\right|,\ S_{n+1}(z), \widetilde{S}_{n+1}(z)$ are N_n-starlike functions.

summarize above, for $\varepsilon_n \downarrow 0,$ exist starlike functions $S_0(z), S_1(z), \widetilde{S}_1(z), S_2(z), \widetilde{S}_2(z), \ldots, S_n(z), \widetilde{S}_n(z), \ldots,$ such that

$$\left| f(z) - \sum_{k=0}^{n}\left(S_k(z) + \widetilde{S}_k(z)\right)\right| \leq \varepsilon_n$$

for $\forall \varepsilon,\ \varepsilon_n \downarrow 0,\ \exists n_0,$ s.t. when$n > n_0,\ \varepsilon_n < \varepsilon.$

$$\left| f(z) - \sum_{k=0}^{n}\left(S_k(z) + \widetilde{S}_k(z)\right)\right| \leq \varepsilon_n < \varepsilon.$$