On

Triangles

with rational altitudes,
angle bisectors or medians

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Abstract

This thesis is principally concerned with integer-sided triangles. Such a triangle is called a Heron triangle if its area is also an integer. The main theme of the thesis is a study of the relationships between Heron triangles and triangles with various types of cevians of integer length. Cevians are lines joining the vertices of a triangle to any points on the opposite sides.

The first chapter covers the case of three integer altitudes and initially considers isosceles and Pythagorean examples. We then examine two transformations, the hinge and the pivot, which can be used to classify integer-sided triangles with three integer altitudes. The general parametrization emerges from an analogous one for the set of Heron triangles.

The second chapter deals with the case of three angle bisectors of integer length. Again we search for isosceles and Pythagorean examples but this time it turns out that Pythagorean triangles with three integer angle bisectors do not exist. Next we show that, somewhat surprisingly, any integer-sided triangle with three integer angle bisectors must necessarily have integer area. This leads to a general parametrization of such triangles, by the same technique as was used in Chapter 1.

The third chapter deals with the case of three integer medians. Previously known results are that the semiperimeter must be even and that the primitive triangles occur in related pairs. The search for simple examples leads to negative results, for example, such triangles cannot be isosceles, Pythagorean or have sides in an arithmetic progression.

One of the unsolved problems in this area asks whether there is a nonempty intersection between the set of Heron triangles and the set of triangles with three integer medians. To this end we note that Euler's parametrization of a proper subset of triangles with three integer medians does not include any triangle with integer area.

A complete parametrization is given for triangles with three rational sides and two rational medians. This leads to a method for generating triangles with three rational medians by application of the tangent-chord process to various elliptic curves.

In the fourth chapter we convert the parametrization of Chapter 3 into a computer search routine. This leads to the discovery of several examples of Heron triangles with two integer medians. These all lie on a surface defined by a polynomial function of two variables. The problem of discovering rational points

on this surface turns out to be difficult to attack with any known techniques.

The fifth and final chapter deals with rational concurrent cevians through arbitrary points inside and outside a triangle. The defining equations are considered along with some interesting special cases e.g. cevians through the circumcentre. The relationship between three rational cevians and a rational tiling of the triangle is also discussed.

Acknowledgements

I would like to thank my supervisor Roger B. Eggleton for his enthusiastic encouragement when I was first exploring diophantine problems about triangles - and for his subsequent expert guidance during the preparation of this thesis. I must also thank my acting supervisors; Mike Hayes who supplied some critical insights into Chapter 3; Bruce Richmond whose help was greater than he anticipated and Professor Warren Brisley for his skillful assistance during the final stages.

I am grateful to Kevin Sharp, my high school mathematics teacher of six years duration, for instilling in me a desire to explore and enjoy mathematics. Finally I must thank Judy whose constant gentle nudges helped push this thesis from my mind into the real world.

Any errors that remain in this thesis are the sole responsibility of the author.

Part II

Thesis

Introduction

0.1 Historical Perspective

Historically, triangles are among the first objects documented to have attracted mathematical attention. This attention has frequently emphasised triangles with integer dimensions.

One of the oldest mathematical documents in existence is the Rhind papyrus [13, pp. 169-178]. It was written by the Egyptian scribe A^{c}h-mosè (pronounced Aah-mes) in 4th month of the flood season in the 33rd year of the reign of one of the Apophis kings sometime between 1585 B.C. and 1542 B.C. [1, p. 36]. Furthermore A^{c}h-mosè writes, rather matter-of-factly, that this is just a copy of an earlier work (which he presumably has in front of him) made for king Amenemhet III who reigned from 1844 B.C. to 1797 B.C.

In Problem 51, as translated by J. Peet [14, pp. 91-94], the scribe shows how to calculate the area of a triangular section of land (see Figure 1). It has a base length of 4 khet and a height of 10 khet (where 1 khet = 100 cubits and 1 cubit = 523mm) which makes the block about 209 metres by 523 metres.

While the hieratic script in Figure 1 is unreadable except to an egyptologist, the accompanying figure, drawn by A^{c}h-mosè three and a half millenia ago, can be described by a child. (My five year old daughter reassured me that it is indeed a triangle).

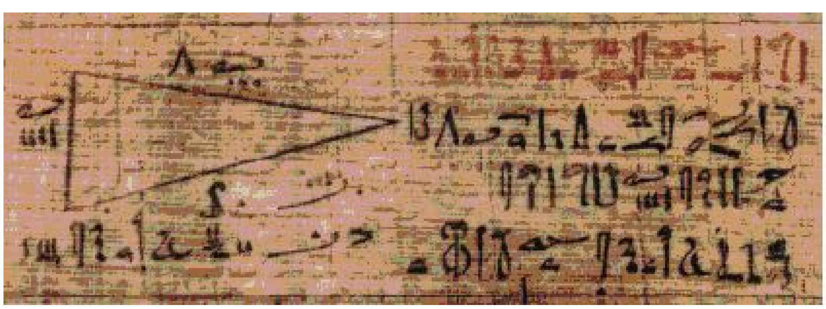

Figure 1: Rhind Papyrus – Problem 51

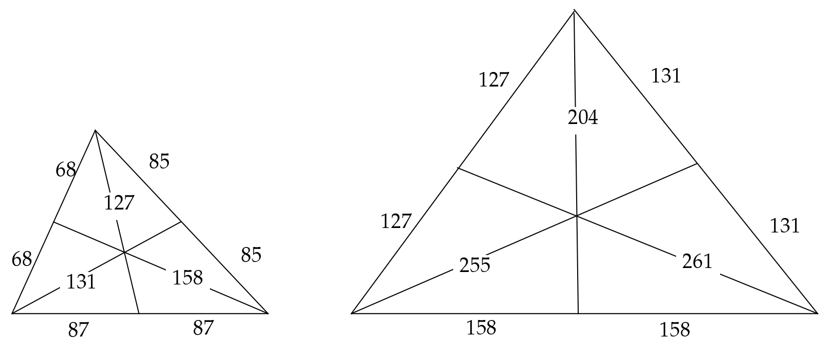

Figure 2: Euler's Related Triangles

The ancient Egyptians were practical people who invariably solved practical problems. The ancient Greeks on the other hand were probably the first to study the triangle (amongst other figures) just for its own sake. Recall Pythagoras and his theorem, Euclid's Elements and Diophantus' Arithmetica. Book 1 of the Elements devotes over half of its propositions to triangles. While in Arithmetica, for example, Problem 16 of Book VI [10, pp. 240-241], asks for a Pythagorean triangle in which the length of the bisector of one of the acute angles is rational. Diophantus obtained the triangle $4(7, 24, 25)$ which has an angle bisector of 35 to the side 96. In Chapter 2 we will derive the general solution to this problem. (Notationally we will specify triangles by just the three sides, in increasing order, with any nontrivial gcd of the sides alone explicitly shown as a scale factor).

In more recent times Giovanni Ceva (1648-1734) became interested in the properties of internal side dividers of triangles. The line segments joining each vertex of a triangle to any point on the opposite side have since become known as cevians [4, pp. 4-5]. We will denote their lengths by $d_1, d_2, d_3 :$ this thesis we will normally consider only primitive triangles in which $\gcd(a, b, c, d_1, d_2, d_3) = 1$. When he was thirty years old Ceva published a remarkably concise condition for the concurrency of the three cevians of a triangle. Though he apparently proved only the necessity of the condition, in more recent times it has also been proved to be sufficient. So we have the formulation: If three cevians divide the three sides of a triangle in the ratios $a_1 : a_2$, $b_1 : b_2$ and $c_1 : c_2$ in a clockwise sense, the cevians are concurrent if and only if $\frac{a_1}{a_2}\frac{b_1}{b_2}\frac{c_1}{c_2} = 1$.

The first mathematician known to have worked on the problem of finding integer-sided triangles with three integer medians seems to have been Euler [6, p. 282] in 1773. Working with simplifying assumptions he produced several such triangles including the two smallest examples namely, $2(68, 85, 87)$ and $2(127, 131, 158)$ (see Figure 2).

These two triangles are related in that the side lengths of the second triangle are twice the median lengths of the first. Euler went on [7, p. 290], in 1778, to investigate triangles with rational vertex-to-centroid lengths (which necessarily implies rational medians) and the following year [7, p. 399] he produced a parametrization to the original problem:

$$\begin{aligned}
a &= m(9m^4 + 26m^2n^2 + n^4) - n(9m^4 - 6m^2n^2 + n^4)\\
b &= m(9m^4 + 26m^2n^2 + n^4) + n(9m^4 - 6m^2n^2 + n^4)\\
c &= 2m(9m^4 - 10m^2n^2 - 3n^4).
\end{aligned}$$

Unfortunately this parametrization turned out to be incomplete as we will show in Chapter 3. Later [8] he produced still more parametrizations of triangles with three integer medians but these all turn out to be essentially the same as the one shown above.

In 1813 N. Fuss [5, p. 210] produced the first examples of triangles with three rational angle bisectors and necessarily rational area e.g. $(14, 25, 25)$. In 1905 H. Schubert [15] thought that he had parametrized all Heron triangles with one integer median and from his parametrization he deduced that no Heron triangle could have two or more integer medians. Dickson showed [5, p. 208] that Schubert's parametrization does not cover all solutions. But the counterexamples given by Dickson only have one integer median, so the existence of Heron triangles with more than one integer median remained open. We take up and advance this particular problem in Chapter 3.

0.2 Personal Perspective

My earliest recollection of triangles, at least of a mathematical nature, is of the delightful proofs of many of the Euclidean propositions during high school. The organised and self-contained nature of these proofs was (and still is) very appealing and likely steered my career into that of a mathematician.

I first looked at a copy of Richard Guy's Unsolved Problems in Number Theory [9] early in 1985 and became interested in several problems such as F25, the largest persistence of a number and F4, the no-three-in-line problem before deciding to make a concerted effort on the innocuous looking D21.

While D21 itself seemed difficult to solve directly there was a rich "sea" of sub-problems in which a mathematician could easily "surf" away the hours. However, following discussions with a colleague, Dr. Roger Eggleton, a more organised and aggressive approach was undertaken. As the work proceeded it became increasingly obvious that the mathematical research was more like an experimental science than the phrase "pure mathematics" would lead us to believe. Constant testing of hypotheses against observations (from the computer searches) and searching for patterns in the data to reveal new hypotheses became the order of the day.

The chapters of this thesis follow similar paths. From the consideration of simple cases, they proceed to the more general formulation of the problem. This often leads to greater generality which subsumes the earlier cases. But the simpler cases are included explicitly because they give an instructive and concrete introduction to the more general problems and solutions presented.

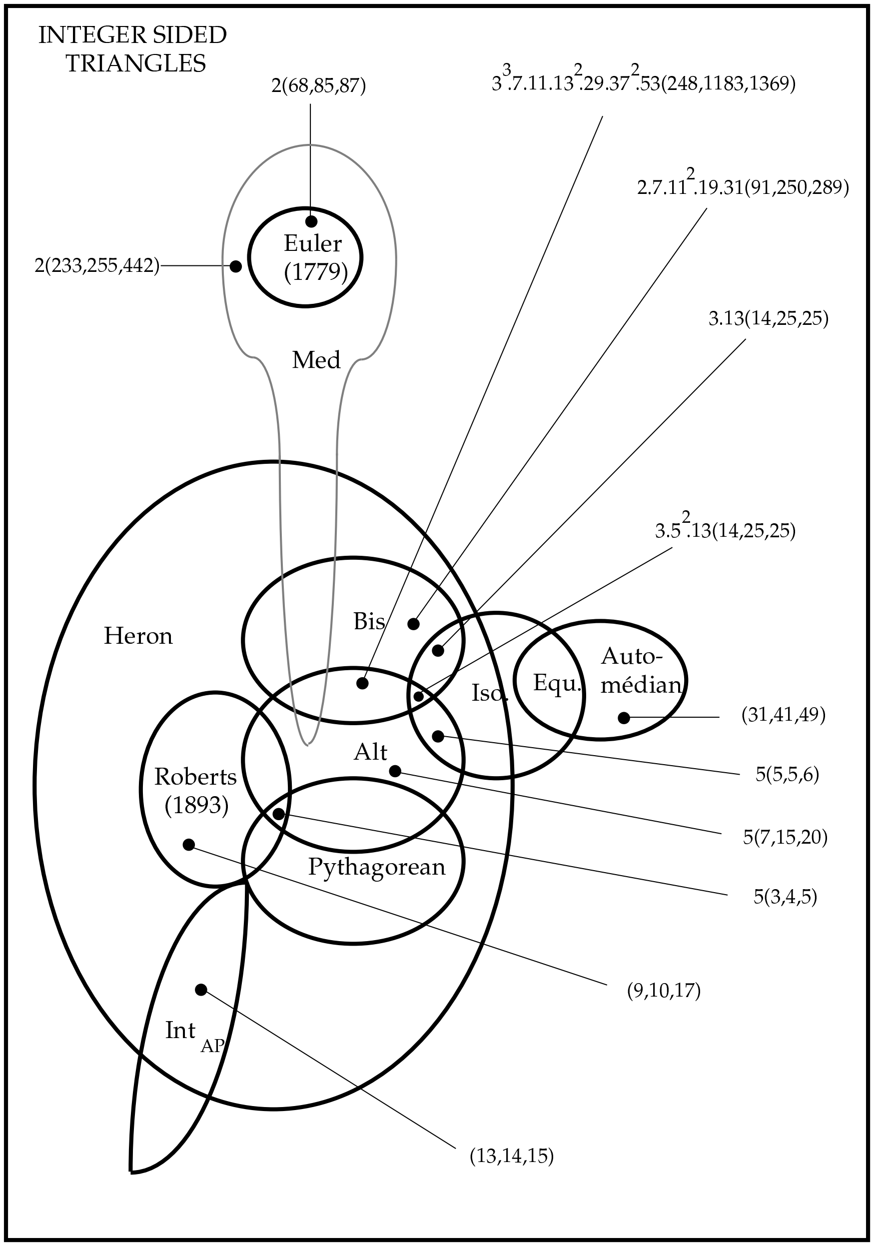

Figure 3: Pictorial Overview

Chapter 1

Three Integer Altitudes

This chapter will mainly consider the subset of integer-sided triangles which have three integer length altitudes. After obtaining the defining equations of the altitudes in terms of the sides the main theme is to derive parametrizations for isosceles triangles, Pythagorean triangles and finally for a general triangle.

1.1 Defining Equations

There are at least two possible methods that can be used to obtain expressions for the lengths of the altitudes of a triangle in terms of the sides. One method uses the expression for area in terms of base and height and then appeals to Heron's formula for the area in terms of the sides. An alternative path is as follows: Denote the lengths of the sides of the triangle by a, b, c and the corresponding altitudes by α, β, γ as in Figure 1.1. Expressing the sine and cosine of $\angle ABC$ in terms of the sides and one altitude and then eliminating $\angle ABC$ gives

$$4c^2\gamma^2 = 4a^2c^2 - (a^2 - b^2 + c^2)^2.$$

By expanding the right hand side to replace the term $4a^2c^2$ by $4a^2b^2$ one obtains a more symmetric equation. Hence the three altitudes satisfy

$$\begin{aligned} \alpha^2 &= \frac{4b^2c^2 - (a^2 - b^2 - c^2)^2}{4a^2} \\ \beta^2 &= \frac{4a^2c^2 - (b^2 - a^2 - c^2)^2}{4b^2} \\ \gamma^2 &= \frac{4a^2b^2 - (c^2 - a^2 - b^2)^2}{4c^2}. \end{aligned} \tag{1.1}$$

Using the relations $a\alpha = b\beta = c\gamma = 2\triangle$, where $\triangle$ is the area of the triangle, one finds the inverse system of equations, expressing the sides in terms of the

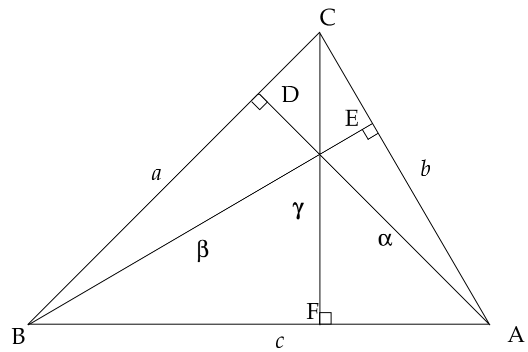

Figure 1.1: Alt Triangle

altitudes:

$$\begin{aligned} a^2 &= \frac{4\alpha^2\beta^4\gamma^4}{4\alpha^4\beta^2\gamma^2 - (\beta^2\gamma^2 - \alpha^2\gamma^2 - \alpha^2\beta^2)^2} \\ b^2 &= \frac{4\alpha^4\beta^2\gamma^4}{4\alpha^2\beta^4\gamma^2 - (\alpha^2\gamma^2 - \alpha^2\beta^2 - \beta^2\gamma^2)^2} \\ c^2 &= \frac{4\alpha^4\beta^4\gamma^2}{4\alpha^2\beta^2\gamma^4 - (\alpha^2\beta^2 - \beta^2\gamma^2 - \alpha^2\gamma^2)^2}. \end{aligned} \tag{1.2}$$

Having found these expressions let us now restrict our attention to integer-sided triangles and analyse some simple cases.
Definition : An alt triangle is a triangle with three integer sides and three integer altitudes. The set of all alt triangles is denoted by Alt.

1.2 Isosceles Restriction

The simplest nontrivial case to attack is that of the isosceles triangle. Without loss of generality, consider the case $a = b$. Then equations (1.1) show $\alpha = \beta$ and they reduce to two equations for α and γ,

$$\begin{aligned} 4\alpha^2 a^2 &= c^2(4a^2 - c^2) \\ 4\gamma^2 &= 4a^2 - c^2. \end{aligned}$$

Combining them reduces the first to $a\alpha = c\gamma$, and the second shows that c must be even. Setting $2C := c$ leads to $C^2 + \gamma^2 = a^2$, which has the primitive solution $a = u^2 + v^2$, $C = 2uv$, $\gamma = u^2 - v^2$ where $u, v \in \mathbb{N}$, $2 \mid uv$ and $\gcd(u, v) = 1$. Now since $\alpha = c\gamma/a$ and $\gcd(u^2 + v^2, 2uv) = 1$ and $\gcd(u^2 + v^2, u^2 - v^2) = 1$, scaling up the parametric solution by α ensures that $\alpha \in \mathbb{N}$, so the general solution in this case is

$$\begin{aligned} a = b &= (u^2 + v^2)^2 \\ c &= 4uv(u^2 + v^2) \\ \alpha = \beta &= 4uv(u^2 - v^2) \\ \gamma &= u^4 - v^4. \end{aligned}$$

Notice that each of the triangles produced by this parametrization generates a related but dissimilar isosceles triangle which also has three integer altitudes. In particular the triangle $5(5, 5, 8)$ generates the triangle $5(5, 5, 6)$. This reflects the algebraic fact that the parameters C and γ can be interchanged. Alternatively, it reflects the geometric fact that each of the isosceles solutions is composed of two copies of the same Pythagorean triangle joined along the same leg. Since this can be done in two possible ways we obtain two sets of solutions.

1.3 Pythagorean Restriction

Consider any integer-sided right-angled triangle referred to as a Pythagorean triangle. It already has two integer altitudes so it is necessary only to coerce the third altitude to become an integer to obtain an alt triangle. For any primitive Pythagorean triangle, (where $a^2 + b^2 = c^2$), the sides are expressible as $a = u^2 - v^2$, $b = 2uv$, $c = u^2 + v^2$ where u and v are relatively prime integers of opposite parity. Then the altitudes are $\alpha = b$, $\beta = a$ and $\gamma = a\alpha/c$. Now as before $\gcd(a, c) = 1$ and $\gcd(\alpha, c) = 1$ so scaling up by c yields the solution

$$\begin{aligned} a = \beta &= u^4 - v^4 \\ b = \alpha &= 2uv(u^2 + v^2) \\ c &= (u^2 + v^2)^2 \\ \gamma &= 2uv(u^2 - v^2). \end{aligned}$$

1.4 Area of Alt Triangles

At this stage one might conjecture that any alt triangle must have integer area. In fact, this turns out to be true but is not entirely obvious since at this point it is conceivable that there are instances in which all the sides and altitudes are odd. To show that this is never the case, via Theorem 1, we need two preliminary lemmas.

Lemma 1 *If $a, b, c, \gamma \in \mathbb{N}$ and c_1 and c_2 are the distances from the foot of the altitude γ to the endpoints of the side c then $c_1, c_2 \in \mathbb{N}$.*

Proof : By definition $c_1^2 = a^2 - \gamma^2$ and $c_2^2 = b^2 - \gamma^2$. So there exists $e_1, e_2 \in \mathbb{N}$ such that $c_1^2 = e_1$ and $c_2^2 = e_2$. But $c_1 + c_2 = c$, so

$$\sqrt{e_1} + \sqrt{e_2} = c.$$

Squaring leads to $2\sqrt{e_1e_2} = c^2 - e_1 - e_2 \in \mathbb{N}$. Hence there exists an $n \in \mathbb{N}$ such that

$$4e_1e_2 = n^2.$$

So $n = 2N$ for some $N \in \mathbb{N}$ and $e_1e_2 = N^2$. If $\gcd(e_1, e_2) = g > 1$ then there exists $E_1, E_2 \in \mathbb{N}$ such that $e_1 = gE_1$, $e_2 = gE_2$ and $\gcd(E_1, E_2) = 1$. Substituting these gives $g^2E_1E_2 = N^2$. Hence g divides N, so $E_1E_2 = m^2$ for some $m \in \mathbb{N}$. By the Fundamental Theorem of Arithmetic, there exists $\varepsilon_1, \varepsilon_2 \in \mathbb{N}$ such that $E_1 = \varepsilon_1^2$ and $E_2 = \varepsilon_2^2$. Since g can be written as $g = k^2s$ where $k, s \in \mathbb{N}$ and s is squarefree, substituting into $c_1 + c_2 = c$ gives

$$k(\varepsilon_1 + \varepsilon_2)\sqrt{s} = c$$

whence $s = 1$ and $c_1, c_2 \in \mathbb{N}$. ■

Lemma 2 *Let* $a, b, c, \alpha, \beta, \gamma \in \mathbb{N}$. *Then* $\gcd(a, b, c, \alpha, \beta, \gamma) \equiv 0 \pmod 2$ *iff* $\gcd(a, b, c) \equiv 0 \pmod 2$.

Proof : $\Longrightarrow$ If $\gcd(a, b, c, \alpha, \beta, \gamma) \equiv 0 \pmod 2$ then 2 divides each of a, b, c and so $\gcd(a, b, c) \equiv 0 \pmod 2$.

$\Longleftarrow$ If $\gcd(a, b, c) \equiv 0 \pmod 2$ then there exist $A, B, C \in \mathbb{N}$ such that $a = 2A$, $b = 2B$, $c = 2C$. Using the same notation as in Lemma 1, Pythagoras's Theorem gives

$$c_1^2 + \gamma^2 = (2A)^2$$

where $c_1 \in \mathbb{N}$ by Lemma 1 and $\gamma \in \mathbb{N}$ by assumption. Consequently c_1 and γ have the same parity. But from the form of Pythagorean triplets at least one of c_1 and γ must be even – hence both are even. By symmetry α and β are even and so $\gcd(a, b, c, \alpha, \beta, \gamma)$ is even. ■

Theorem 1 *If* $a, b, c, \alpha, \beta, \gamma \in \mathbb{N}$ *and* $\gcd(a, b, c, \alpha, \beta, \gamma) = 1$ *then exactly one of* a, b *and* c *is even.*

Proof : The α equation in (1.1) yields the modulo 2 congruence

$$(a^2 - b^2 - c^2)^2 \equiv 0 \pmod 2.$$

Hence $a + b + c \equiv 0 \pmod 2$ and so an odd number of the sides is even. But the possibility that $\gcd(a, b, c)$ is even contradicts our assumption about the primitivity of the six parameters, by Lemma 2. Hence exactly one side is even. ■

Corollary 1 *All triangles in* **Alt** *have integer area.*

Proof : Without loss of generality let the even side be c. Then $c\gamma/2 \in \mathbb{N}$. ■

Corollary 2 *If* $\gcd(a, b, c, \alpha, \beta, \gamma) = 1$ *then at least two of* α, β, γ *are even.*

Proof : Since $\triangle = a\alpha/2 = b\beta/2 = c\gamma/2 \in \mathbb{N}$ then if any two sides are odd the corresponding two altitudes must be even. ■

1.5 Related Alt Triangles

A program written to generate all primitive solutions to equations (1.1) with largest side less than or equal to 1000 turned up 17 such triangles. Scrutinising the results in Table 1.1 leads one to the conclusion that the two related isosceles alt triangles mentioned earlier could generate a third distinct, scalene alt triangle.

By Lemma 1 any altitude of an alt triangle decomposes that triangle into two Pythagorean triangles. So an alt triangle has associated with it at most 6 Pythagorean triangles which will be referred to as the π-set. The triangles ADB and CFB in Figure 1.1 belong to the same similarity class, so they must each be integer-scaled versions of the same primitive Pythagorean triangle. Since the same is true for other pairs of right triangles in Figure 1.1 the π-set of any alt triangle contains at most three distinct primitive Pythagorean triangles. To clarify the connection between alt triangles alluded to in the above paragraph imagine a hinge at the foot of each altitude (i.e. D, E and F). Rotate each pair of Pythagoreans either side of an altitude about the hinge, keeping them coplanar, until the old bases meet and become a new altitude (after appropriate rescaling). This leads to three potentially new alt triangles each of which have the same π-set. With the original triangle this yields a set of four related triangles.

Semiperimeter	Sides			Altitudes			Area
s	a	b	c	α	β	γ	$\triangle$
30	15	20	25	20	15	12	150
40	25	25	30	24	24	20	300
45	25	25	40	24	24	15	300
105	35	75	100	60	28	21	1050
195	65	156	169	156	65	60	5070
234	130	169	169	156	120	120	10140
340	136	255	289	255	136	120	17340
425	272	289	289	255	240	240	34680
325	169	169	312	120	120	65	10140
544	289	289	510	240	240	136	34680
700	175	600	625	600	175	168	52500
800	350	625	625	600	336	336	105000
825	275	625	750	600	264	220	82500
1015	580	609	841	609	580	420	176610
1040	260	845	975	780	240	208	101400
1365	845	910	975	840	780	728	354900
1300	625	975	1000	936	600	585	292500

Table 1.1: Alt Triangles

Let $\{(a, b, c)\}$ represent the similarity class of a triangle with sides (a, b, c) and altitudes α, β, γ respectively. Also let f_α denote the rotation and appropriate rescaling about the altitude α to produce a related triangle and similarly for f_β and f_γ. Clearly, 'hinging' about the altitude α transforms $\{(a,b,c)\}$ into the related $\{(a\alpha, ba_2, ca_1)\}$. The new altitudes of the triangle $(a\alpha, ba_2, ca_1)$ are $\alpha' = a_1a_2$, $\beta' = \beta a_1$, $\gamma' = \gamma a_2$. These transformations can be written explicitly as

$$\begin{aligned} f_\alpha\{(a,b,c)\} &= \{(a\alpha, ba_2, ca_1)\} \\ f_\beta\{(a,b,c)\} &= \{(ab_1, b\beta, cb_2)\} \\ f_\gamma\{(a,b,c)\} &= \{(ac_2, bc_1, c\gamma)\} \end{aligned}$$

where a_1, a_2, b_1, b_2, c_1, c_2 are the base segments as in Lemma 1. For example, hinging $(a,b,c) = 5(5,5,6)$ about the altitude $\gamma = 20$ leads to the triangle $5(5,5,8)$ as before. But hinging it about $\beta = 24$ and rescaling leads to the alt triangle $5(15,20,7)$, while hinging and rescaling about $\alpha = 24$ leads to $5(20,15,7)$. Now hinging a new triangle about the same altitude leads back to the original triangle since

$$\begin{aligned} f_\alpha\{f_\alpha\{(a,b,c)\}\} &= f_\alpha\{(a\alpha, ba_2, ca_1)\} \\ &= \{(a\alpha a_1a_2, ba_2a_2', ca_1a_1')\} \end{aligned}$$

where

$$\begin{aligned} a_1' &= \sqrt{(ba_2)^2 - (a_1a_2)^2} = a_2\sqrt{b^2 - a_1^2} = a_2\alpha \\ a_2' &= \sqrt{(ca_1)^2 - (a_1a_2)^2} = a_1\sqrt{c^2 - a_2^2} = a_1\alpha. \end{aligned}$$

So

$$\begin{aligned} f_\alpha\{f_\alpha\{(a,b,c)\}\} &= \{(a\alpha a_1a_2, ba_2a_1\alpha, ca_1a_2\alpha)\} \\ &= \{(a,b,c)\}. \end{aligned}$$

Hence $(f_\alpha)^2 = 1$. Similarly $(f_\beta)^2 = 1$ and $(f_\gamma)^2 = 1$. Next consider the composition $f_\beta f_\alpha$. From the definitions

$$\begin{aligned} f_\beta\{f_\alpha\{(a,b,c)\}\} &= f_\beta\{(a\alpha, ba_2, ca_1)\} \\ &= \{(a\alpha b_1', ba_2\beta a_1, ca_1b_2')\}. \end{aligned}$$

where

$$\begin{aligned} b_1' &= \sqrt{(ca_1)^2 - (\beta a_1)^2} = a_1\sqrt{c^2 - \beta^2} = a_1b_1 \\ b_2' &= \sqrt{(a\alpha)^2 - (\beta a_1)^2} = a_1\sqrt{(b\beta)^2 - (\beta a_1)^2} = \beta\alpha. \end{aligned}$$

So

$$\begin{aligned} f_\beta\{f_\alpha\{(a,b,c)\}\} &= \{(a\alpha a_1b_1, ba_2\beta a_1, ca_1\beta\alpha)\} \\ &= \{\frac{\alpha\beta a_1}{\gamma}\left(\frac{ab_1\gamma}{\beta}, \frac{ba_2\gamma}{\alpha}, c\gamma\right)\} \\ &= \{\frac{\alpha\beta a_1}{\gamma}(ac_2, bc_1, c\gamma)\} \\ &= f_\gamma\{(a,b,c)\}. \end{aligned}$$

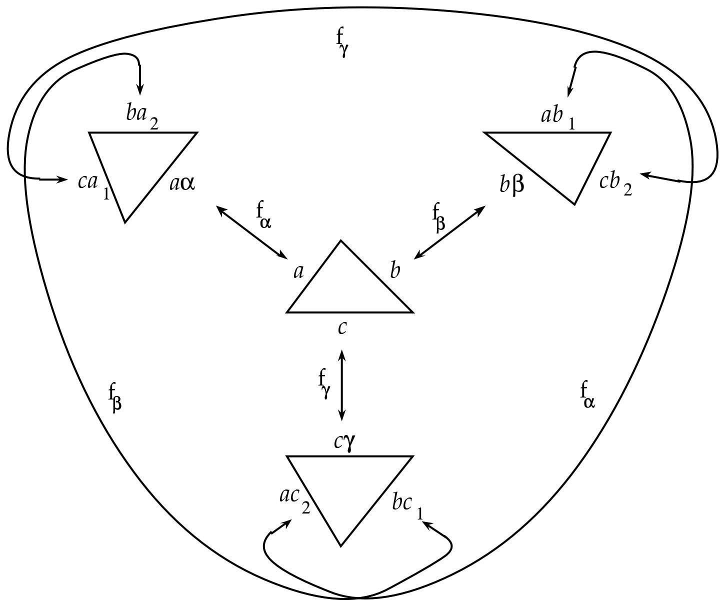

Figure 1.2: Four related alt triangles

Hence $f_\beta f_\alpha = f_\gamma$. Similarly $f_\alpha f_\beta = f_\gamma$, and $f_\alpha f_\gamma = f_\gamma f_\alpha = f_\beta$ while $f_\beta f_\gamma = f_\gamma f_\beta = f_\alpha$. So under composition these transformations form a group acting on similarity classes of alt triangles. The group table is

$\circ$	1	f_α	f_β	f_γ
1	1	f_α	f_β	f_γ
f_α	f_α	1	f_γ	f_β
f_β	f_β	f_γ	1	f_α
f_γ	f_γ	f_β	f_α	1

and the group structure is isomorphic to $C_2 \times C_2$. The relationship between the similarity classes of triangles is shown in Figure 1.2.

Allowing the hinge at the foot of each altitude to become a three dimensional pivot increases the number of related alt triangles from 4 to infinity. Now there are eight ways in which any pair of Pythagorean triangles about a given altitude can meet at their "legs" to form, after appropriate rescaling, a potentially new alt triangle. For the "hinge" transformation the π-sets of the new alt triangles were identical to the original π-set which led to the "early closure" of the set of related alt triangles. However, the "pivot" transformation generates alt triangles with different π-sets and does not close the related set of alt triangles - in fact it becomes infinite in size.

Begin with two arbitrary Pythagorean triangles (a, b, c) and (d, e, f) where c

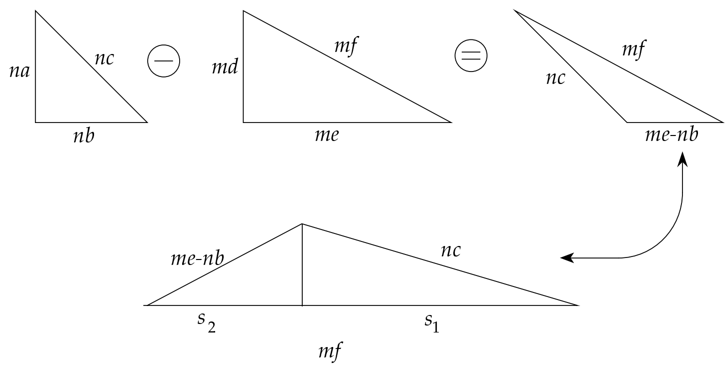

Figure 1.3: Generating a Heron triangle

and f are the hypotenuses. If they are dissimilar the legs of each can be paired in four ways then either adding or subtracting the areas of the Pythagoreans results in the aforementioned eight alt triangles. For example matching side a with side d means that there exist integers m and n such that $na = md$. Supposing, without loss of generality, that $nb < me$ leads to an alt triangle of $(nc, mf, me - nb)$ by subtracting areas and $(nc, mf, me + nb)$ by adding areas (see Figure 1.3).

In the "subtraction case" the altitude to the side mf is given by $2\triangle/mf$ which is just $(me - nb)d/f$. If mf is split into the two base segments s_1 and s_2, which are integers by Lemma 1, then the two Pythagoreans either side of this altitude are similar to $(s_1 f, d(me - nb), ncf)$ and $(s_2 f, d(me - nb), f(me - nb))$. Using $na = md$ and Pythagoras' Theorem shows that these are similar to $(ad + be, ae - bd, cf)$ and (e, d, f) i.e. one "new" and one "old". Interchanging a with b leads to the two Pythagoreans $(ad + be, ae - bd, cf)$ and $(bd + ae, be - ad, cf)$. It turns out that any other pairing of the two original Pythagoreans will only produce alt triangles which have as π-sets any three of $\{(a, b, c), (d, e, f), (ad + be, ae - bd, cf), (bd + ae, be - ad, cf)\}$. As a specific example begin with the alt triangle $65(13, 14, 15)$ which has the π-set of $\{(3, 4, 5), (5, 12, 13), (33, 56, 65)\}$. Pairing up the first two Pythagorean triangles in the above eight ways leads to eight alt triangles with their associated π-sets (see Table 1.2).

Clearly the appearance of the Pythagorean triangle $(16, 63, 65)$ which is not a member of the original π-set shows that the process becomes self-propagating. Indeed, the union of the π-sets of all the alt triangles produced from each pair of the three Pythagoreans in the π-set of $65(13, 14, 15)$ is

$$\{(3, 4, 5), (5, 12, 13), (33, 56, 65), (16, 63, 65), (36, 323, 325), (116, 837, 845)\}.$$

Finally since the same process can be applied to any pair of the Pythagoreans in a recursive manner the union of the π-sets contain only those Pythagoreans with hypotenuses of the form $5^\alpha 13^\beta$ for non-negative integers α and β. For the two arbitrary initial Pythagorean triangles the hypotenuses in the π-set are of the form $c^\alpha f^\beta$.

Legs	Alt Triangle	π-set
4,12	$(13, 14, 15)$	$\{(3, 4, 5), (5, 12, 13), (33, 56, 65)\}$
	$(4, 13, 15)$	$\{(3, 4, 5), (5, 12, 13), (16, 63, 65)\}$
3,12	$(13, 20, 21)$	$\{(3, 4, 5), (5, 12, 13), (16, 63, 65)\}$
	$(11, 13, 20)$	$\{(3, 4, 5), (5, 12, 13), (33, 56, 65)\}$
4,5	$(25, 52, 63)$	$\{(3, 4, 5), (5, 12, 13), (16, 63, 65)\}$
	$(25, 33, 52)$	$\{(3, 4, 5), (5, 12, 13), (33, 56, 65)\}$
3,5	$(25, 39, 56)$	$\{(3, 4, 5), (5, 12, 13), (33, 56, 65)\}$
	$(16, 25, 39)$	$\{(3, 4, 5), (5, 12, 13), (16, 63, 65)\}$

Table 1.2: π-sets of the $(13, 14, 15)$ triangle

1.6 General Parametrization

By virtue of the equations $a\alpha = b\beta = c\gamma = 2\triangle$, all Heron triangles must have three rational altitudes while by Theorem 1, all alt triangles have integer area. Consequently we can scale up Heron triangles to produce alt triangles and be sure that no alt triangle is excluded by this process.

Carmichael [3, p. 12] showed that all Heron triangles have sides proportional to a, b and c where

$$\begin{aligned} a &= n(m^2 + k^2) \\ b &= m(n^2 + k^2) \\ c &= (m + n)(mn - k^2) \end{aligned} \tag{1.3}$$

for some integers m, n, k such that $mn > k^2$. Note that different sets of triples (m, n, k) can produce the same primitive Heron triple e.g.

$$\begin{aligned} (m, n, k) = (2, 1, 1) &\Longrightarrow (a, b, c) = (5, 4, 3) \\ (m, n, k) = (3, 1, 1) &\Longrightarrow (a, b, c) = 2(5, 3, 4) \\ (m, n, k) = (3, 2, 1) &\Longrightarrow (a, b, c) = 5(4, 3, 5). \end{aligned}$$

To eliminate this degeneracy it is useful to invert equations (1.3) to obtain m, n, k in terms of a, b, c.

Lemma 3 *For any Heron triple (a, b, c) the parameters (m, n, k) of equations 1.3 are given by*

$$\begin{aligned} m &= (a - b + c)(a + b + c) \\ n &= (b - a + c)(a + b + c) \\ k &= 4\triangle. \end{aligned}$$

Proof : Three useful linear combinations of equations (1.3) are

$$\begin{aligned} a+b+c &= 2mn(m+n) \\ a+b-c &= 2k^2(m+n) \\ a-b &= (m-n)(mn-k^2). \end{aligned}$$

Dividing the first two of these yields

$$mn = k^2\left(\frac{a+b+c}{a+b-c}\right)$$

and substituting this into the third equation leads to

$$m-n = \frac{(a+b-c)(a-b)}{2ck^2}.$$

The second equation yields $m+n = \frac{a+b-c}{2k^2}$ hence solving these last two for m and n leads to

$$\begin{aligned} m &= \frac{(a+b-c)(a-b+c)}{4ck^2} \\ n &= \frac{(a+b-c)(-a+b+c)}{4ck^2}. \end{aligned}$$

Substituting these two expressions into the expression for the product mn yields k in terms of a, b and c only i.e.

$$k^6 = \frac{(a+b-c)^3(a-b+c)(-a+b+c)}{16c^2(a+b+c)}.$$

Using this to eliminate k from the expressions for m and n one obtains

$$\begin{aligned} m^3 &= \frac{(a-b+c)^2(a+b+c)}{4c(-a+b+c)} \\ n^3 &= \frac{(-a+b+c)^2(a+b+c)}{4c(a-b+c)}. \end{aligned}$$

Multiplying the expression for k^6 by $(a+b+c)/(a+b+c)$ and rearranging leads to

$$k^3 = \frac{\triangle(a+b-c)}{c(a+b+c)}.$$

Scaling up these last three expressions by the factor $4c(a+b+c)^2(a-b+c)(b+c-a)$ to make them integers leads to

$$\begin{aligned} m^3 &= (a-b+c)^3(a+b+c)^3 \\ n^3 &= (b-a+c)^3(a+b+c)^3 \\ k^3 &= (4\triangle)^3 \end{aligned}$$

from which we obtain the required result. ■

Selecting any particular Heron triangle (a, b, c) it is possible to permute the sides in 6 ways to produce 6 potentially distinct corresponding sets of the parameters (m, n, k). To select exactly one member from this set of six one need simply choose $a \geq b \geq c$. This inequality leads to

$$a - b + c \geq b - a + c.$$

So by Lemma 3 we have $m \geq n$. Furthermore $b \geq c$ implies by equations (1.3) that $k^2 \geq m^2 n/(2m + n)$. To exclude any scale multiples of smaller Heron triangles reappearing a final constraint is that $\gcd(m, n, k) = 1$. Hence we have proved:

Theorem 2 *Exactly one member of each similarity class of Heron triangles is obtained from Carmichael's parametrization by applying the constraints*

$$\gcd(m, n, k) = 1, m \geq n \geq 1$$

$$mn > k^2 \geq \frac{m^2 n}{2m + n}.$$

Now to obtain a general parametrization for alt triangles substitute (1.3) into the equations $a\alpha = b\beta = c\gamma = 2\triangle$ which results in

$$\begin{aligned} \alpha &= \frac{2km(m+n)(mn-k^2)}{m^2+k^2} \\ \beta &= \frac{2kn(m+n)(mn-k^2)}{n^2+k^2} \\ \gamma &= 2kmn. \end{aligned}$$

To ensure that α and β are both integers scale up the triangle by the factor $(m^2 + k^2)(n^2 + k^2)$. So the Heron triangle defined by equations (1.3) gives rise to the alt triangle defined by

$$\begin{aligned} a &= n(m^2+k^2)^2(n^2+k^2) \\ b &= m(m^2+k^2)(n^2+k^2)^2 \\ c &= (m+n)(mn-k^2)(m^2+k^2)(n^2+k^2) \\ \alpha &= 2km(m+n)(mn-k^2)(n^2+k^2) \\ \beta &= 2kn(m+n)(mn-k^2)(m^2+k^2) \\ \gamma &= 2kmn(m^2+k^2)(n^2+k^2). \end{aligned}$$

A question that now arises is the following. Is the scale factor from primitive Heron triangle to primitive alt triangle an integer? Consider all alt triangles such that $\gcd(a, b, c, \alpha, \beta, \gamma) = 1$ and let $g := \gcd(a, b, c)$. Then $g \equiv 1 \pmod 2$ by Lemma 2. Also by Theorem 1 exactly one of a, b and c is even so without loss of generality let $2A := a$ hence $g \mid A$. Now $\triangle = a\alpha/2 = A\alpha$ while from equations

(1.1) $g \mid \alpha$ so that $g^2 \mid \triangle$. Consequently the greatest common divisor of a, b and c is also a divisor of the area and can be removed, leaving a primitive Heron triangle of the form $(a/g, b/g, c/g, \triangle/g^2)$. In summary, there is a one-to-one correspondence between the similarity classes of alt triangles and the similarity classes of Heron triangles given by the parametrization above. Furthermore alt triangles are related by their decomposition into related sets of Pythagorean triangles.

Chapter 2

Three Integer Angle Bisectors

2.1 Defining Equations

In this chapter we will consider integer-sided triangles with integer length angle bisectors. Denote the bisectors by p, q, r and, as before, denote the sides by a, b, c as in Figure 2.1. The sine rule applied to triangles BFC and AFC leads to

$$\frac{\overline{BF}}{\sin C/2} = \frac{r}{\sin B} \qquad \text{and} \qquad \frac{c - \overline{BF}}{\sin C/2} = \frac{r}{\sin A}.$$

Eliminating the length $\overline{BF}$ from these two expressions yields

$$\frac{c}{r} = \left(\frac{1}{\sin B} + \frac{1}{\sin A}\right)\sqrt{\frac{1-\cos C}{2}}.$$

Since the sines and cosines of the angles can be expressed as functions of the sides (via the cosine rule) the angle bisector r can be expressed in terms of the sides of the triangle. By analogous reasoning on the other two pairs of triangles one obtains the three defining equations for the angle bisectors

$$\begin{aligned} p^2 &= \frac{bc(b+c-a)(a+b+c)}{(b+c)^2} \\ q^2 &= \frac{ac(a+c-b)(a+b+c)}{(a+c)^2} \\ r^2 &= \frac{ab(a+b-c)(a+b+c)}{(a+b)^2}. \end{aligned} \tag{2.1}$$

Definition : A bis triangle is a triangle with three integer sides and three integer angle bisectors. The set of all bis triangles will be denoted by Bis.

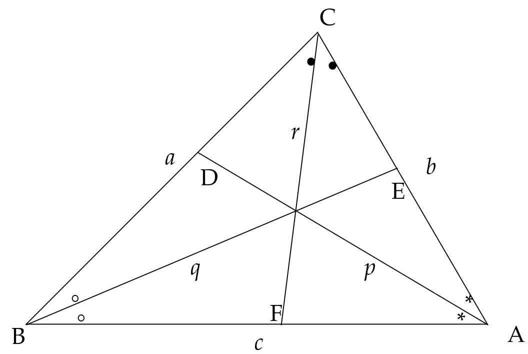

Figure 2.1: Bis Triangle

2.2 Isosceles Restriction

As in Chapter 1, it is convenient to begin with the relatively simple task of deciding the existence or otherwise of isosceles bis triangles. If we require that case $a = b$, then $p = q$ by equations (2.1), which reduce to

$$\begin{aligned} 4r^2 &= 4a^2 - c^2 \\ p^2 &= c^2(2a^2 + ac)/(a + c)^2. \end{aligned}$$

From the first of these c must be even so let $2C := c$. This leads to the Pythagorean equation $r^2+C^2 = a^2$, which has the general solution of $a = u^2+v^2$, $C = u^2 - v^2$, $r = 2uv$ where $u, v \in \mathbb{N}$, $2 \mid uv$ and $\gcd(u, v) = 1$. Substituting these into the second equation of the above pair leads to

$$p = \frac{2u(2u^2 - 2v^2)}{(3u^2 - v^2)}\sqrt{u^2 + v^2}.$$

Since we require p to be an integer the expression under the radical must be the square of an integer, say w, and so $u^2 + v^2 = w^2$. Again resorting to the general solution of the Pythagorean equation leads to $u = 2xy$, $v = x^2 - y^2$, $w = x^2 + y^2$ where $x, y \in \mathbb{N}$, $2 \mid xy$ and $\gcd(x, y) = 1$, or the corresponding equations with u and v interchanged. Scaling up the triangle by the factor $3u^2 - v^2$ gives us the general parametrization of isosceles bis triangles:

$$\begin{aligned} a = b &= (u^2 + v^2)(3u^2 - v^2) \\ c &= (2u^2 - 2v^2)(3u^2 - v^2) \\ p = q &= 2uw(2u^2 - 2v^2) \\ r &= 2uv(3u^2 - v^2) \end{aligned} \tag{2.2}$$

where $(u, v, w) = (2xy, x^2 - y^2, x^2 + y^2)$ or $(x^2 - y^2, 2xy, x^2 + y^2)$. Notice that C and r could have been interchanged earlier which would have led to a different expression for p, namely

$$p = \frac{4uv(u+v)}{(u^2 + 4uv + v^2)} \sqrt{2u^2 + 2v^2}.$$

Now the requirement that p be an integer leads to $2u^2 + 2v^2 = w^2$. Clearly w must be even, so define W by $2W := w$. Then $u^2 + v^2 = 2W^2$ and u and v have the same parity. Defining U and V by $U + V := u$ and $U - V := v$ leads to $U^2 + V^2 = W^2$. Solving this in the usual way yields

$$\begin{aligned} u &= x^2 + 2xy - y^2 \\ v &= y^2 + 2xy - x^2 \text{ or } x^2 - 2xy - y^2 \\ w &= 2x^2 + 2y^2. \end{aligned}$$

This time scale up by the factor $u^2 + 4uv + v^2$, to give the parametrization

$$\begin{aligned} a = b &= (u^2 + v^2)(u^2 + 4uv + v^2) \\ c &= 4uv(u^2 + 4uv + v^2) \\ p = q &= 4uvw(u + v) \\ r &= (u^2 - v^2)(u^2 + 4uv + v^2). \end{aligned} \tag{2.3}$$

Substituting the expressions for u, v and w into equations (2.3) shows that this parametrization is just four times the one in equations (2.2). So finally the primitive isosceles bis triangles are given explicitly by
(1) If $x^2 - y^2 < 2xy$ then

$$\begin{aligned} a = b &= (x^2 + y^2)^2(14x^2y^2 - x^4 - y^4) \\ c &= 2(6x^2y^2 - x^4y^4)(14x^2y^2 - x^4 - y^4) \\ p = q &= 8xy(x^2 + y^2)(6x^2y^2 - x^4 - y^4) \\ r &= 4xy(x^2 - y^2)(14x^2y^2 - x^4 - y^4) \end{aligned}$$

(2) and if $x^2 - y^2 > 2xy$ then

$$\begin{aligned} a = b &= (x^2 + y^2)^2(3x^4 - 10x^2y^2 + 3y^4) \\ c &= 2(x^4 - 6x^2y^2 + y^4)(3x^4 - 10x^2y^2 + 3y^4) \\ p = q &= 4(x^4 - y^4)(x^4 - 6x^2y^2 + y^4) \\ r &= 4xy(x^2 - y^2)(3x^4 - 10x^2y^2 + 3y^4). \end{aligned}$$

Note that only one of the above pair will produce a bis triangle for any particular $x, y \in \mathbb{N}$. This leads to no obvious classification for bis triangles, as happened for alt triangles (and as will happen for triangles with three integer medians).

2.3 Pythagorean Restriction

Do there exist any Pythagorean bis triangles? The answer in the negative is obtained from Theorem 3. However in the process of proving this we obtain a general parametrization of Pythagorean triangles with one integer angle bisector, which turns out to be a solution to Diophantus' Problem 16 Book VI.

Theorem 3 *If $a^2 + b^2 = c^2$ then at most one of the angle bisectors of the triangle (a, b, c) can have rational length.*

Proof : Substituting $a^2 + b^2 = c^2$ into the defining equations (2.1) gives

$$
\begin{aligned}
p^2 &= 2b^2c/(b+c) \\
q^2 &= 2a^2c/(a+c) \\
r^2 &= 2a^2b^2/(a+b)^2.
\end{aligned}
$$

But the third equation leads to

$$\frac{r^2(a+b)^2}{a^2b^2} = 2$$

which cannot be solved in integers since $\sqrt{2}$ is irrational. Hence $(a, b, c) \notin \mathbf{Bis}$. Now since (a, b, c) form a Pythagorean triangle let us assume without loss of generality that $a = u^2 - v^2$, $b = 2uv$, $c = u^2 + v^2$ where $2 \mid uv$ and $\gcd(u, v) = 1$. Then p and q are given by

$$
\begin{aligned}
p &= \frac{(u^2 - v^2)\sqrt{u^2 + v^2}}{u} \\
q &= \frac{2uv\sqrt{2u^2 + 2v^2}}{u + v}.
\end{aligned}
$$

For both p and q to be integers we require that $u^2 + v^2$ and $2u^2 + 2v^2$ be simultaneously squares of integers. But this is impossible as $u^2 + v^2 = w^2$ and $2u^2 + 2v^2 = x^2$ imply that $2w^2 = x^2$. So finally letting $u = 2UV$, $v = U^2 - V^2$, $w = U^2 + V^2$ where $2 \mid UV$ and $\gcd(U, V) = 1$ leads to the parametrization of Pythagorean triangles with one integer angle bisector,

$$
\begin{aligned}
a &= 2UV(U^2 - V^2)^2 \\
b &= 8U^2V^2(U^2 - V^2) \\
c &= 2UV(U^2 + V^2)^2 \\
q &= (U^2 + V^2)(6U^2V^2 - U^2 - V^2)
\end{aligned}
$$

providing the required result. ∎

2.4 Sides in an Arithmetic Progression

For completeness regarding the set intersections of Figure 2.1 we will now consider bis triangles in which the sides are in an arithmetic progression. Let $(a, b, c) := (e - d, e, e + d)$. Then from equations (2.1) we get

$$\begin{aligned} 4e^2q^2 &= (e^2 - d^2)\{(e-d)^2 + (e+d)^2 + 2(e^2 - d^2) - e^2\} \\ 4q^2 &= 3(e^2 - d^2). \end{aligned}$$

Clearly $3 \mid q$ so set $3Q := q$ to give the equation $d^2 + 3(2Q)^2 = e^2$ whose general solution is

$$\begin{aligned} d &= u^2 - 3v^2 \\ e &= u^2 + 3v^2 \\ 2Q &= 2uv. \end{aligned}$$

Now consider the p and r equations from (2.1). They give

$$\begin{aligned} (2e + d)^2p^2 &= e(e + d)(3e^2 + 6ed) \\ (2e - d)^2r^2 &= e(e - d)(3e^2 - 6ed). \end{aligned}$$

Substituting for d and e in terms of u and v leads to

$$\begin{aligned} (3u^2 + 3v^2)^2p^2 &= 32(u^2 + 3v^2)^2(2u^2)(u^2 - v^2) \\ (u^2 - 9v^2)2r^2 &= 32(u^2 + 3v^2)^2(2v^2)(-u^2 + 9v^2). \end{aligned}$$

So for p and q to be rational we require that $2(u^2 - v^2)$ and $2(9v^2 - u^2)$ are simultaneously squares. So letting $x^2 := 2(u^2 - v^2)$, and $y^2 := 2(9v^2 - u^2)$, we obtain after rearrangement

$$\begin{aligned} x^2 + y^2 &= (4v)^2 \\ 9x^2 + y^2 &= (4u)^2. \end{aligned}$$

By a result of Collins [5, p. 475] this last pair of quadratic forms cannot be squares together. Note that this was also proved by Diophantus and became his Proposition 15. We also give an independent proof in Lemma 4.

2.5 Roberts Triangles

In 1893, C.A. Roberts [5, p. 198] showed that any triangle with sides given by $(a, b, c) = (x^2 + 2y^2, x^2 + 4y^2, 2x^2 + 2y^2)$ has integer area. We shall refer to them as Roberts triangles. As we shall show in Chapter 3, the motivation for considering Roberts' triangles is that no such triangle has three integer medians. It turns out that a Roberts triangle cannot have three integer angle bisectors either.

Substitute Roberts' parametrization into equations (2.1) to give

$$\begin{aligned} 9p^2 &= 16(x^2+4y^2)(x^2+y^2) \\ (3x^2+4y^2)q^2 &= 16x^2(x^2+2y^2)^2(x^2+y^2) \\ (2x^2+6y^2)r^2 &= 16y^2(x^2+2y^2)^2(x^2+4y^2). \end{aligned}$$

Now rearranging these and taking the square root we find that

$$\begin{aligned} p &= \frac{4\sqrt{x^2+4y^2}\sqrt{x^2+y^2}}{3} \\ q &= \frac{4x(x^2+2y^2)\sqrt{x^2+y^2}}{3x^2+4y^2} \\ r &= \frac{2y(x^2+2y^2)\sqrt{x^2+4y^2}}{2x^2+6y^2}. \end{aligned}$$

So if we require p, q and r to be rational we must have x^2+4y^2 and x^2+y^2 simultaneously squares of integers. This is also impossible, again by Collins' result [5, p. 475].

2.6 Area of Bis Triangles

W. Rutherford [5, p. 212] showed that triangles in which all three angle bisectors have rational length must also have rational area. Here we will review and extend this result to show that bis triangles have integer area. Of course this does not mean that all Heron triangles have rational angle bisectors - the set of Pythagorean triangles is an obvious counterexample.

Theorem 4 *The area of any bis triangle is rational.*

Proof : If we rewrite equations (2.1) in terms of the semiperimeter s then we get

$$\begin{aligned} p^2(b+c)^2 &= 4bcs(s-a) \\ q^2(a+c)^2 &= 4acs(s-b) \\ r^2(a+b)^2 &= 4abs(s-c). \end{aligned}$$

The product of these three expressions gives

$$p^2(b+c)^2q^2(a+c)^2r^2(a+b)^2 = 64a^2b^2c^2s^2\triangle^2$$

so $\triangle = \frac{pqr(b+c)(a+c)(a+b)}{4abc(a+b+c)}$. Hence the area is clearly rational when $a, b, c, p, q, r \in \mathbb{N}$. ■

Theorem 5 *If* $(a, b, c) \in$ **Bis** *then the perimeter of the triangle must be even.*

Proof : Let $P := a + b + c$. Assuming that P is odd leads to two cases.
Case (i): $a, b, c \equiv 1 \pmod 2$. From equations (2.1) we find that

$$p(b + c) \equiv bc \mod 2.$$

But this is impossible since $b + c$ is even while bc is odd.
Case (ii): $a, b \equiv 0 \pmod 2$, $c \equiv 1 \pmod 2$. Now suppose $2^m \parallel a$ (that is, the largest power of two dividing a is 2^m) and $2^n \parallel b$, where without loss of generality we take $m \geq n \geq 1$. Then let $\alpha := a/2^m$ and $\beta := b/2^n$, where α and β are both odd. Substituting into equations (2.1) gives

$$\begin{aligned} p^2(2^n\beta + c)^2 &= 2^n\beta c(2^m\alpha + 2^n\beta + c)(-2^m\alpha + 2^n\beta + c) \\ q^2(2^m\alpha + c)^2 &= 2^m\alpha c(2^m\alpha + 2^n\beta + c)(2^m\alpha - 2^n\beta + c). \end{aligned}$$

Since c is odd these imply that $2^n \parallel p^2$ and $2^m \parallel q^2$ so that $2 \mid n$ and $2 \mid m$. Dividing out the powers of two from both equations leads to

$$\begin{aligned} 1 &\equiv \beta c \pmod 4 \\ 1 &\equiv \alpha c \pmod 4. \end{aligned}$$

Now let $A := a/2^n$ where A can be even or odd. Then the last of equations (2.1) implies that

$$\begin{aligned} r^2(A + \beta)^2 &= A\beta(2^n(A + \beta) + c)(2^n(A + \beta) - c) \\ r^2(A + \beta)^2 &\equiv -A\beta \pmod 4. \end{aligned}$$

This last equation will turn out to be impossible by consideration of the different cases of parity of r and $A+\beta$. If both r and $A+\beta$ are odd then A must be even, since β is odd. This leads to a contradiction as $r^2(A + \beta)^2 \equiv 1 \pmod 4$ while the right hand side is $-A\beta \equiv 0$ or $2 \pmod 4$. Next if r is odd and $A+\beta$ is even then A must be odd so $r^2(A + \beta)^2 \equiv 0 \pmod 4$ and $-A\beta \equiv 1$ or $3 \pmod 4$. These two cases show that r must be even so A must be divisible by 4. In fact, if $2^k \parallel r$ then $2^{2k} \parallel A$ and $m = n + 2^k$ where $k \geq 1$. Now set $\rho := r/2^k$ and using α and β as defined earlier we obtain from equations (2.1)

$$\begin{aligned} \rho^2(2^{2k}\alpha + \beta)^2 &= \alpha\beta(2^m\alpha + 2^n\beta + c)(2^m\alpha + 2^n\beta - c) \\ 1 &\equiv -\alpha\beta \pmod 4. \end{aligned}$$

Recall that $\beta c \equiv \alpha c \equiv 1 \pmod 4$ which implies that $\alpha \equiv \beta \pmod 4$ so that we have $-\alpha\beta \equiv -\alpha^2 \equiv -1 \pmod 4$, the final contradiction for Case (ii). ■

Corollary 3 *The area of any bis triangle is an integer.*

Proof: Since the semiperimeter of any bis triangle must be an integer by Theorem 5 and the area is rational, by Rutherford's result, Heron's formula for the area of a triangle yields the required result. ■

It is not difficult to implement a program to search for bis triangles and using Corollary 3 one can disregard any triangles without integer area. The results of one such search are listed in Table 2.1, and since the numbers are so large bis triangles are listed with the greatest common divisor of the sides removed. They seem to occur less frequently than alt triangles did. For example the smallest bis triangle, $(546, 975, 975)$ which is just a scaled version of N. Fuss' example mentioned in the Introduction, is the only one with all sides less than 1000.

Semiperimeter	Sides			Angle Bisectors			Area
s	a	b	c	p	q	r	$\triangle$
32	14	25	25	24	$\frac{560}{39}$	$\frac{560}{39}$	168
189	84	125	169	$\frac{975}{7}$	$\frac{26208}{253}$	$\frac{12600}{209}$	5040
224	125	154	169	$\frac{48048}{323}$	$\frac{2600}{21}$	$\frac{30800}{279}$	9240
288	169	169	238	$\frac{74256}{407}$	$\frac{74256}{407}$	120	14280
297	150	169	275	$\frac{15015}{74}$	$\frac{3168}{17}$	$\frac{2340}{29}$	11088
315	91	250	289	$\frac{20400}{77}$	$\frac{4641}{38}$	$\frac{27300}{341}$	10920
385	231	250	289	$\frac{1700}{7}$	$\frac{11781}{52}$	$\frac{92400}{481}$	27720
450	289	289	322	$\frac{164220}{611}$	$\frac{164220}{611}$	240	38640
352	77	289	338	$\frac{17680}{57}$	$\frac{48048}{415}$	$\frac{10472}{183}$	9240
483	289	338	399	$\frac{248976}{737}$	$\frac{101745}{344}$	$\frac{2652}{11}$	47880

Table 2.1: Triangles with three rational Angle Bisectors

2.7 General Parametrization

Since all bis triangles have integer area we can use Carmichael's parametrization, as was done in Chapter 1, to obtain a parametrization of bis triangles. Rewriting equations (1.3) here, for convenience of reference, we have

$$\begin{aligned} a &= n(m^2 + k^2) \\ b &= m(n^2 + k^2) \\ c &= (m + n)(mn - k^2). \end{aligned}$$

From equations (2.1) we have

$$\begin{aligned} p^2 &= bc \left(1 - \frac{a^2}{(b+c)^2}\right) \\ q^2 &= ac \left(1 - \frac{b^2}{(a+c)^2}\right) \\ r^2 &= ab \left(1 - \frac{c^2}{(a+b)^2}\right). \end{aligned}$$

Meanwhile equations (1.3) imply

$$\frac{a}{b+c} = \frac{m^2+k^2}{2mn+m^2-k^2}$$
$$\frac{b}{a+c} = \frac{n^2+k^2}{2mn+n^2-k^2}$$
$$\frac{c}{a+b} = \frac{mn-k^2}{mn+k^2}.$$

Substituting the latter set of equations into the former leads to

$$p = \frac{2m(m+n)(mn-k^2)}{2mn+m^2-k^2}\sqrt{n^2+k^2}$$
$$q = \frac{2n(m+n)(mn-k^2)}{2mn+n^2-k^2}\sqrt{m^2+k^2}$$
$$r = \frac{2mnk}{mn+k^2}\sqrt{(n^2+k^2)(m^2+k^2)}.$$

So p, q and r are rational if and only if n^2+k^2 and m^2+k^2 are simultaneously perfect squares. This leads to four potentially different cases depending on which leg of a Pythagorean triple is represented by each of the parameters m, n and k. These cases are now considered in turn.

Case (i): Assume that

$m = u^2 - v^2$, $k = 2uv$, where $u, v \in \mathbb{N}$, $2 \mid uv$ and $\gcd(u,v) = 1$;

$n = x^2 - y^2$, $k = 2xy$, where $x, y \in \mathbb{N}$, $2 \mid xy$ and $\gcd(x,y) = 1$.

This requires that $uv = xy := \alpha\beta\gamma\delta$ say, where $u = \alpha\beta$, $v = \gamma\delta$, $x = \alpha\gamma$, $y = \beta\delta$. Substituting these relations into Carmichael's equations (1.3) yields

$$\begin{aligned} a &= (\alpha^2\gamma^2 - \beta^2\delta^2)(\alpha^2\beta^2 + \gamma^2\delta^2)^2 \\ b &= (\alpha^2\beta^2 - \gamma^2\delta^2)(\alpha^2\gamma^2 + \beta^2\delta^2)^2 \\ c &= (\beta^2+\gamma^2)(\alpha^2-\delta^2)[(\alpha^2\beta^2 - \gamma^2\delta^2)(\alpha^2\gamma^2 - \beta^2\delta^2) - (2\alpha\beta\gamma\delta)^2] \end{aligned} \tag{2.4}$$

which produces a triangle with three rational angle bisectors for any choice of $\alpha, \beta, \gamma, \delta \in \mathbb{N}$.

Case (ii): Assume that

$m = 2uv$, $k = u^2 - v^2$, where $u, v \in \mathbb{N}$, $2 \mid uv$ and $\gcd(u,v) = 1$;

$n = 2xy$, $k = x^2 - y^2$, where $x, y \in \mathbb{N}$, $2 \mid xy$ and $\gcd(x,y) = 1$.

This requires that $u^2 - v^2 = x^2 - y^2$. But the transformations

$$u = U + V, v = U - V, x = X + Y, y = X - Y$$

lead to m, n and k having the same form as in Case (i), so this leads to the same parametrization.

Case (iii): Assume that

$m = 2uv$, $k = u^2 - v^2$, where $u, v \in \mathbb{N}$, $2 \mid uv$ and $\gcd(u, v) = 1$;

$n = x^2 - y^2$, $k = 2xy$, where $x, y \in \mathbb{N}$, $2 \mid xy$ and $\gcd(x, y) = 1$.

Now equating the two k's requires that $u^2 - v^2 = 2xy$, so u and v must have the same parity. Let $u = U + V$, $v = U - V$ and $x = 2X$. Then $UV = Xy := \alpha\beta\gamma\delta$ say, where $U = \alpha\beta$, $V = \gamma\delta$, $X = \alpha\gamma$, $y = \beta\delta$. Consequently, $u = \alpha\beta + \gamma\delta$, $v = \alpha\beta - \gamma\delta$, $X = 2\alpha\gamma$, and $y = \beta\delta$. Substituting these relations back into equations (1.3) gives the second parametrization

$$\begin{aligned} a &= 4(4\alpha^2\gamma^2 - \beta^2\delta^2)(\alpha^2\beta^2 + \gamma^2\delta^2)^2 \\ b &= 2(\alpha^2\beta^2 - \gamma^2\delta^2)(4\alpha^2\gamma^2 + \beta^2\delta^2)^2 \\ c &= [2(\alpha^2\beta^2 - \gamma^2\delta^2) + 4(\alpha^2\gamma^2 - \beta^2\delta^2)] \\ &\quad \times [2(\alpha^2\beta^2 - \gamma^2\delta^2)(4\alpha^2\gamma^2 - \beta^2\delta^2) - (4\alpha\beta\gamma\delta)^2]. \end{aligned} \tag{2.5}$$

Case (iv): Assume that

$m = u^2 - v^2$, $k = 2uv$, where $u, v \in \mathbb{N}$, $2 \mid uv$ and $\gcd(u, v) = 1$;

$n = 2xy$, $k = x^2 - y^2$, where $x, y \in \mathbb{N}$, $2 \mid xy$ and $\gcd(x, y) = 1$.

Since equations (1.3) are symmetric in m and n, this case turns out to be the same as case (iii) by interchanging m and n. To produce bis triangles from these parametrizations it only remains to scale up equations (2.4) and (2.5) by $(mn + k^2)(2mn + n^2 - k^2)(2mn + m^2 - k^2)$.

Chapter 3

Three Integer Medians

3.1 Defining Equations

In this chapter we consider integer-sided triangles in which the cevians bisecting the opposite sides are of integer length. In other words, this chapter treats the case of three integer medians. To obtain the defining equations we note that the cosine rule for triangle ADC (see Figure 3.1) gives

$$\overline{AD}^2 = \overline{CD}^2 + \overline{AC}^2 - 2\overline{AC}\;\overline{CD}\cos C,$$

while for triangle ABC we get

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\overline{BC}\;\overline{AC}\cos C.$$

Eliminating $\cos C$ between these two equations produces an expression for the median in terms of the sides of the triangle, $4\overline{AD}^2 = 2\overline{AC}^2 + 2\overline{AB}^2 - \overline{BC}^2$. Hence if the lengths of the sides and median are integers then $\overline{BC}$ must be even. It follows similarly that all three sides have even length. For the rest of this chapter we will denote the lengths of the sides by $2a$, $2b$, $2c$ and the lengths of the medians by k, l, m. By repeating the above process we find that the medians are defined by

$$\begin{aligned} k^2 &= 2b^2 + 2c^2 - a^2 \\ l^2 &= 2c^2 + 2a^2 - b^2 \\ m^2 &= 2a^2 + 2b^2 - c^2. \end{aligned} \tag{3.1}$$

These equations can be rearranged to give the sides in terms of the medians

$$\begin{aligned} 9a^2 &= 2l^2 + 2m^2 - k^2 \\ 9b^2 &= 2m^2 + 2k^2 - l^2 \\ 9c^2 &= 2k^2 + 2l^2 - m^2. \end{aligned} \tag{3.2}$$

Definition : A med triangle is an integer-sided triangle with three integer medians. We will use Med to denote the set of all med triangles.

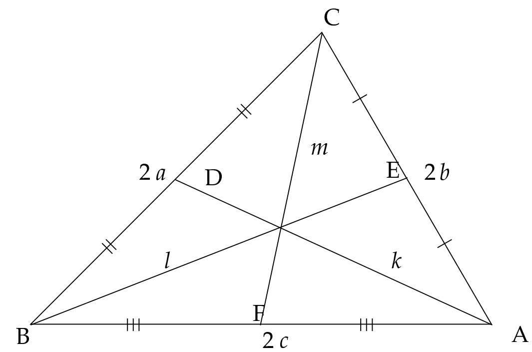

Figure 3.1: Med Triangle

3.2 Even Semiperimeter

One of the first patterns I noticed in searching for med triangles was that the semiperimeter always seemed to be even. I assumed that this was always the case and proceeded to incorporate this pattern into the computer searches for med triangles. It was not until some time later that I thought I should try to prove this "fact". It turned out as follows.

Theorem 6 *If* $a, b, c, k, l, m \in \mathbb{N}$ *and* $\gcd(a, b, c, k, l, m) = 1$ *then one and only one of the half-sides is even.*

Proof : Since there are only four distinct cases for the parity of the half-sides we will eliminate three of them leaving the one we want as follows:
Case (i) : If $a, b \equiv 0 \pmod 2$ and $c \equiv 1 \pmod 2$ then by equation (3.1)

$$k^2 \equiv 2(b^2 + c^2) - a^2 \equiv 2 \pmod 4$$

which is impossible.
Case (ii) : If $a, b, c \equiv 1 \pmod 2$ then as before

$$k^2 \equiv 2(b^2 + c^2) - a^2 \equiv 3 \pmod 4$$

which is also impossible.
Case (iii) : Finally if all three half-sides are even then by equations (3.1) all the medians are even, contradicting the assumption that we have a primitive triangle. ∎

Corollary 4 *The semiperimeter of a med triangle is even.*

Proof : This is immediate from Theorem 6, since the semiperimeter is given by $s = (\overline{BC} + \overline{AC} + \overline{AB})/2 = a + b + c$. ∎

Corollary 5 *Exactly one of the medians is even.*

Proof : Without loss of generality suppose that a is even. Then k must be even, by equation (3.1). ■

Corollary 6 *Exactly one of a, b, c, k, l, m is divisible by four.*

Proof : If we assume that $a = 2\alpha$, $b = 2\beta + 1$, $c = 2\gamma + 1$ while $k = 2\kappa$, $l = 2\lambda + 1$, $m = 2\mu + 1$ then substitution into the first of equations (3.1) gives us

$$4\kappa^2 + 4\alpha^2 = 8\beta^2 + 8\beta + 2 + 8\gamma^2 + 8\gamma + 2.$$

So $\kappa^2 + \alpha^2 = 2\beta^2 + 2\beta + 2\gamma^2 + 2\gamma + 1$ i.e. $\kappa^2 + \alpha^2 \equiv 1 \pmod 2$. So κ and α have opposite parity and the result follows. ■

3.3 Paired Triangles

As mentioned in the introduction Euler was already aware that there is a natural correspondence between pairs of med triangles. Suppose that we already have a med triangle $(2a, 2b, 2c)$ with medians k, l, m. Now if we consider a new triangle with sides $(2k, 2l, 2m)$ then its medians k', l', m' are given by

$$\begin{aligned} {k'}^2 &= 2l^2 + 2m^2 - k^2 \\ {l'}^2 &= 2m^2 + 2k^2 - l^2 \\ {m'}^2 &= 2k^2 + 2l^2 - m^2. \end{aligned}$$

But since k, l and m are expressible in terms of the sides of the original triangle a, b and c we have

$$\begin{aligned} {k'}^2 &= 2(2c^2 + 2a^2 - b^2) + 2(2a^2 + 2b^2 - c^2) - (2b^2 + 2c^2 - a^2) = 9a^2 \\ {l'}^2 &= 2(2a^2 + 2b^2 - c^2) + 2(2b^2 + 2c^2 - a^2) - (2c^2 + 2a^2 - b^2) = 9b^2 \\ {m'}^2 &= 2(2b^2 + 2c^2 - a^2) + 2(2c^2 + 2a^2 - b^2) - (2a^2 + 2b^2 - c^2) = 9c^2. \end{aligned}$$

So the new medians k', l', m' are respectively $3a$, $3b$, $3c$ and hence are integers. So $(2a, 2b, 2c) \in \mathbf{Med}$ implies that $(2k, 2l, 2m) \in \mathbf{Med}$. If we repeat the above process on the triangle $(2k, 2l, 2m)$ we obtain another med triangle, namely $(6a, 6b, 6c)$, which is just a scaled version of the original triangle. Hence this process generates only one new dissimilar med triangle.

3.4 Negative Results

While the various programs I had written were searching for med triangles with integer area I considered the possibility that some subclasses of Heron triangles might not contain any med triangles at all. The results of this approach to solving D21 [9] are contained in the next few sections.

3.4.1 Isosceles

Do there exist isosceles med triangles? Because the current section deals with negative results it would be foolish to maintain any air of suspense. The answer is No! This is an interesting result since referring to Figure 3, just before the Introduction, we see that examples of isosceles alt and isosceles bis triangles are known. In fact in Chapters 1 and 2 we found the general parametrizations for all such triangles. My first proof of the non-existence of isosceles med triangles relied on much case work and the method of infinite descent. Subsequently I found the following more concise method.

Theorem 7 *No med triangle can be isosceles.*

Proof : If $a = b$ in equations (3.1) then $k = l$, and we have just two equations for the medians i.e.

$$k^2 = a^2 + 2c^2$$
$$m^2 = 4a^2 - c^2.$$

The second equation shows that m and c have the same parity while Theorem 6 shows that c must be even (otherwise two sides would be even). So let $c := 2C$ and $m := 2M$ which leads to

$$k^2 = a^2 + 8C^2$$
$$M^2 = a^2 - C^2.$$

If we consider the second of this pair together with the sum of the two equations we have, in the terminology of Dickson [5, p. 475], a "discordant" pair of quadratic forms,

$$k^2 = M^2 + 9C^2$$
$$a^2 = M^2 + C^2$$

i.e. have no common solutions. ■

In our attempt to make this thesis as self-contained as possible the following lemma will display an independent proof that these two equations have no common solution in integers.

Lemma 4 *If $x, y \in \mathbb{N}$ then the expressions $x^2 + y^2$ and $x^2 + 9y^2$ cannot both be simultaneously squares of integers.*

Proof : Let us take $x^2 + y^2 = z^2$ and $x^2 + 9y^2 = t^2$. These two diophantine equations can be solved by the standard Pythagorean parametrization. If u and v are the parameters for the first solution set while r and s are the parameters for the second solution set then $z = u^2 + v^2$ and $t = r^2 + s^2$ but x and y can take four distinct combinations of the parametric forms. We now consider these cases and eliminate them one by one.

Case (i): Assume that

$x = 2uv$, $y = u^2 - v^2$, where $u, v \in \mathbb{N}$ and $\gcd(u, v) = 1$, $2 \mid uv$; and

$x = 2rs$, $3y = r^2 - s^2$, where $r, s \in \mathbb{N}$ and $\gcd(r, s) = 1$, $2 \mid rs$.

Then equating x's gives $uv = rs := \alpha\beta\gamma\delta$ say, where $u := \alpha\beta$, $v := \gamma\delta$, $r := \alpha\gamma$ and $s := \beta\delta$. Thus comparing the expressions for y shows that we require

$$3(\alpha^2\beta^2 - \gamma^2\delta^2) = \alpha^2\gamma^2 - \beta^2\delta^2$$
$$\alpha^2(3\beta^2 - \gamma^2) = \delta^2(3\gamma^2 - \beta^2).$$

Now $\gcd(u, v) = 1$ implies that $\gcd(\alpha, \delta) = 1$ which means that there is some integer ε for which we have

$$3\beta^2 - \gamma^2 = \delta^2\varepsilon$$
$$3\gamma^2 - \beta^2 = \alpha^2\varepsilon. \tag{3.3}$$

Combining these two we find that

$$8\beta^2 = (\alpha^2 + 3\delta^2)\varepsilon$$
$$8\gamma^2 = (3\alpha^2 + \delta^2)\varepsilon. \tag{3.4}$$

So $\varepsilon \mid 8$ otherwise $\varepsilon \mid \beta$ and $\varepsilon \mid \gamma$ would force $\varepsilon = 1$. Now $\varepsilon = 1$, 2, 4 or 8 will each lead to a contradiction hence eliminating this first case.

If $\varepsilon = 1$ then by equation (3.4) α and δ are both odd but then the equation $\alpha^2 + 3\delta^2 \equiv 0 \pmod 8$ is insoluble.

If $\varepsilon = 2$ then by equations (3.3) and (3.4) α, β, γ and δ are all odd which implies that u, v, r and s are all odd. But this contradicts the original assumptions.

If $\varepsilon = 4$ then by equation (3.3) β and γ are both odd but then, as before, the equation $3\beta^2 - \gamma^2 \equiv 0 \pmod 4$ is insoluble.

If $\varepsilon = 8$ then by equation (3.3) β and γ are both odd. So that finally the equation $3\beta^2 - \gamma^2 \equiv 0 \pmod 8$ is insoluble.

Case (ii): Assume that

$x = u^2 - v^2$, $y = 2uv$, where $u, v \in \mathbb{N}$ and $\gcd(u, v) = 1$, $2 \mid uv$; and

$x = 2rs$, $3y = r2 - s2$, where $r, s \in \mathbb{N}$ and $\gcd(r, s) = 1$, $2 \mid rs$.

Now the requirement $2rs = u^2 - v^2$ is a contradiction as u and v must have opposite parity.
Case (iii): Assume that

$x = 2uv$, $y = u^2 - v^2$, where $u, v \in \mathbb{N}$ and $\gcd(u, v) = 1$, $2 \mid uv$; and

$x = r^2 - s^2$, $3y = 2rs$, where $r, s \in \mathbb{N}$ and $\gcd(r, s) = 1$, $2 \mid rs$.

As before we obtain an easy contradiction by equating the x's.
Case (iv): Assume that

$$x = u^2 - v^2,\ y = 2uv, \text{ where } u, v \in \mathbb{N} \text{ and } \gcd(u, v) = 1,\ 2 \mid uv; \text{ and}$$

$$x = r^2 - s^2,\ 3y = 2rs, \text{ where } r, s \in \mathbb{N} \text{ and } \gcd(r, s) = 1,\ 2 \mid rs.$$

So $3uv = rs := \alpha\beta\gamma\delta$ say, where either

$$3u := \alpha\beta, v := \gamma\delta, r := \alpha\gamma, s := \beta\delta$$

or

$$u := \alpha\beta, 3v := \gamma\delta, r := \alpha\gamma, s := \beta\delta.$$

Note that if we interchange α with δ and β with γ that the second identification is equivalent to the first. So considering the first set leads to

$$\begin{aligned}(\alpha^2\beta^2 - 9\gamma^2\delta^2) &= 9\alpha^2\gamma^2 - 9\beta^2\delta^2\\ \alpha^2(\beta^2 - 9\gamma^2) &= \delta^2(9\gamma^2 - 9\beta^2).\end{aligned}$$

Now, just as in the first case, $\gcd(u, v) = 1$ implies that $\gcd(\alpha, \delta) = 1$ which means that there is some integer ε for which

$$\begin{aligned}\beta^2 - 9\gamma^2 &= \delta^2\varepsilon\\ 9\gamma^2 - 9\beta^2 &= \alpha^2\varepsilon.\end{aligned} \tag{3.5}$$

Combining these two we find that

$$\begin{aligned}-8\beta^2 &= (\alpha^2 + \delta^2)\varepsilon\\ -72\gamma^2 &= (\alpha^2 + 9\delta^2)\varepsilon.\end{aligned} \tag{3.6}$$

Now $\gcd(\varepsilon, \gamma) = 1$ and the second of (3.6) implies that $\varepsilon \mid -72$. We can now use infinite descent to eliminate the two subcases $3 \mid \alpha$ and $3 \nmid \alpha$.

- If $3 \nmid \alpha$ then equations (3.6) imply that $9 \mid \varepsilon$ so substituting $\varepsilon := 9\eta$ into (3.5) leads to

$$\begin{aligned}\beta^2 - 9\gamma^2 &= 9\delta^2\eta\\ \gamma^2 - \beta^2 &= \alpha^2\eta.\end{aligned} \tag{3.7}$$

If $2 \mid \eta$ then β and γ are both odd and since at least one of α or δ must be odd we see that $\eta \equiv 0 \pmod 8$. Together with $9\eta \mid -72$ this shows that $\eta = -8$. Substitution into (3.6) gives us the pair of equations

$$\begin{aligned}\alpha^2 + \delta^2 &= B^2\\ \alpha^2 + 9\delta^2 &= \gamma^2\end{aligned}$$

where $B^2 := \beta^2/9 < (\alpha^2\beta^2/9 + \gamma^2\delta^2) = z^2$. This last pair of equations is equivalent to the first pair in the statement of the lemma. So any solution

would imply the existence of a smaller solution (as $B < z$) and hence by infinite descent there can be no non-trivial solution.
If $\eta = -1$ then eliminating β from equations (3.7) gives $\alpha^2 + 9\delta^2 = 8\gamma^2$ which means that α and δ are both odd. But then $\alpha^2 + 9\delta^2 \equiv 2 \pmod 8$ leads to a contradiction.

• If $3 \mid \alpha$ then equation (3.6) implies that $3 \nmid \varepsilon$ and so $\varepsilon \mid 8$. Substituting $\alpha = 3\theta$ into equations (3.5) reduces them to

$$\begin{aligned} \beta^2 - 9\gamma^2 &= \delta^2\varepsilon \\ \gamma^2 - \beta^2 &= \theta^2\varepsilon. \end{aligned} \tag{3.8}$$

Now as before if $2 \mid \varepsilon$ then β and γ are both odd and since at least one of θ or δ must be odd we see that $\varepsilon \equiv 0 \pmod 8$ hence $\varepsilon = -8$. Substitution into (3.8) gives us the familiar pair of equations

$$\begin{aligned} \delta^2 + \theta^2 &= \gamma^2 \\ \delta^2 + 9\theta^2 &= \beta^2. \end{aligned}$$

This time $\gamma^2 < (\alpha^2\beta^2/9 + \gamma^2\delta^2) = z^2$ and so we have no solution again by infinite descent.

If $\varepsilon = -1$ then eliminating β from equations (3.8) gives us $\theta^2 + \delta^2 = 8\gamma^2$ which means that θ and δ are both odd. But then $\theta^2 + \delta^2 \equiv 2 \pmod 8$ leads to a contradiction which finally eliminates case (iv). ■

3.4.2 Pythagorean

The next subset of Heron triangles I chose to consider was the set of integer-sided triangles with a right angle i.e. Pythagorean triangles (sometimes referred to as Pythagorean triples). The following theorem proves that no such triangle can have three integer medians.

Theorem 8 *If $a, b, c \in \mathbb{N}$ such that $a^2 + b^2 = c^2$ then the triangle $(2a, 2b, 2c)$ has exactly one integer median.*

Proof : If we set $a^2 + b^2 = c^2$ in equations (3.1) we find that

$$\begin{aligned} k^2 &= 4b^2 + a^2 \\ l^2 &= 4a^2 + b^2 \\ m^2 &= a^2 + b^2. \end{aligned}$$

So clearly the median m is an integer for any Pythagorean triangle. But if we take, without loss of generality, $a = u^2 - v^2$, $b = 2uv$, and $c = u^2 + v^2$ then the equations for k and l become

$$\begin{aligned} k^2 &= u^4 + 14u^2v^2 + v^4 \\ l^2 &= u^4 - u^2v^2 + v^4. \end{aligned}$$

Mordell shows [12, pp. 20-21] that these pair of equations have only the solutions $(u^2, v^2) = (1, 0), (1, 1)$ and $(0, 1)$ which all lead to degenerate triangles. ■

3.4.3 Arithmetic Progression

The set of integer-sided triangles with sides in an arithmetic progression also turns out to have no member in common with the set of Med triangles. Notice that in the following proof of Theorem 9 no mention is made of the rationality or otherwise of the area of the triangle. This explains why, in Figure 3, the set $\mathrm{Int}_{\mathrm{AP}}$ is not wholly contained within the set of Heron triangles and remains disjoint from **Med**.

Theorem 9 *No med triangle can be have its sides in an arithmetic progression.*

Proof : If $(a, b, c) = (p - q, p, p + q)$ where $\gcd(p, q) = 1$, p is even and q is odd then equations (3.1) become

$$\begin{aligned} k^2 &= 3p^2 + 6pq + q^2 \\ l^2 &= 3p^2 + 4q^2 \\ m^2 &= 3p^2 - 6pq + q^2 \end{aligned} \tag{3.9}$$

Solutions to the second of these are

(i) $p = 2xy$, $2q = x^2 - 3y^2$, $\gcd(x, y) = 1$, $2 \nmid xy$

(ii) $p = 2xy$, $2q = 3x^2 - y^2$, $\gcd(x, y) = 1$, $2 \nmid xy$

Substituting set (i) into (3.9) leads to

$$\begin{aligned} 4k^2 &= x^4 + 24x^3y + 42x^2y^2 - 72xy^3 + 9y^4 \\ 4m^2 &= x^4 - 24x^3y + 42x^2y^2 + 72xy^3 + 9y^4 \end{aligned}$$

Any rational solution (x_0, y_0) to the first equation corresponds to two rational solutions $(-x_0, y_0)$ and $(x_0, -y_0)$ of the second equation and vice versa. So common solutions to both quartic equations can only occur when $x_0 = -x_0$ or $y0 = -y0$ i.e. $x_0y_0 = 0$. This leads to only degenerate triangles. Substituting set (ii) into (3.9) leads to

$$\begin{aligned} 4k^2 &= 9x^4 + 72x^3y + 42x^2y^2 - 24xy^3 + y^4 \\ 4m^2 &= 9x^4 - 72x^3y + 42x^2y^2 + 24xy^3 + y^4 \end{aligned}$$

As above, the only common rational solutions to both equations have $xy = 0$ which result in degenerate triangles. ■

Note that the equation $4k^2 = x^4 + 24x^3y + 42x^2y^2 - 72xy^3 + 9y^4$ does have rational solutions e.g. $(x, y, k) = (1, 1, 1)$, $(3, 1, 15)$, $(7, 5, 97)$, or $(1, 13, 163)$. Using Mordell's transformation (as on p.47) this is equivalent to the cubic equation

$$T^2 = s^3 - 147s + 610$$

with corresponding rational solutions

$$(s, T) = (11, 18), (41, 252), (419/52, 6678/53), (833/132, 5004/133).$$

Using the tangent-chord process the third rational point generates an infinite number of rational solutions on the elliptic curve and hence there exists an infinite number of triangles with sides in an arithmetic progression and two integer medians.

3.4.4 Automédian

If the medians of a triangle are proportional to the sides then the triangle is defined to be automédian. Shortly, we will prove a necessary and sufficient condition for a triangle to be automédian. This will then be used to show that the set of isosceles automédian triangles is equivalent to the set of equilateral triangles hence clarifying Figure 3 a little more. The above condition will also show that no integer-sided automédian triangle can have an integer median or integer area which will justify our consideration of this type.

Theorem 10 *A triangle with sides $2a$, $2b$, $2c$ is automédian if and only if one of $a^2+c^2=2b^2$, $a^2+b^2=2c^2$, $b^2+c^2=2a^2$ is true.*

Proof : If $a^2+c^2=2b^2$ then substitution into equations (3.1) gives the three expressions $k^2=3c^2$, $l^2=3b^2$, $m^2=3a^2$ and so the medians are proportional to the sides. Similarly so for the other two conditions. On the other hand if $k/c=l/b=m/a=x$ then equations (3.1) become

$$\begin{aligned} x^2c^2 &= 2b^2+2c^2-a^2 \\ x^2b^2 &= 2c^2+2a^2-b^2 \\ x^2a^2 &= 2a^2+2b^2-c^2. \end{aligned}$$

(Note that solving these equations for x leads to $x^4=9$ and so the definition of automédian triangles is much more restrictive than first impressions would imply.) Eliminating x from the first two of these equations gives

$$c^2(2c^2+2a^2-b^2)=b^2(2b^2+2c^2-a^2).$$

Expanding and rearranging this one obtains

$$(2c^2+b^2)(c^2+a^2-2b^2)=0.$$

Clearly this implies that $c^2+a^2=2b^2$. If $k/a=l/c=m/b$ then $b^2+c^2=2a^2$ and if $k/b=l/a=m/c$ then $a^2+b^2=2c^2$. While the other three permutations lead to equilateral triangles or degenerate triangles (zero side), both of which identically satisfy the conditions as stated in the theorem. ■

Corollary 7 *All isosceles automédian triangles are equilateral.*

Proof: If $a=c$ then from Theorem 10 we have $2a^2=2b^2$ so that $a=b$ as well. The remaining two cases are similar. ■

Corollary 8 *An automédian triangle with integer sides cannot have an integer median.*

Proof: If a, b, c are integers then k, l, m are all irrational since $k/c = l/b = m/a = \sqrt{3}$. ■

Corollary 9 *An automédian triangle with integer sides a, b, c cannot have integer area.*

Proof: The area of a triangle is given by Heron's formula

$$16\triangle^2 = 2a^2c^2 + 2b^2c^2 + 2a^2b^2 - a^4 - b^4 - c^4.$$

Eliminating c by Theorem 10 gives

$$16\triangle^2 = 4a^2b^2 - 2a^4 - b^4.$$

First assume that a solution exists with $\gcd(a, b) = 1$. From the last expression we must have $2 \mid b$ so letting $b = 2B$ then

$$8\triangle^2 = 8a^2B^2 - a^4 - 8B^4.$$

But now $2 \mid a$ which is a contradiction. ■

3.4.5 Roberts Triangles

Recall, from Chapter 2, that a Roberts triangle which has sides $(a, b, c) = (x^2 + 2y^2, x^2 + 4y^2, 2x^2 + 2y^2)$ for some integers x and y also has integer area. More explicitly, using Heron's formula for the area of a triangle one finds that $\triangle = 2xy(x^2 + 2y^2)$. Notice that the above expressions for the sides lead to $a + b = 3c$. In fact it turns out that all Heron triangles whose sides satisfy this relationship must be of Roberts' form. The main result in this section is that no Roberts' triangle can belong to the set of Med triangles.

Theorem 11 *A Roberts' triangle cannot have three integer medians.*

Proof : Considering only the similarity classes of Roberts' triangles let the sides be $(2a, 2b, 2c) = (X^2 + 2Y^2, X^2 + 4Y^2, 2X^2 + 2Y^2)$ where $\gcd(a, b, c) = 1$. Then clearly X must be even. Setting $X = 2x$ and $Y = y$, say and dividing out the factor of 2 shows that the half-sides have the same parametric form as the sides, $(a, b, c) = (2x^2 + y^2, 2x^2 + 2y^2, 4x^2 + y^2)$ where $\gcd(x, y) = 1$. Note that y must be odd by Theorem 6. Substituting this into equations (3.1) we obtain

$$\begin{aligned} k^2 &= (6x^2 - 3y^2)^2 + (8xy)^2 \\ l^2 &= 4x^2(9x^2 + 4y^2) \\ m^2 &= y^2(16x^2 + 9y^2). \end{aligned}$$

Now focusing attention on the latter two equations, the requirement that all three medians be integers means there exist integers p and q which satisfy

$$p^2 = (3x)^2 + (2y)^2$$
$$q^2 = (4x)^2 + (3y)^2.$$

As usual, applying the solution for Pythagorean triples and the additional constraint that y is odd leads to only one possible simultaneous solution:

$3x = r^2 - s^2$, $2y = 2rs$, where $r, s \in \mathbb{N}$ and $\gcd(r, s) = 1$, $2 \nmid rs$;

$4x = 2uv$, $3y = u^2 - v^2$, where $u, v \in \mathbb{N}$ and $\gcd(u, v) = 1$, $2 \mid uv$.

Since r and s have the same parity let $r = R + S$ and $s = R - S$ which leads to

$3x = 4RS$, $2y = R^2 - S^2$, $\gcd(R, S) = 1$, $2 \mid RS$,

$4x = 2uv$, $3y = u^2 - v^2$, $\gcd(u, v) = 1$, $2 \mid uv$.

Now we must have $8RS = 3uv$ so setting both to $\alpha\beta\gamma\delta$ gives

(i) $8R = \alpha\beta$, $S = \gamma\delta$, $3u = \alpha\gamma$, $v = \beta\delta$,

(ii) $R = \alpha\beta$, $8S = \gamma\delta$, $3u = \alpha\gamma$, $v = \beta\delta$,

(iii) $8R = \alpha\beta$, $S = \gamma\delta$, $u = \alpha\gamma$, $3v = \beta\delta$,

(iv) $R = \alpha\beta$, $8S = \gamma\delta$, $u = \alpha\gamma$, $3v = \beta\delta$.

Without loss of generality one need only consider case (i). Substituting this into $3(R^2 - S^2) = u^2 - v^2$ leads to

$$\alpha^2(27\beta^2 - 64\gamma^2) = \delta^2(1728\gamma^2 - 576\beta^2).$$

Now $\gcd(R, S) = 1$ implies that $\gcd(\alpha, \delta) = 1$ so there must exist an integer, ε say, such that

$$1728\gamma^2 - 576\beta^2 = \alpha^2\varepsilon$$
$$27\beta^2 - 64\gamma^2 = \delta^2\varepsilon.$$

Thus solving these two equations for β and γ leads to

$$153\beta^2 = (\alpha^2 + 27\delta^2)\varepsilon$$
$$1088\gamma^2 = (3\alpha^2 + 64\delta^2)\varepsilon.$$

Since $\gcd(\varepsilon, \gamma) = 1$ implies that $\varepsilon \mid 1088 = 2^6 \cdot 17$ while $\gcd(\varepsilon, \beta) = 1$ implies that $\varepsilon \mid 153 = 3^2 \cdot 17$ we have $\varepsilon \mid 17$. If $\varepsilon = \pm 1$ or ± 17 then the latter equation implies that $8 \mid \alpha$. In all four cases the former equation reduces to $\pm\beta^2 \equiv 3\delta^2 \pmod 8$. But δ is odd as $\gcd(\alpha, \delta) = 1$ and α is even. Hence $\delta^2 \equiv 1 \pmod 8$ implies that $\pm\beta^2 \equiv 3 \pmod 8$ which is impossible. ∎

3.5 Euler's Parametrizations

In a paper written in 1779 Euler showed that if the half-sides of a triangle are given by the expressions

$$\begin{aligned}2a &= 2\alpha(e-f)\\ 2b &= \alpha(d+e)+\beta(d-e)\\ 2c &= \alpha(d+e)-\beta(d-e)\end{aligned}$$

where $d = 16\alpha^2\beta^2$, $e = (\alpha^2+\beta^2)(9\alpha^2+\beta^2)$ and $f = 2(9\alpha^4-\beta^4)$ for integers α and β then the medians are integers. This parametrization appeared several times in Euler's works. In a posthumous paper of 1849 he noted that

$$\begin{aligned}a &= (m+n)p-(m-n)q\\ b &= (m-n)p+(m+n)q\\ c &= 2mp-2nq\end{aligned} \tag{3.10}$$

where $p = (m^2+n^2)(9m^2-n^2)$ and $q = 2mn(9m^2+n^2)$ for integers m and n is sufficient to make the three medians integers as well. With a little algebraic manipulation it is not hard to show that these two formulations are essentially the same. Interchanging a with m and b with n yields the correspondences $(a,b,c)_{\{1849\}} = 2(b,c,-a)_{\{1779\}}$.

To show that the parametrization (3.10) does not generate every med triangle one needs to solve for the ratio m/n in terms of a, b and c.

$$\begin{aligned}a+b+c &= 4mp\\ a+b-c &= 4nq\\ a-b+c &= 2(p-q)(m+n).\end{aligned}$$

Eliminating the parameters p and q in these expressions leads to a quadratic in m and n

$$(a+b-c)m^2+2(a-b)mn-(a+b+c)n^2 = 0$$

whence

$$\frac{m}{n} = \frac{b-a\pm\sqrt{2a^2+2b^2-c^2}}{a+b-c}.$$

The med triangle $2(233, 255, 442)$ yields $m/n = 5$ or $-93/23$. For the first value substitute $m = 5k$ and $n = k$, where $k \in \mathbb{N}$, into the c equation in set (3.10). This leads to $c = 53720k^5$ but then $k^5 = 442/53720$ is not solvable for integer k. Similarly, substituting $m = -93k$ and $n = 23k$ results in $c = -116557658136k^5$ and as before $k^5 = -442/116557658136$ is not solvable for integer k. To complete the current line of argument the related med triangle also needs to be checked. If $(a,b,c) = 2(208, 659, 683)$ then $m/n = 25/4$ or $-31/23$. Substituting $m = 25k$ and $n = 4k$ leads to $k^5 = 683/170742850$ which, as expected, is insoluble for integer k. Similarly $m = -31k$ and $n = 23k$ leads to $k^5 = -683/148085512$. Hence neither of these two med triangles can be

Semiperimeter	Sides			Medians			Eulerian
s	a	b	c	k	l	m	
240	136	170	174	158	131	127	yes
646	226	486	580	523	367	244	no
680	290	414	656	529	463	142	yes
798	318	628	650	619	404	377	yes
930	466	510	884	683	659	208	no
1122	654	772	818	725	632	587	no
1200	554	892	954	881	640	569	yes
1764	932	982	1614	1252	1223	515	no
2262	1162	1548	1814	1583	1312	1025	no
3248	1620	2198	2678	2312	1921	1391	yes
3496	1018	2646	3328	2963	2075	1118	no
4100	1754	2612	3834	3161	2680	1129	no
4350	2446	3048	3206	2879	2410	2251	no
4408	1678	3280	3858	3481	2482	1751	no
5152	2802	3556	3946	3485	2924	2521	no
6240	3810	3930	4740	3915	3825	3060	no

Table 3.1: Triangles with three integer medians

generated by Euler's parametrization. Eulerian and non eulerian med triangles are listed in Table 3.1.

Another negative result is that no med triangle from Euler's parametrization can have integer area. If the triangle is a primitive one, i.e. $\gcd(a,b,c,k,l,m) = 1$, then a and b are odd by Theorem 6. This implies that $\gcd(m,n) = 1$ and that m and n have opposite parity. From Heron's formula and equations (3.10) the area is given by

$$\begin{aligned}\triangle^2 &= 4mp\,[2(m-n)p + 2(m-n)q]\,[2(m+n)p - 2(m+n)q]\,4nq \\ &= 64mnpq(m^2-n^2)(p^2-q^2) \\ &= 128m^2n^2(34m^4-n^4)(m^4-n^4)(p^2-q^2).\end{aligned}$$

Hence if the area is to be an integer then $2(34m^4-n^4)(m^4-n^4)(p^2-q^2) = z^2$ for some integer z. Note that $\gcd(2, 34m^4-n^4, m^4-n^4, p^2-q^2) = 1$ since the latter three terms are all odd. This clearly leads to a contradiction since $z^2 \equiv 2 \pmod 4$ is impossible.

3.6 Two Median General Parametrization

To date no med triangle has been discovered which also has integer area so the approach used in Chapters 1 and 2 of applying Carmichael's equations is unlikely to produce a parametrization of even a subset of med triangles. A different approach is needed. Factorizing equations (3.1) over the field $\mathbb{Q}(\sqrt{2})$

gives the following for k:

$$\begin{aligned}(k-b\sqrt{2})(k+b\sqrt{2}) &= -(a-c\sqrt{2})(a+c\sqrt{2})\\ \frac{k+b\sqrt{2}}{a+c\sqrt{2}} &= \frac{-(a-c\sqrt{2})(k+b\sqrt{2})}{(k^2-2b^2)}\\ k+b\sqrt{2} &= \left(\frac{r}{t}+\frac{s}{t}\sqrt{2}\right)(a+c\sqrt{2})\end{aligned}$$

where $r = 2cb-ak$, $s = ck-ab$, $t = k^2-2b^2$. Now the norm of any $a := x+y\sqrt{2}$ in $\mathbb{Q}(\sqrt{2})$ is given by $\text{Norm}(a) := x^2-2y^2$. It turns out that the norm of the factor $\frac{r}{t}+\frac{s}{t}\sqrt{2}$ is just -1 since

$$\begin{aligned}\text{Norm}\left(\frac{r}{t}+\frac{s}{t}\sqrt{2}\right) &= \frac{(2cb-ak)^2-2(ck-ab)^2}{(k^2-2b^2)^2}\\ &= \frac{a^2(k^2-2b^2)-2c^2(k^2-2b^2)}{(k^2-2b^2)^2}\\ &= \frac{a^2-2c^2}{k^2-2b^2}\\ &= -1.\end{aligned}$$

In other words $k+b\sqrt{2}$ is just some unit of $\mathbb{Q}(\sqrt{2})$ times $a+c\sqrt{2}$. To find all units which have a norm of -1 we simply solve the equation:

$$\frac{r}{t}+\frac{s}{t}\sqrt{2} = -1.$$

Thus $r^2+t^2 = 2s^2$ and so r and t have the same parity. Letting $2R := r+t$ and $2T := r-t$ leads to $R^2+T^2 = s^2$. Using the general Pythagorean triple gives

$$\begin{aligned}r &= 2p_1q_1+p_1^2-q_1^2\\ s &= p_1^2+q_1^2\\ t &= \pm(2p_1q_1-p_1^2+q_1^2)\end{aligned}\tag{3.11}$$

where $\gcd(p_1,q_1) = 1$ and $2 \mid p_1q_1$. Since the same analysis can be applied to the l and m equations from (3.1) then defining u,v,w and x,y,z analogously to r,s,t gives

$$\begin{aligned}k+b\sqrt{2} &= \left(\frac{r}{t}+\frac{s}{t}\sqrt{2}\right)(a+c\sqrt{2})\\ l+c\sqrt{2} &= \left(\frac{u}{w}+\frac{v}{w}\sqrt{2}\right)(b+a\sqrt{2})\\ m+a\sqrt{2} &= \left(\frac{x}{z}+\frac{y}{z}\sqrt{2}\right)(c+b\sqrt{2})\end{aligned}\tag{3.12}$$

where $\text{Norm}\left(\frac{r}{t}+\frac{s}{t}\sqrt{2}\right) = \text{Norm}\left(\frac{u}{w}+\frac{v}{w}\sqrt{2}\right) = \text{Norm}\left(\frac{x}{z}+\frac{y}{z}\sqrt{2}\right) = -1$. So as in equations (3.11) the variables u,v,w and x,y,z are expressible in terms of the parameters p_2, q_2 and p_3, q_3. Equating the irrational parts of equation (3.12) leads to

$$\begin{aligned} rc + sa &= tb \\ ua + vb &= wc \\ xb + yc &= za. \end{aligned} \tag{3.13}$$

Solving the first two of these three for the ratios a/b and c/b gives

$$\begin{aligned} \frac{a}{b} &= \frac{tw - rv}{sw + ru} \\ \frac{c}{b} &= \frac{tu + sv}{sw + ru}. \end{aligned}$$

So any integer-sided triangle with two integer medians has sides proportional to

$$\begin{aligned} a &= tw - rv \\ b &= sw + ru \\ c &= tu + sv. \end{aligned}$$

Substituting for the parameters r, s, t and u, v, w as given by equations like (3.11) the sides are

$$\begin{aligned} a &= -4p_1q_1p_2^2 + 2(2p_1q_1 - p_1^2 + q_1^2)p_2q_2 + 2(q_1^2 - p_1^2)q_2^2 \\ b &= (2p_1q_1 - 2q_1^2)p_2^2 + 2(2p_1q_1 + 2p_1^2)p_2q_2 + 2(q_1^2 - p_1q_1)q_2^2 \\ c &= (2p_1q_1 + 2q_1^2)p_2^2 + 2(2p_1q_1 - p_1^2 + q_1^2)p_2q_2 + 2(p_1^2 - p_1q_1)q_2^2 \end{aligned} \tag{3.14}$$

where $p_1, q_1, p_2, q_2 \in \mathbb{N}$. Since a, b, c can be either positive or negative by equations (3.1) one may take, without loss of generality, the positive sign in equations (3.11). However u, v, w can also take both positive and negative values. Comparing the result of the eight possible permutations of the signs of u, v, w on the form of a, b, c leads to only four distinct types, namely The last three forms are

u	v	v	a	b	c
$+$	$+$	$+$	$tw - rv$	$sw + ru$	$tu + sv$
$+$	$+$	$-$	$-tw - rv$	$-sw + ru$	$tu + sv$
$+$	$-$	$+$	$tw + rv$	$sw + ru$	$tu - sv$
$+$	$-$	$-$	$-tw + rv$	$-sw + ru$	$tu - sv.$

all equivalent to the first form i.e. to (3.14) by the respective interchanges

$$\begin{aligned} (p_1, q_1, p_2, q_2) &\Longrightarrow (-q_1, p_1, -q_2, p_2) \\ (p_1, q_1, p_2, q_2) &\Longrightarrow (p_1, q_1, -q_2, p_2) \\ (p_1, q_1, p_2, q_2) &\Longrightarrow (-q_1, p_1, p_2, q_2). \end{aligned}$$

Considering the more general case of rational sided triangles with rational medians the arguments above readily lead to

Theorem 12 *If the rational numbers a, b and c are the lengths of sides of a triangle with at least two rational medians then*

$$\begin{aligned} a &= t\left[-2\phi\theta^2 + (-\phi^2 + 2\phi + 1)\theta + (-\phi^2 + 1)\right] \\ b &= t\left[(\phi - 1)\theta^2 + 2(\phi^2 + \phi)\theta + (-\phi + 1)\right] \\ c &= t\left[(\phi + 1)\theta^2 + (-\phi^2 + 2\phi + 1)\theta + (\phi^2 - \phi)\right] \end{aligned}$$

where $\theta, \phi, t \in \mathbb{Q}$.

Proof : Since any pair of the equations (3.13) can substitute for any other pair by symmetry, set $\phi := p_1/q_1$ and $\theta := p_2/q_2$ in equations (3.14) to give the above result. ■

3.7 Med Triangles

Using Theorem 12 as a starting point one could proceed in at least two relevant directions given a parametrization of rational-sided triangles with two rational medians. Either constrain the third median of a triangle to be rational length or constrain the area to be rational. The second case is taken up in Chapter 4 while here the first case leads to med triangles.

Reverting the discussion back to integer-sided triangles: force the "m" median of equations (3.12) to be of integer length. Begin with the third equation of set (3.13) namely

$$xb + yc = za.$$

The other constraint on the median is that Norm $\left(\frac{x}{z} + \frac{y}{z}\sqrt{2}\right) = -1$ which has solutions

$$\begin{aligned} x &= 2p_3q_3 + p_3^2 - q_3^2 \\ y &= p_3^2 + q_3^2 \\ z &= \pm(2p_3q_3 - p_3^2 + q_3^2). \end{aligned}$$

Substituting these into the above gives

$$(a + b + c)p_3^2 + 2(b - a)p_3q_3 + (-a - b + c)q_3^2 = 0.$$

Finally substituting the parametrization (3.14) into this last equation leads to

$$\alpha p_3^2 + \beta p_3q_3 + \gamma q_3^2 = 0$$

where

$$\begin{aligned} \alpha &:= (a + b + c)/4 = (3p_1q_1 + q_1^2)p_2q_2 + (q_1^2 - p_1q_1)q_2^2 \\ \beta &:= (2b - 2a)/4 = (3p_1q_1 - q_1^2)p_2^2 + (3p_1^2 - q_1^2)p_2q_2 + (p_1^2 - p_1q_1)q_2^2 \\ \gamma &:= (-a - b + c)/4 = (p_1q_1 + q_1^2)p_2^2 + (-p_1^2 - p_1q_1)p_2q_2 + (p_1^2 - q_1^2)q_2^2. \end{aligned}$$

Now the ratio p_3/q_3 is rational if and only if the discriminant of the quadratic in this ratio is the square of some integer, η say. Hence expanding $\beta^2-4\alpha\gamma=\eta^2$ in terms of the parameters p_1,q_1,p_2,q_2 and then rewriting these in terms of θ and ϕ leads to a sixth degree polynomial in two variables which must be a rational square, namely

$$\begin{aligned}\eta^2 &= \theta^4(3\phi-1)^2+2\theta^3(9\phi^3-9\phi^2-11\phi-1)+3\theta^2(3\phi^4+6\phi^3+2\phi^2+2\phi-1)\\ &\quad+2\theta(\phi-1)(3\phi^3-8\phi^2-11\phi-2)+(\phi-1)^2(\phi+2)^2.\end{aligned} \tag{3.15}$$

Thus any rational solutions of equation (3.15) will provide rational numbers θ, ϕ, η and hence integers p_1,q_1,p_2,q_2 which when substituted into the parametrization (3.14) will produce a triangle with three integer medians. For example the rational solution $(\theta,\phi,\eta)=(11/2,2,765/4)$ corresponds to $(p_1,q_1,p_2,q_2)=(2,1,11,2)$ which in turn corresponds to $2(127,131,158)\in$ **Med**.

Notice that substituting $\theta=-1$ in equation (3.15) leads to $\eta^2=4\phi^2(\phi+3)^2$ so that η is trivially rational for any rational choice of ϕ. And similarly substituting $\phi=1$ leads to $\eta^2=4\theta^2(\theta-3)^2$ which means η is rational for any rational θ. These two cases lead to degenerate triangles since substituting $p_1=-q_1$ into the parametrization (3.14) leads to $a+c=b$ while $p_2=q_2$ leads to $b+c=a$.

3.7.1 Tiki Surface

Since the equation (3.15) is so closely associated with med triangles it makes sense to investigate its properties a little more fully. Consider the function of two variables $f(\theta,\phi)$ defined by

$$\begin{aligned}f(\theta,\phi) = \theta^4&(3\phi-1)^2\\ &+2\theta^3(9\phi^3-9\phi^2-11\phi-1)\\ &+3\theta^2(3\phi^4+6\phi^3+2\phi^2+2\phi-1)\\ &+2\theta(\phi-1)(3\phi^3-8\phi^2-11\phi-2)\\ &+(\phi-1)^2(\phi+2)^2.\end{aligned}$$

then the problem of finding rational triples which satisfy (3.15) is equivalent to finding rational θ, and ϕ which make $f(\theta,\phi)$ the square of some rational number.

Firstly note that $f(-\phi,-\theta)=f(\theta,\phi)$ which implies that $f(\theta,\phi)$ is symmetric about the line $\phi=-\theta$ in the $\theta\phi$-plane. Now if $\phi=-\theta$ then the function simplifies to

$$g(\theta):=f(\theta,-\theta)=-24\theta^5+52\theta^4-16\theta^3-24\theta^2+8\theta+4.$$

Hence $g'(\theta)=-120\theta^4+208\theta^3-48\theta^2-48\theta+8=-8(\theta-1)^2(15\theta^2+4\theta-1)$ which is zero when $\theta=1,\frac{-2\pm\sqrt{19}}{15}$. The points corresponding to these values of θ in the $\theta\phi$-plane turn out to be three of the 'critical points' of the surface. To find all of the critical points of $f(\theta, \phi)$ one needs to determine the simultaneous zeros of the two partial derivatives i.e. points (θ, ϕ) satisfying

$$\frac{\partial f}{\partial \theta}(\theta, \phi) = 0 = \frac{\partial f}{\partial \phi}(\theta, \phi).$$

The partial derivatives are given by

$$\begin{aligned}
\frac{\partial f}{\partial \theta}(\theta, \phi) = {} & 4\theta^3(3\phi - 1)^2 \\
& + 6\theta^2(9\phi^3 - 9\phi^2 - 11\phi - 1) \\
& + 6\theta(3\phi^4 + 6\phi^3 + 2\phi^2 + 2\phi - 1) \\
& + 2(\phi - 1)(3\phi^3 - 8\phi^2 - 11\phi - 2) \\
\frac{\partial f}{\partial \phi}(\theta, \phi) = {} & 6\theta^4(3\phi - 1) \\
& + 2\theta^3(27\phi^2 - 18\phi - 11) \\
& + 3\theta^2(12\phi^3 + 18\phi^2 + 4\phi + 2) \\
& + 6\theta(4\phi^3 - 11\phi^2 - 2\phi + 3) \\
& + 2(\phi - 1)(\phi + 2)(2\phi + 1).
\end{aligned}$$

Consider these two partial derivatives to be elements of the polynomial ring $(\mathbb{Z}[\phi])[\theta]$. They have a common zero if and only if $Res_\theta(f_\theta, f_\phi)$, the so-called resultant polynomial, is zero (cf [11, pp. 58-60]). The resultant of the two partial derivatives is the following determinant

$$Res_\theta(f_\theta, f_\phi) = \begin{vmatrix}
a_0 & a_1 & a_2 & a_3 & 0 & 0 & 0 \\
0 & a_0 & a_1 & a_2 & a_3 & 0 & 0 \\
0 & 0 & a_0 & a_1 & a_2 & a_3 & 0 \\
0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 \\
b_0 & b_1 & b_2 & b_3 & b_4 & 0 & 0 \\
0 & b_0 & b_1 & b_2 & b_3 & b_4 & 0 \\
0 & 0 & b_0 & b_1 & b_2 & b_3 & b_4
\end{vmatrix}$$

where a_i and b_i are the coefficients of f_θ, f_ϕ respectively, i.e. $f_\theta = \sum_{i=0}^{3} a_i\theta^{3-i}$ and $f_\phi = \sum_{i=0}^{4} b_i\theta^{4-i}$. Expanding this determinant leads to the nineteen degree polynomial

$$\begin{aligned}
Res_\theta(f_\theta, f_\phi) = {} & 21336\phi^2(3645\phi^{17} - 2187\phi^{16} + 79056\phi^{15} - 76464\phi^{14} \\
& - 223344\phi^{13} + 311904\phi^{12} + 177912\phi^{11} - 458856\phi^{10} \\
& - 2934\phi^9 + 310506\phi^8 - 54880\phi^7 - 97056\phi^6 \\
& + 23880\phi^5 + 12600\phi^4 - 3432\phi^3 - 456\phi^2 + 97\phi + 9).
\end{aligned}$$

The four critical points found earlier along the line $\phi = -\theta$ must also be roots of $Res_\theta(f_\theta, f_\phi)$ since both partial derivatives are zero there. By inspection one notes that $\phi = 1$ is a double root, $\phi = 1/3$ is a single root and $\phi = \pm\sqrt{3}/3$ are conjugate roots which means that the resultant can be factorized to

$$Res_\theta(f_\theta, f_\phi) = 21336\phi^2(\phi+1)^2(15\phi^2 - 4\phi - 1)(\phi - 1)2(3\phi - 1)(3\phi^2 - 1)r_8(\phi)$$

where $r_8(\phi) = 27\phi^8 + 648\phi^6 - 216\phi^5 - 330\phi^4 + 720\phi^3 + 256\phi^2 - 88\phi - 9$. It turns out that the octic factor, $r_8(\phi)$, is irreducible in $\mathbb{Q}[\phi]$. Consider the transformation $\phi \mapsto \phi + 1$. Then new coefficients c_i are defined by $r_8(\phi+1) = \sum_{i=0}^{8} c_i \phi^{8-i}$ where

$$r_8(\phi+1) = 3^3\phi^8 + 2^3 \cdot 3^3\phi^7 + 2^3 \cdot 3^3 \cdot 13\phi^6 + 2^6 \cdot 3^4\phi^5 + 2^3 \cdot 3 \cdot 5^2\phi^4 + 2^6 \cdot 3 \cdot 61\phi^3 + 2^4 \cdot 547\phi^2 + 2^6 \cdot 67\phi + 2 \cdot 3^3 \cdot 19.$$

Since $2 \mid c_i$ for $i = 1\ldots 8$ while $2 \nmid c_0$ and $2^2 \nmid c_8$ then $r_8(\phi+1)$ is irreducible in $\mathbb{Q}[\phi]$ by Eisensteins test. Hence $r_8(\phi)$ is irreducible in $\mathbb{Q}[\phi]$. Using iterative approximation one finds that the four real roots of $r_8(\phi)$ occur at $\phi \approx -0.876255, -0.5294, -0.086116, 0.279237$. These points define a quartic equation, $p_4(\phi)$, which is approximately

$$p_4(\phi) = \phi^4 + 1.21253\phi^3 + 0.168381\phi^2 - 0.123388\phi - 0.011155.$$

Dividing this quartic into $r_8(\phi)$ leads to another quartic, $q_4(\phi)$, where

$$q_4(\phi) = 27(\phi^4 - 1.21253\phi^3 + 25.3019\phi^2 - 38.3519\phi + 29.8819).$$

Applying Viète's method to find the roots of $q_4(\phi)$ leads to the four approximations $\phi \approx 4.96561 \exp^{1\pm 0.48844 i\pi}, 1.10085 \exp^{\pm 0.246651 i\pi}$. Since all the roots of $Res_\theta(f_\theta, f_\phi)$ are now known so are the positions of all the critical points of $f(\theta, \phi)$. Furthermore the type of critical point is determined by the second derivative test.

θ	ϕ	$f_{\theta\theta}$	$f_{\theta\phi}$	$f_{\phi\phi}$	$f_{\theta\phi}^2 - f_{\theta\theta}f_{\phi\phi}$	type
1	-1	72	72	72	0	sing.
-1	0	18	-36	72	0	sing.
0	1	72	-36	18	0	sing.
$-\sqrt{3}/3$	$-\sqrt{3}/3$	11.7	16	-43.7	768	saddle
$\sqrt{3}/3$	$\sqrt{3}/3$	-43.7	16	11.7	768	saddle
$\frac{-2-\sqrt{19}}{15}$	$\frac{2+\sqrt{19}}{15}$	38.2	-32.4	38.2	-413.5	min.
$\frac{-2+\sqrt{19}}{15}$	$\frac{2-\sqrt{19}}{15}$	-6.18	18.5	-6.18	306.9	saddle
0.086116	-0.876255	-24.2	-35.0	6.3	1380.3	saddle
-0.279237	-0.5294	-9.88	1.22	-29.8	-293.1	max.
0.876255	-0.086116	6.3	-35.0	-24.2	1380.3	saddle
0.5294	0.279237	-29.8	1.22	-9.88	-293.1	max.

Table 3.2: Critical Points of the Tiki surface

The second partial derivatives are given by

$$\begin{aligned}
f_{\theta\theta} &= 12\theta^2(3\phi-1)^2 + 12\theta(9\phi^3 - 9\phi^2 - 11\phi - 1) \\
&\quad + 6(3\phi^4 + 6\phi^3 + 2\phi^2 + 2\phi - 1) \\
f_{\theta\phi} &= 24\theta^3(3\phi - 1) + 6\theta^2(27\phi^2 - 18\phi - 11) + 6\theta(12\phi^3 + 18\phi^2 + 4\phi + 2) \\
&\quad + 6(4\phi^3 - 11\phi^2 - 2\phi + 3) \\
f_{\phi\phi} &= 18\theta^4 + 2\theta^3(54\phi - 18) + 3\theta^2(36\phi^2 + 36\phi + 4) + 6\theta(12\phi^2 - 22\phi - 2) \\
&\quad + 6(2\phi^2 + 2\phi - 1).
\end{aligned}$$

Recall that

if $f_{\theta\phi}^2 - f_{\theta\theta}f_{\phi\phi} > 0$ then the critical point is a saddle otherwise -

if $f_{\theta\theta}$ and $f_{\phi\phi}$ are greater than zero it is a local minimum and

if $f_{\theta\theta}$ and $f_{\phi\phi}$ are less than zero it is a local maximum.

If $f(\theta, \phi)$ is also zero the critical point is a singularity. The coordinates of the critical points are listed in Table 3.2 along with their type. Using the first partial derivatives of $f(\theta, \phi)$ the level curves of the surface can be plotted as shown in Figure 3.2. This view is a somewhat deceptive indicator of the true nature of the surface (as Figure 3.3 shows) since the maxima and minima are not as pronounced as the level curves suggest. It was from one of the first contour plots of the surface $f(\theta, \phi)$ in the region $|\ \theta\ | \leq 1$, $|\ \phi\ | \leq 1$ that the name 'Tiki' first emerged due to the resemblance to a popular Maori symbol. Consider the function of three variables $F(\theta, \phi, \eta) := f(\theta, \phi) - \eta^2$, the zeros of which correspond to the surface of (3.15). Clearly

$$\begin{aligned}
\frac{\partial F}{\partial \theta}(\theta, \phi, \eta) &= \frac{\partial f}{\partial \theta}(\theta, \phi) \\
\frac{\partial F}{\partial \phi}(\theta, \phi, \eta) &= \frac{\partial f}{\partial \phi}(\theta, \phi) \\
\frac{\partial F}{\partial \eta}(\theta, \phi, \eta) &= -2\eta.
\end{aligned}$$

So the critical points of $F(\theta, \phi, \eta)$ correspond to the critical points of $f(\theta, \phi)$ when η is zero. Similarly with the singularities.

3.7.2 Plotting Med Triangles on Tiki Surface

The aim in this section is to determine θ and ϕ in terms of a, b and c so that for any given med triangle a point can be plotted in the corresponding position of the $\theta\phi$-plane.

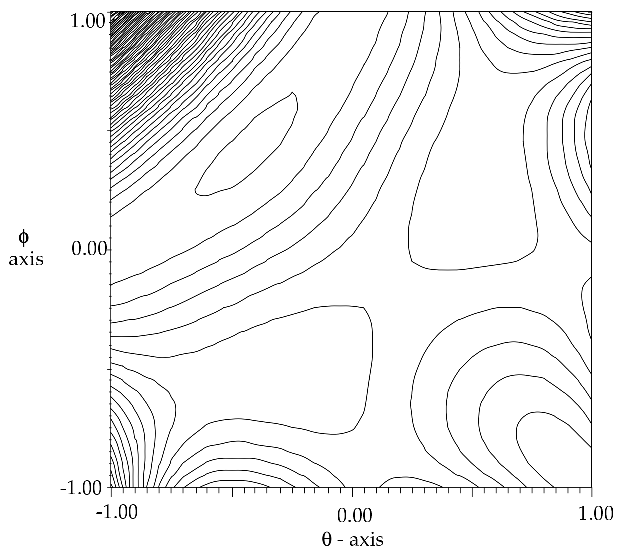

Figure 3.2: Level curves of Tiki surface

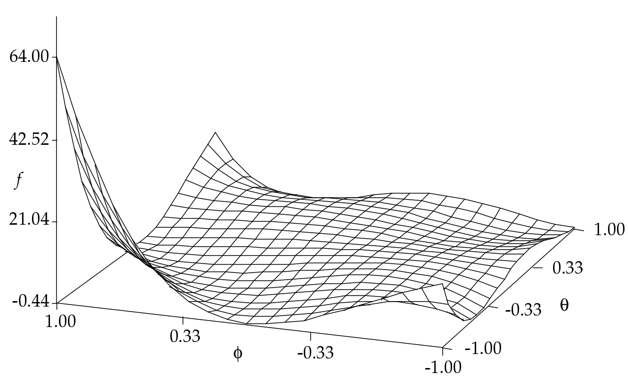

Figure 3.3: Three dimensional view of Tiki surface

Recall that all integer-sided triangles with two integer length medians are given by

$$\begin{aligned} a &= t\left[-2\phi\theta^2 + (-\phi^2 + 2\phi + 1)\theta + (-\phi^2 + 1)\right] \\ b &= t\left[(\phi - 1)\theta^2 + 2(\phi^2 + \phi)\theta + (-\phi + 1)\right] \\ c &= t\left[(\phi + 1)\theta^2 + (-\phi^2 + 2\phi + 1)\theta + (\phi^2 - \phi)\right]. \end{aligned}$$

One easily obtains

$$\begin{aligned} a + b + c &= 2t\left[(3\theta - 1)\phi + (\theta + 1)\right] \\ a + b - c &= 2t\left[(\theta - 1)\phi^2 + (-\theta^2 + \theta)\phi + (1 - \theta^2)\right] \\ a - b + c &= 2t\left[-2\theta\phi^2 + (\theta - \theta^2)\phi + (\theta^2 + \theta)\right]. \end{aligned}$$

Eliminating ϕ^2 from the last two equations and then ϕ with the help of the expression for $a + b + c$ leads to a quadratic in θ. Hence

$$2\theta(a+b-c)+(\theta-1)(a-b+c) = 2t\theta(1-\theta)\left[(3\theta - 1)\phi + (\theta + 1)\right] = \theta(1-\theta)(a+b+c)$$

or

$$(a + b + c)\theta^2 + (2a - 2c)\theta + (-a + b - c) = 0.$$

Solving the quadratic for θ gives

$$\theta = \frac{c - a \pm \sqrt{2a^2 + 2c^2 - b^2}}{a + b + c}.$$

Similarly by eliminating θ^2 and then θ from the above three equations one obtains the analogous expression for ϕ in terms of a, b and c namely

$$\phi = \frac{b - c \pm \sqrt{2b^2 + 2c^2 - a^2}}{a + b + c}.$$

Using equations (3.1) leads to

$$\begin{aligned} \theta &= \frac{c - a \pm l}{a + b + c} \\ \phi &= \frac{b - c \pm k}{a + b + c}. \end{aligned} \tag{3.16}$$

If the parameters (a, b, c) correspond to any med triangle then the above expressions are rational for any of the six permutations of a, b and c. So one med triangle will correspond to 24 distinct points in the $\theta\phi$-plane. The three triangle inequalities in terms of θ and ϕ define regions in the $\theta\phi$-plane which cannot contain points corresponding to med triangles.
Case (i) $a + b \le c$ iff $\{(\theta - 1)\phi^2 + (-\theta^2 + \theta)\phi + (1 - \theta^2)\} \le 0$. Solving the quadratic for ϕ in terms of θ leads to the inequality $\theta < 1$ which implies that $\phi \le \theta + 1$.

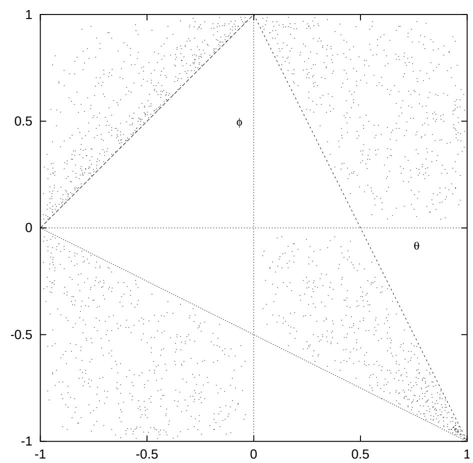

Figure 3.4: Med triangles in the $\theta\phi$-plane

Case (ii) $b+c \leq a$ iff $\{(2\theta+2)\phi^2+(4\theta^2+2\theta-2)\phi\} \leq 0$. As before $\phi < 0$ implies that $\phi \leq -2\theta+1$ while $\phi > 0$ implies that $\phi \leq -2\theta+1$.
Case (iii) $c+a \leq b$ iff $\{-2\theta\phi^2+(\theta-\theta^2)\phi+(\theta^2+\theta)\} \leq 0$. Hence $\theta < 0$ implies that $2\phi \leq -\theta-1$ while $\theta > 0$ implies that $2\phi \leq -\theta-1$.

The boundaries of the six regions in the positive quadrant of the $\theta\phi$-plane are implicitly defined by the curves $a = b$, $b = c$ and $c = a$. Substituting for θ and ϕ leads to the three equations

$$(-3\phi+1)\theta^2 + (-3\phi^2+1)\theta + (-\phi^2+\phi) = 0$$
$$-2\theta^2 + (3\phi^2-1)\theta + (-\phi^2+1) = 0$$
$$(-3\phi-1)\theta^2 + (-2\phi^2+\phi+1) = 0.$$

The point of intersection of the three curves $a = b$, $b = c$ and $c = a$ occurs at $(\theta, \phi) = \left(\sqrt{3}/3, \sqrt{3}/3\right)$ (see Figure 3.5) which corresponds to a critical point of the Tiki surface, as substitution of $a = b = c$ into the equations for θ and ϕ readily shows.

3.7.3 Generating New Med Triangles

The intersection of the Tiki surface (3.15) with any plane of the form $\theta = k$ or $\phi = k$, for constant k, leads to an elliptic curve. Choosing the constant to correspond to a med triangle provides a rational point on the elliptic curve from which one can generate 'new' rational points and hence 'new' med triangles.

Consider the case $(p_1, q_1, p_2, q_2) = (1, 2, 8, 9)$ which corresponds to the med triangle $2(491, 466, 807)$ when substituted into (3.14). This in turn corresponds to the rational point $(\theta, \phi, \eta) = (8/9, 1/2, 515/324)$ on equation (3.15). So the

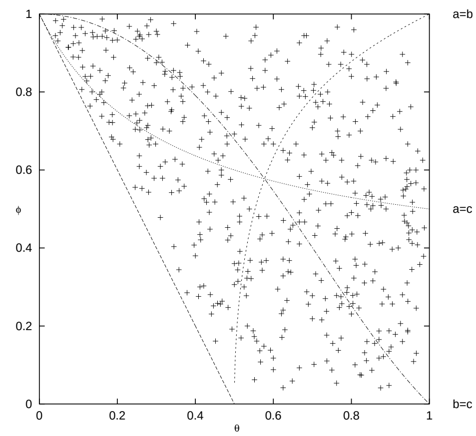

Figure 3.5: Med triangles in the positive quadrant

intersection of the Tiki surface and the plane $\phi = 1/2$ is the quartic curve

$$\eta^2 = (4\theta^4 - 244\theta^3 + 69\theta^2 + 146\theta + 25)/24.$$

There exists a unique quadratic curve which 'touches' the above quartic at $\theta = 8/9$ (with multiplicity three) and intersects the quartic again at a distinct rational point. If the quadratic and its derivatives are

$$\begin{aligned} \eta_p &= a_p\theta^2 + b_p\theta + c_p \\ \eta_p' &= 2a_p\theta + b_p \\ \eta_p'' &= 2a_p. \end{aligned}$$

While from the quartic curve

$$\begin{aligned} 2\eta\eta' &= (16\theta^3 - 732\theta^2 + 138\theta + 146)/24 \\ 2\eta\eta'' + 2{\eta'}^2 &= (48\theta^2 - 1464\theta + 138)/24. \end{aligned}$$

So when $\theta = \frac{8}{9}$ then $\eta = \frac{515}{324}$, $\eta' = -\frac{108791}{18540}$, $\eta'' = -\frac{5981042989}{136590875}$ and hence the coefficients of the quadratic are

$$a_p = -\frac{11962085978}{546363500}, \quad b_p = \frac{18059920297}{546363500}, \quad c_p = -\frac{5733289607}{546363500}.$$

To intersect the quartic with the quadratic consider the zeros of $g(\theta)$ where

$$g(\theta) := \eta^2 - \eta_p^2.$$

Recall that $\theta = 8/9$ is a multiplicity three intersection so

$$g(\theta) = (\theta - 8/9)^3(P_2\theta - Q_2)$$

say, where by equating these two forms for g one finds

$$P_2 = -\frac{2^4 \cdot 3^9 \cdot 38851 \cdot 2922226057}{(2 \cdot 5^3 \cdot 103^3)^2}$$
$$Q_2 = -\frac{2^9 \cdot 3^9 \cdot 53 \cdot 59 \cdot 151 \cdot 727 \cdot 13337}{(2^2 \cdot 5^3 \cdot 103^3)^2}.$$

So the new rational point is the fourth zero of g, namely

$$\theta = \frac{Q_2}{P_2} = \frac{36625821758584}{113531404540507}.$$

Setting $(p_1, q_1, p_2, q_2) = (1, 2, 36625821758584, 113531404540507)$ in equation (3.14) leads to the new med triangle with half-sides

$$a = 4804\ 49439\ 15111\ 19825\ 00700\ 36514$$
$$b = 2426\ 65920\ 06925\ 26285\ 88830\ 27903$$
$$c = 6240\ 96030\ 79770\ 62497\ 83214\ 45539,$$

and medians

$$k = 4358\ 22175\ 92717\ 60765\ 24502\ 46367$$
$$l = 10870\ 91665\ 73618\ 90661\ 59312\ 12805$$
$$m = 8160\ 47514\ 09485\ 87997\ 98039\ 67108.$$

To convert the quartic to a cubic curve first make the transformation $N := 2\eta$ and $\Theta := \theta - 61/4$ to give

$$N^2 = \Theta^4 - 22050/4^2\Theta^2 - 444960/4^2\Theta - 40366431/4^4.$$

Next set $x := 4\Theta$ and $y := 16N$ to obtain

$$y^2 = x^4 - 6(3675)x^2 - 4(444960)x - 40366431.$$

The crucial step occurs via the Mordell transformation [12, p. 139] namely, $2x(s - 3675) = t + 444960$ and $y = 2s - x^2 + 3675$ which leads to the elliptic curve

$$E \quad : \quad t^2 = 4s^3 - 150444s + 9595800.$$

The rational point $(\theta, \eta) = \left(\frac{8}{9}, \frac{515}{324}\right)$ of the quartic curve is transformed to a rational point $(s, t) = \left(-\frac{13133}{9^2}, \pm\frac{3000368}{9^3}\right)$ on the elliptic curve. If $t^2 = 0$ then E becomes $(s-75)(s^2+75s-31986) = 0$ and so $s = 75, (-75 \pm 9\sqrt{1649})/2$. Hence the only rational point of order 2 on $E(\mathbb{Q})$ is $(s, t) = (75, 0)$. Define $T := t/2$ which leads to

$$T^2 = s^3 - 37611s + 2398950.$$

From elliptic curve theory [16, p. 221] the points of finite order on this curve must satisfy either $T = 0$ or $T^2 \mid 4(-37611)^3+27(2398950)^2$ i.e. $T^2 \mid 2^{16}3^{12}17{\cdot}97$

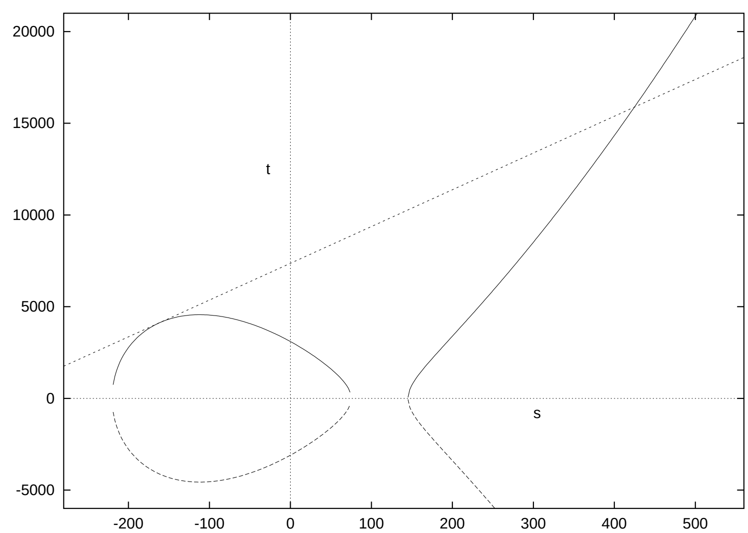

Figure 3.6: The elliptic curve $t^2 = 4s^3 - 150444s + 9595800$

or $T \mid 2^8 3^6$. A computer search using Newton's method to solve the resulting 63 cubic equations lead to four 'candidate' torsion points, namely $(s,T) = (147, \pm 216)$ and $(s,T) = (-213, \pm 864)$. However, using the group law doubling formula for the elliptic curve one can show that none of the above four points generate only integral points. Hence $(s,t) = (75,0)$ is the only torsion point while $(s,t) = \left(-\frac{13133}{9^2}, \pm\frac{3000368}{9^3}\right)$ must be a point of infinite order on $E(\mathbb{Q})$. Each new rational point that is generated from the latter corresponds to a new med triangle and so the tangent-chord process produces an infinite number of med triangles on this curve alone.

3.8 Euler's Parametrization Revisited

Recall Euler's partial parametrization of med triangles cf. (3.10)

$$\begin{aligned} a &= \alpha(-9\alpha^4 + 10\alpha^2\beta^2 + 3\beta^4)/2 \\ b &= \left[\alpha(9\alpha^4 + 26\alpha^2\beta^2 + \beta^4) - \beta(9\alpha^4 - 6\alpha^2\beta^2 + \beta^4)\right]/4 \\ c &= \left[\alpha(9\alpha^4 + 26\alpha^2\beta^2 + \beta^4) + \beta(9\alpha^4 - 6\alpha^2\beta^2 + \beta^4)\right]/4. \end{aligned}$$

Then from the equations (3.1) one can obtain the medians

$$\begin{aligned} k &= \beta(27\alpha^4 + 10\alpha^2\beta^2 - \beta^4)/2 \\ l &= \left[\alpha(27\alpha^4 - 18\alpha^2\beta^2 + 3\beta^4) + \beta(9\alpha^4 + 26\alpha^2\beta^2 + \beta^4)\right]/4 \\ m &= \left[\alpha(27\alpha^4 - 18\alpha^2\beta^2 + 3\beta^4) - \beta(9\alpha^4 + 26\alpha^2\beta^2 + \beta^4)\right]/4. \end{aligned}$$

The inversion formulae for θ and ϕ i.e. (3.16) lead to

$$\theta = \frac{c-a\pm l}{a+b+c} = \frac{9\alpha^4+32\alpha^3\beta+10\alpha^2\beta^2-4\alpha\beta^3+\beta^4}{4\alpha\beta(9\alpha^2+\beta^2)}$$
$$\phi = \frac{b-c\pm k}{a+b+c} = \frac{(9\alpha^2-\beta^2)(\alpha^2+\beta^2)}{\alpha\beta(9\alpha^2+\beta^2)}.$$

Thus one obtains a relationship between the parameters defining Euler's subset of **Med** triangles and the parameters from Theorem 12 describing all integer-sided triangles with at least two rational medians. In fact, setting $\lambda = \frac{a}{b}$ means that Euler's parametrization in terms of a rational parameter becomes

$$\begin{aligned} a &= 9\lambda^5 - 9\lambda^4 + 26\lambda^3 + 6\lambda^2 + \lambda - 1 \\ b &= 9\lambda^5 + 9\lambda^4 + 26\lambda^3 - 6\lambda^2 + \lambda + 1 \\ c &= 18\lambda^5 - 20\lambda^3 - 6\lambda. \end{aligned} \tag{3.17}$$

It is tempting to consider the existence of a parametrization, lying somewhere between Theorem 12 and equations (3.17), which describes all integer-sided triangles with three rational medians.

Chapter 4

Heron Triangles with Integer Medians

4.1 Two Median Heron Triangles

Heron of Alexandria has his name associated with the formula for the area of a triangle in terms of the lengths of the sides. If a triangle has, as usual, sides of length (a, b, c) and an area of $\triangle$ then Heron's formula is

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$ is the semiperimeter. As a consequence of this any triangle with rational sides and rational area is invariably called a "Heron triangle". Recall from Chapter 1 that there is a one-to-one correspondence between the similarity classes of alt triangles and Heron triangles. Hence in this thesis the definition of a Heron triangle is restricted to that of an integer-sided triangle with integer area.

The question alluded to in the previous chapter would read as follows.

"Can any Heron triangle have two integer medians?"

Certainly H. Schubert [15] was of the opinion that no such triangle could exist. Such triangles do, in fact exist and an account of Schubert's error will be given in a subsequent section. If a triangle has sides (a, b, c) - (not half-sides) then in equation (3.14) one has a parametrization of all integer-sided triangles with at least two integer length medians, namely

$$\begin{aligned} a &= -4p_1q_1p_2^2 + 2(2p_1q_1 - p_1^2 + q_1^2)p_2q_2 + 2(q_1^2 - p_1^2)q_2^2 \\ b &= (2p_1q_1 - 2q_1^2)p_2^2 + 2(2p_1q_1 + 2p_1^2)p_2q_2 + 2(q_1^2 - p_1q_1)q_2^2 \\ c &= (2p_1q_1 + 2q_1^2)p_2^2 + 2(2p_1q_1 - p_1^2 + q_1^2)p_2q_2 + 2(p_1^2 - p_1q_1)q_2^2 \end{aligned} \tag{4.1}$$

Rewriting this in terms of rational parameters $\phi = \frac{p_1}{q_1}$ and $\theta = \frac{p_2}{q_2}$ as in Theorem

12 gives

$$
\begin{aligned}
a &= 2t\left[-2\phi\theta^2 + (-\phi^2 + 2\phi + 1)\theta + (-\phi^2 + 1)\right] \\
b &= 2t\left[(\phi - 1)\theta^2 + 2(\phi^2 + \phi)\theta + (-\phi + 1)\right] \\
c &= 2t\left[(\phi + 1)\theta^2 + (-\phi^2 + 2\phi + 1)\theta + (\phi^2 - \phi)\right]
\end{aligned}
\tag{4.2}
$$

Substituting the parametrization (4.2) into Heron's formula leads to a degree eleven polynomial in two variables which must be the square of a rational number. Firstly

$$
\begin{aligned}
s &= 2(3\phi + 1)\theta + 2(1 - \phi) \\
s - a &= 2\phi\left[2\theta^2 + (\phi + 1)\theta + (\phi - 1)\right] \\
&= 2\phi(\theta + 1)(2\theta + \phi - 1) \\
s - b &= 2\theta\left[-2\phi^2 + (1 - \theta)\phi + (\theta + 1)\right] \\
&= 2\theta(1 - \phi)(\theta + 2\phi + 1) \text{ and} \\
s - c &= 2\left[(\theta - 1)\phi^2 + 2(-\theta^2 + \theta)\phi + (1 - \theta^2)\right] \\
&= 2(1 - \theta)(1 + \phi)(\theta - \phi + 1).
\end{aligned}
$$

Hence

$$
\gamma^2 = \theta\phi(1 - \theta^2)(1 - \phi^2)(3\theta\phi + \theta - \phi + 1)(2\theta + \phi - 1)(\theta + 2\phi + 1)(\theta - \phi + 1)
\tag{4.3}
$$

where $\gamma = \frac{\triangle}{4}$. Note that (4.3) is identically a rational square whenever $\theta = 0, \pm 1$ or $\phi = 0, \pm 1, 1 - 2\theta, 1 + \theta, \frac{1+\theta}{1-3\theta}, \frac{1-\theta}{2}$. So any non-trivial rational values of θ and ϕ which lead to a rational γ will correspond to a Heron triangle with two integer medians.

4.2 Even Semiperimeter

The triangle $2(25, 31, 51)$ has two integer medians while the area is irrational and the semiperimeter is odd. The triangle $(91, 250, 289)$ has integer area but only one integer median and the semiperimeter is odd. However, combining both conditions leads to the following theorem.

Theorem 13 *Any Heron triangle with two integer medians has an even semiperimeter.*

Proof: For a triangle (a, b, c) let k and l be the medians of integer length so that

$$
\begin{aligned}
4k^2 &= 2b^2 + 2c^2 - a^2 \\
4l^2 &= 2a^2 + 2c^2 - b^2.
\end{aligned}
$$

So a and b must be even but from Heron's formula

$$(4\triangle)^2 = (a+b+c)(b+c-a)(a+c-b)(a+b-c)$$

whence c must also be even. Letting $a = 2A$, $b = 2B$, $c = 2C$ leads to

$$k^2 = 2B^2 + 2C^2 - A^2$$
$$l^2 = 2A^2 + 2C^2 - B^2.$$

Case (i) : If $A, B \equiv 0 \pmod 2$ and $C \equiv 1 \pmod 2$ then $l^2 \equiv 2 \pmod 4$ which is impossible.
Case (ii) : If $A, C \equiv 0 \pmod 2$ and $B \equiv 1 \pmod 2$ then $l^2 \equiv 3 \pmod 4$ which is impossible.
Case (iii) : If $A, B, C \equiv 1 \pmod 2$ then $l^2 \equiv 3 \pmod 4$ which is impossible.
So either all of A, B, C are even or exactly one of A, B, C is even. ■

4.3 Area Divisibility

Another constraint on the set of Heron triangles with two integer medians is given in the following theorem but was motivated by an inspection of the areas of the triangles uncovered by a computer search.

Theorem 14 *If a Heron triangle has two integer medians then its area is divisible by 120.*

Proof: Equation (4.3) in terms of the parameters p_1, q_1, p_2, q_2 is

$$\triangle^2 = 16^2 p_1 q_1 p_2 q_2 (p_1 - q_1)(p_1 + q_1)(p_2 - q_2)(p_2 + q_2) L_1 L_2 L_3 L_4 \tag{4.4}$$

where

$$L_1 = p_1(3p_2 + q_2) - q_1(p_2 - q_2)$$
$$L_2 = 2p_1 q_2 + q_1(p_2 - q_2)$$
$$L_3 = p_1 q_2 + q_1(2p_2 + q_2)$$
$$L_4 = p_1 q_2 - q_1(p_2 - q_2).$$

Firstly consider equation (4.4) modulo 3. If $p_1 \equiv 0 \pmod 3$ then $\triangle^2 \equiv 0 \pmod 3$ and hence $\triangle \equiv 0 \pmod 3$. Similarly if $q_1 \equiv 0 \pmod 3$ then $\triangle^2 \equiv 0 \pmod 3$ and hence $\triangle \equiv 0 \pmod 3$. But if p_1 and q_1 are both non-zero modulo 3 then clearly $p_1 \pm q_1 \equiv 0 \pmod 3$ which again implies that $\triangle \equiv 0 \pmod 3$. So the area is divisible by three.
Secondly consider equation (4.4) modulo 5. If p_1, q_1, p_2, or $q_2 \equiv 0 \pmod 5$ then as before $\triangle^2 \equiv 0 \pmod 5$ and hence $\triangle \equiv 0 \pmod 5$. Next consider $q_1 \equiv \pm 1, \pm 2 \pmod 5$. If $p_1 \equiv \pm q_1 \pmod 5$ then $\triangle^2 \equiv 0 \pmod 5$ which again leads to $\triangle \equiv 0 \pmod 5$. Similar reasoning for p_2, and q_2 leads to the only

remaining cases of $q_1 \equiv \pm 1, \pm 2 \pmod 5$, $p_1 \not\equiv \pm q_1 \pmod 5$ and $q_2 \equiv \pm 1, \pm 2 \pmod 5$, $p_2 \not\equiv \pm q_2 \pmod 5$. These can be further reduced by noting that

$$\begin{aligned} \triangle^2(-p_1, -q_1, p_2, q_2) &= \triangle^2(p_1, q_1, p_2, q_2) \\ \triangle^2(p_1, q_1, -p_2, -q_2) &= \triangle^2(p_1, q_1, p_2, q_2) \\ \triangle^2(-p_1, -q_1, -p_2, -q_2) &= \triangle^2(p_1, q_1, p_2, q_2). \end{aligned}$$

The remaining cases all lead to $\triangle \equiv 0 \pmod 5$, as shown in Table 4.1, so the

p_1	q_1	p_2	q_2	$i : L_i \equiv 0 \pmod 5$
2	1	2	1	2
2	1	−2	1	4
−2	1	2	1	1
−2	1	−2	1	3
1	2	2	1	1
1	2	−2	1	3
−1	2	2	1	2
−1	2	−2	1	4
2	1	1	2	4
2	1	−1	2	2
−2	1	1	2	3
−2	1	−1	2	1
1	2	1	2	3
1	2	−1	2	1
−1	2	1	2	4
−1	2	−1	2	2

Table 4.1: Terms in equation 4.4 congruent to zero modulo 5

area is divisible by five.
Finally consider $(\triangle/16)^2$ modulo 2. If any of p_1, q_1, p_2, or $q_2 \equiv 0 \pmod 2$ then $\triangle/16 \equiv 0 \pmod 2$. The remaining cases is all of $p_1, q_1, p_2, q_2 \equiv 1 \pmod 2$ which leads to $\triangle/16 \equiv 0 \pmod 2$. So $32 \mid \triangle$ but since $\gcd(a, b, c, \triangle) = 4$, by equations (4.1) and (4.4), the area is divisible by at most eight. Since $\gcd(3, 5, 8) = 1$ then the product $3 \cdot 5 \cdot 8 = 120 \mid \triangle$. ■

4.4 Schubert's Oversight

Using Theorems 12, 13 and 14 one can readily implement a reasonably efficient computer search for Heron triangles with two integer medians. To date, only six such triangles have been discovered and they are listed in Table 4.2 below and displayed in Figure 4.2. The first three triangles of Table 4.2 were independently discovered by Randall L. Rathbun and by Arnfried Kemnitz. Schubert's argument that no triangle like those in Table 4.2 could exist went as follows. He considered a Heron parallelogram as in Figure 4.1. By the sine rule

Sides			Medians			Area
a	b	c	k	l	m	$\triangle$
52	102	146	—	97	35	1680
582	1252	1750	—	1144	433	221760
2482	7346	8736	7975	—	3314	8168160
22514	28768	29582	—	22002	21177	302793120
27632	30310	57558	43874	—	3589	95726400
371258	3647350	3860912	3751059	2048523	—	569336866560

Table 4.2: Heron triangles with two integer medians

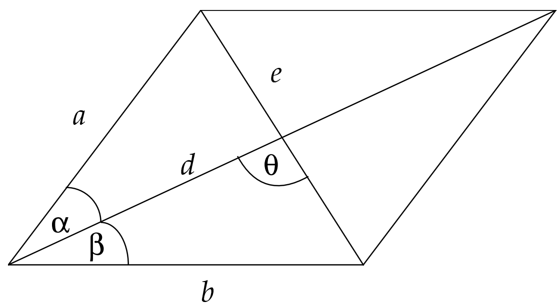

Figure 4.1: Heron Parallelogram

$b\sin(\theta+\beta) = a\sin(\theta-\alpha)$, while consideration of areas leads to $ad\sin\alpha = bd\sin\beta$. Eliminating a/b between these two equations leads to

$$\frac{\sin\beta}{\sin\alpha} = \frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)}$$

or

$$2\cot\theta = \cot\alpha - \cot\beta. \tag{4.5}$$

Since $\tan\frac{A}{2} = \frac{1-\cos A}{\sin A}$ and the sine and cosine of all the angles above are rational the tangent of all the half-angles must be rational as well. So let

$$\tan\frac{\alpha}{2} = \frac{n}{m}, \quad \tan\frac{\beta}{2} = \frac{q}{p}, \quad \tan\frac{\theta}{2} = \frac{y}{x}.$$

Now the $\tan 2A$ identity leads to $\cot A = \frac{1-\tan 2(A/2)}{2\tan(A/2)}$ and so equation (4.5) becomes

$$2\left(\frac{x^2-y^2}{xy}\right) = \frac{m^2-n^2}{mn} - \frac{p^2-q^2}{pq}$$

or

$$2(x^2-y^2)mnpq = xy(mp+nq)(mq-np). \tag{4.6}$$

Schubert deduced that the only solutions to equation (4.6) were

$$(x, y) = (mq, np) \quad \text{or} \quad (mp, nq).$$

This led him to the conclusion that no Heron triangle could have more than one integer median. However, if $(m, n, p, q, x, y) = (35, 6, 84, 5, 7, 40)$ then clearly $x \neq mq$ and $x \neq mp$ but the parameters do satisfy equation (4.6) and so are a clear counterexample to Schubert's conclusion. Furthermore this set of parameters corresponds to the first entry of Table 4.2 shown previously. It is useful to determine the values of Schubert's parameters for all of the Heron-2-median triangles. Notice that there is a connection between the first, second,

semi-perimeter	smallest medians $(m/n, p/q, x/y)$	largest median $(m/n, p/q, x/y)$
150	$\left(\frac{2}{3}, \frac{4}{1}, \frac{3}{8}\right)$	$\left(\frac{\mathbf{35}}{\mathbf{6}}, \frac{84}{5}, \frac{7}{40}\right)$
1792	$\left(\frac{\mathbf{105}}{\mathbf{176}}, \frac{360}{77}, \frac{32}{99}\right)$	$\left(\frac{\mathbf{35}}{\mathbf{6}}, \frac{18}{1}, \frac{10}{63}\right)$
9282	$\left(\frac{231}{260}, \frac{2431}{420}, \frac{17}{55}\right)$	$\left(\frac{728}{51}, \frac{17}{1}, \frac{48}{91}\right)$
40432	$\left(\frac{4845}{1736}, \frac{357}{95}, \frac{1360}{1767}\right)$	$\left(\frac{1395}{476}, \frac{620}{153}, \frac{63}{85}\right)$
57750	$\left(\frac{75}{98}, \frac{\mathbf{176}}{\mathbf{105}}, \frac{539}{800}\right)$	$\left(\frac{\mathbf{3080}}{\mathbf{111}}, \frac{14504}{275}, \frac{147}{1850}\right)$
3939760	$\left(\frac{1344}{605}, \frac{\mathbf{3080}}{\mathbf{111}}, \frac{363}{4736}\right)$	$\left(\frac{255189}{5312}, \frac{165585}{3256}, \frac{36480}{70301}\right)$

Table 4.3: Heron-2-median triangles w.r.t. Schubert's Parameters

fifth and sixth triangles of Table 4.3 shown by the outlined parameters. Recall that each pair of parameters is just the tangent of the corresponding half-angle. Thus equal ratios of parameters correspond to congruent internal angles. While the inversely related ratios of parameters, for the second and fifth triangles, correspond to supplementary internal angles (see Figure 4.2).

4.5 Defining Surface

Consider the surface defined by the equation $z = f(\theta, \phi)$ where

$$\begin{aligned} f(\theta, \phi) = \theta\phi(1-\theta^2)(1-\phi^2)(3\theta\phi + \theta - \phi + 1) \\ \times (2\theta + \phi - 1)(\theta + 2\phi + 1)(\theta - \phi + 1) \end{aligned} \tag{4.7}$$

If $f(\theta, \phi)$ is the square of a rational number for some rational choice of (θ, ϕ) then this corresponds to a Heron triangle with two integer medians. Note that $f(\theta, \phi) = f(-\phi, -\theta)$ and so the surface is symmetric about the line $\phi = -\theta$. This makes it relatively easy to obtain the critical points of the surface along the line of symmetry. Define g as follows

$$g(\theta) := f(\theta, -\theta) = \theta^2(1-\theta)^5(1+\theta)^2(1+3\theta)(1+2\theta).$$

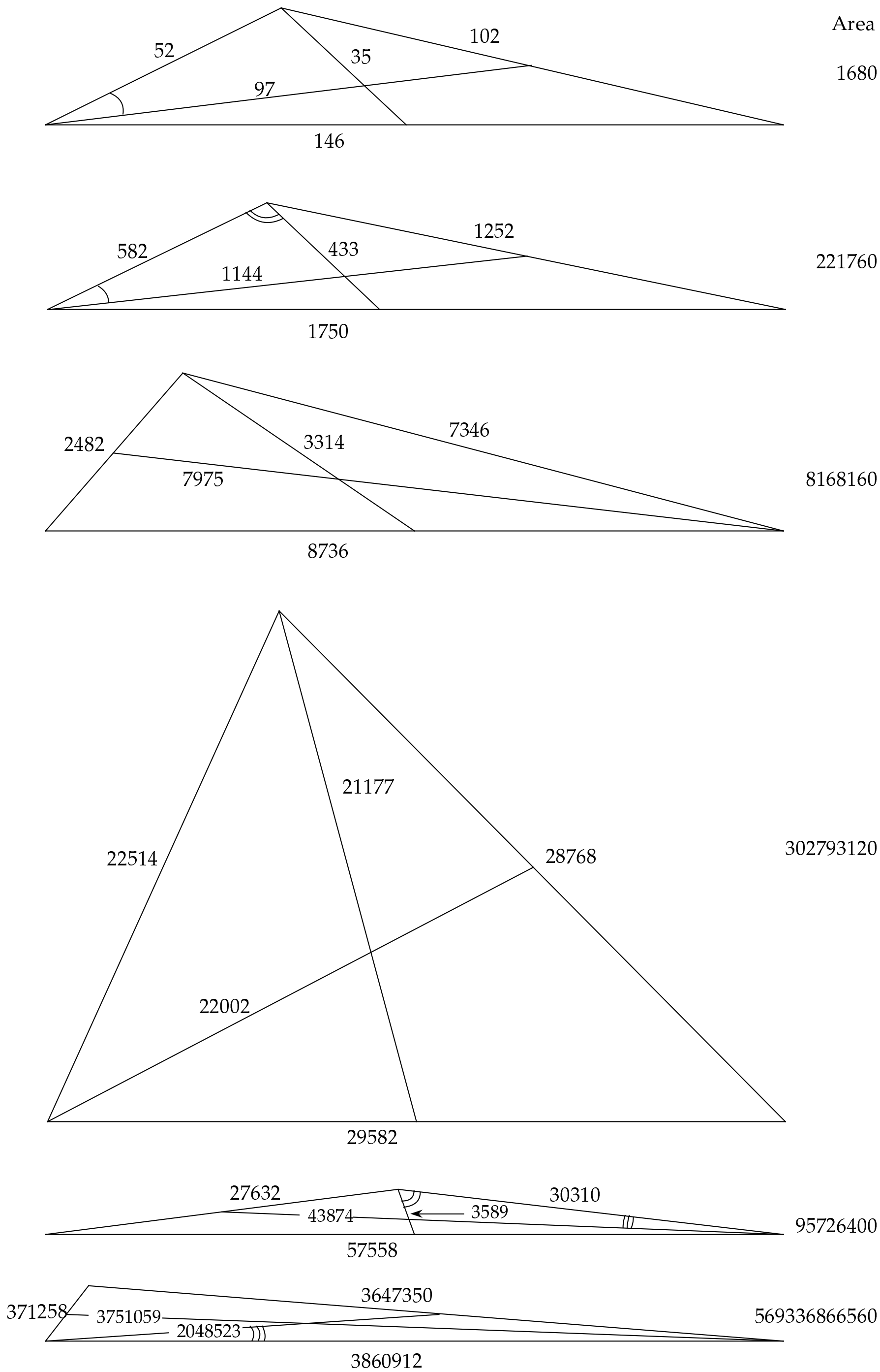

Figure 4.2: Heron Triangles with two integer medians

Then the derivative $g'(\theta) = -2\theta(1-\theta)^4(1+\theta)(33\theta^4+34\theta^3-6\theta-1)$ is zero when

$$q = 0, \pm 1, -0.810387, -0.432087, -0.206637, 0.418807.$$

These correspond to seven of the critical points of the Heron-2-median surface defined by equation (4.7). Now consider the partial derivatives of $f(\theta,\phi)$, namely,

$$\begin{aligned}
f_\theta(\theta,\phi) = \phi(1-\phi^2)[&-14(1+3\phi)\theta^6\\
&-6(5+10\phi+9\phi^2)\theta^5\\
&+5(-1+3\phi-3\phi^2+9\phi^3)\theta^4\\
&+8(3+5\phi+6\phi^2-5\phi^3+3\phi^4)\theta^3\\
&-6(-2-1\phi+2\phi^3+\phi^4)\theta^2\\
&-2(1+3\phi^2-10\phi^3+6\phi^4)\theta\\
&+(-1+\phi+3\phi^2-5\phi^3+2\phi^4)]\\
f_\phi(\theta,\phi) = &-2(1+6\phi-3\phi^2-12\phi^3)\theta^7\\
&+1(-5-20\phi-12\phi^2+40\phi^3+45\phi^4)\theta^6\\
&-(1-6\phi+6\phi^2-24\phi^3-15\phi^4+54\phi^5)\theta^5\\
&-2(-3-10\phi-9\phi^2+40\phi^3+15\phi^4-30\phi^5+21\phi^6)\theta^4\\
&+2(2+2\phi-6\phi^2-12\phi^3-5\phi^4+12\phi^5+7\phi^6)\theta^3\\
&+(-1-6\phi^2+40\phi^3-15\phi^4-60\phi^5+42\phi^6)\theta^2\\
&-(1-2\phi-12\phi^2+24\phi^3+5\phi^4-30\phi^5+14\phi^6)\theta.
\end{aligned}$$

All of the critical points of the Heron-2-median surface are then defined by the (θ,ϕ) values for which $f_\theta(\theta,\phi) = 0 = f_\phi(\theta,\phi)$ i.e. $Res_\theta(f_\theta, f_\phi) = 0$ where

$$Res_\theta(f_\theta,f_\phi) = \begin{vmatrix}
a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6\\
b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & 0 & 0 & 0 & 0 & 0\\
0 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & 0 & 0 & 0 & 0\\
0 & 0 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & 0 & 0 & 0\\
0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & 0 & 0\\
0 & 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & 0\\
0 & 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7
\end{vmatrix}$$

and the a_i, b_i are just the coefficients of f_θ, f_ϕ written as elements of $(\mathbb{Z}[\phi])[\theta]$. Expanding this determinant leads to the 101 degree polynomial

$$\begin{aligned} Res_\theta(f_\theta, f_\phi) = {} & 2^{16}3^{12}\phi^{26}(\phi-1)^{29}(\phi+1)^{26}(\phi-3)(\phi-2)(2\phi+1)^2 \\ & \times (33\phi^4 - 34\phi^3 + 6\phi - 1)r_{12}(\phi) \end{aligned}$$

where

$$\begin{aligned} r_{12}(\phi) = {} & 41503\phi^{12} - 77616\phi^{11} - 170576\phi^{10} + 166712\phi^9 \\ & + 334603\phi^8 - 8728\phi^7 - 245920\phi^6 - 113656\phi^5 \\ & + 49485\phi^4 + 46888\phi^3 + 4080\phi^2 - 2400\phi - 375. \end{aligned}$$

It turns out that $r_{12}(\phi)$ has only ten real roots while the quartic factor has four real roots (c.f. the polynomial $g'(\theta)$ earlier). The second partial derivatives are

$$\begin{aligned} f_{\theta\theta}(\theta,\phi) = {} & \phi(1-\phi^2)[-84(1+3\phi)\theta^5 \\ & - 30(5+10\phi+9\phi^2)\theta^4 \\ & + 20(-1+3\phi-3\phi^2+9\phi^3)\theta^3 \\ & + 24(3+5\phi+6\phi^2-5\phi^3+3\phi^4)\theta^2 \\ & - 12(-2-1\phi+2\phi^3+\phi^4)\theta \\ & - 2(1+3\phi^2-10\phi^3+6\phi^4)] \\ f_{\phi\theta}(\theta,\phi) = {} & -14(1+6\phi-3\phi^2-12\phi^3)\theta^6 \\ & + 6(-5-20\phi-12\phi^2+40\phi^3+45\phi^4)\theta^5 \\ & - 5(1-6\phi+6\phi^2-24\phi^3-15\phi^4+54\phi^5)\theta^4 \\ & - 8(-3-10\phi-9\phi^2+40\phi^3+15\phi^4-30\phi^5+21\phi^6)\theta^3 \\ & + 6(2+2\phi-6\phi^2-12\phi^3-5\phi^4+12\phi^5+7\phi^6)\theta^2 \\ & + 2(-1-6\phi^2+40\phi^3-15\phi^4-60\phi^5+42\phi^6)\theta \\ & - (1-2\phi-12\phi^2+24\phi^3+5\phi^4-30\phi^5+14\phi^6) \\ f_{\phi\phi}(\theta,\phi) = {} & -12(1-\phi-6\phi^2)\theta^7 \\ & - 4(5+6\phi-30\phi^2-45\phi^3)\theta^6 \\ & - (-6+12\phi-72\phi^2-60\phi^3+270\phi^4)\theta^5 \\ & - 4(-5-9\phi+60\phi^2+30\phi^3-75\phi^4+63\phi^5)\theta^4 \\ & + 4(1-6\phi-18\phi^2-10\phi^3-30\phi^4+21\phi^5)\theta^3 \\ & + 12(-\phi+10\phi^2-5\phi^3-25\phi^4+21\phi^5)\theta^2 \\ & - 2(-1-12\phi+36\phi^2+10\phi^3-75\phi^4+42\phi^5)\theta. \end{aligned}$$

Using these partial derivatives one can determine the nature of the critical points of the Heron-2-median surface which are given in the Table 9. As in Chapter 3, the partial derivatives can be used to plot the contours of the surface defined by equation (4.7) - see Figure 4.3 below.

θ	ϕ	$f_{\theta\theta}$	$f_{\theta\phi}$	$f_{\phi\phi}$	$f_{\theta\phi}^2 - f_{\theta\theta}f_{\phi\phi}$	type
1	−1	0	0	0	0	sing.
0.418807	−0.418807	−2.35	−0.89	−2.35	−4.76	max.
0.859967	−0.223668	7.29	0.74	3.11	−22.14	min.
0	0	0	−1	0	1	sing.
0.5	0	0	2.53	2.53	6.4	sing.
1	0	0	−16	0	256	sing.
−0.206637	0.206637	−0.68	0.29	−0.68	−0.37	max.
0.232571	0.244937	2.08	0.59	0.96	−1.66	min.
−0.432087	0.432087	1.38	−1.08	1.38	−0.73	min.
0.820022	0.677835	−48.9	−0.10	−18.7	−916	max.
−0.810387	0.810387	−14.4	0.80	−14.4	−209	max.
−1	1	0	−64	0	4096	sing.
0	1	0	0	0	0	sing.
1	1	0	128	0	16384	sing.
0.872588	1.610751	603	−107	137	−71325	min.
1	2	2592	−1296	0	10^6	sing.
−0.879142	2.388918	7934	1144	683	-4×10^6	min.
−1	3	41472	10368	0	10^8	sing.

Table 4.4: Critical Points of Heron-2-median surface

4.6 Plotting Heron-2-Median Triangles on the $\theta\phi$-plane

Using the relationship for θ and ϕ in terms of a, b and c determined in Chapter 3 leads to a correspondence between Heron-2-median triangles and points in the $\theta\phi$-plane. These points are shown in Figure 4.5 below (cf Figure 3.5). Notice that each Heron-2-median triangle corresponds to exactly two points in the positive quadrant and that the lines through those pairs of points intersect at $(\theta, \phi) = (1, -1)$.

s	symbol	(θ, ϕ)	(θ, ϕ)	Equation of line
150	×	$\left(\frac{1}{3}, \frac{2}{5}\right)$	$\left(\frac{1}{15}, \frac{24}{25}\right)$	$10\phi + 21\theta = 11$
1792	+	$\left(\frac{5}{16}, \frac{3}{7}\right)$	$\left(\frac{7}{128}, \frac{27}{28}\right)$	$77\phi + 160\theta = 83$
9282	△	$\left(\frac{13}{21}, \frac{85}{91}\right)$	$\left(\frac{40}{51}, \frac{21}{221}\right)$	$13\phi + 66\theta = 53$
40432	▽	$\left(\frac{25}{56}, \frac{12}{19}\right)$	$\left(\frac{243}{532}, \frac{217}{361}\right)$	$19\phi + 56\theta = 37$
57750	⋇	$\left(\frac{11}{21}, \frac{3}{77}\right)$	$\left(\frac{32}{375}, \frac{1369}{1375}\right)$	$11\phi + 24\theta = 13$
3939760	♦	$\left(\frac{285}{296}, \frac{37}{40}\right)$	$\left(\frac{48223}{49247}, \frac{513}{6655}\right)$	$5\phi + 259\theta = 254$

Table 4.5: Points (θ, ϕ) corresponding to Heron-2-med triangles

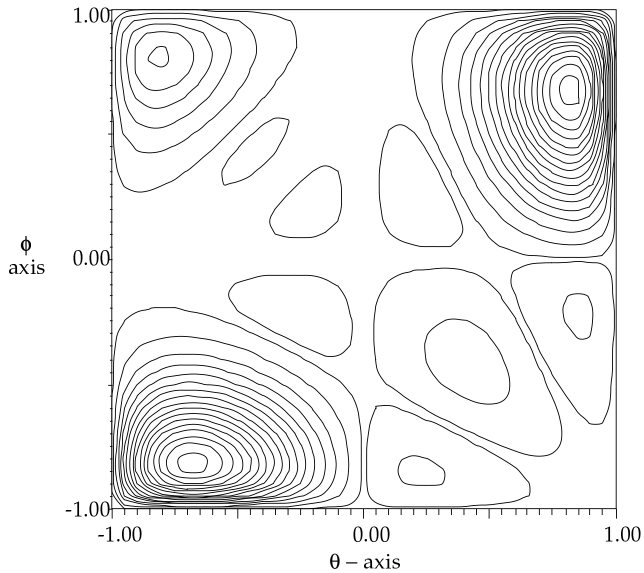

Figure 4.3: Heron-2-median Contours

4.7 Elliptic Curves on Surface

Recall from Chapter 3 that there is an infinite number of med triangles since the Tiki surface contained an elliptic curve with a rational point of infinite order (i.e. the curve has rank > 0). Attempting the same approach on the surface defined by equation (4.7) one could initially consider the intersection of the surface with planes defined by $a\theta + b\phi + c = 0$ where a, b and c are all rational. These planes are equivalent to either $\phi = a\theta + c$ or $\theta = c$.

First consider the planes of the form $\theta = c$. Thus equation (4.3) becomes

$$\begin{aligned}\gamma^2 &= c(1-c^2)\phi(1-\phi^2)[(3c-1)\phi + (c+1)] \\ &\quad \times [\phi + (2c-1)][2\phi + (c+1)][-\phi + (c+1)]\end{aligned} \tag{4.8}$$

which may lead to an elliptic curve in one of the two following ways. Let $p_n(\phi)$ denote a general polynomial of degree n in the single variable ϕ with rational coefficients. Then either

$$\gamma^2 = [p_2(\phi)]^2\, p_3(\phi) \text{ when } c \neq 1/3 \text{ or}$$
$$\gamma^2 = [p_1(\phi)]^2\, p_4(\phi) \text{ when } c = 1/3.$$

Notice that the case $c = 1/3$ reduces the degree of the resulting equation in ϕ from seven to six. In fact one obtains

$$3^7\gamma^2 = 2^6\phi(1-\phi)(1+\phi)(3\phi-1)(3\phi+2)(-3\phi+4)$$

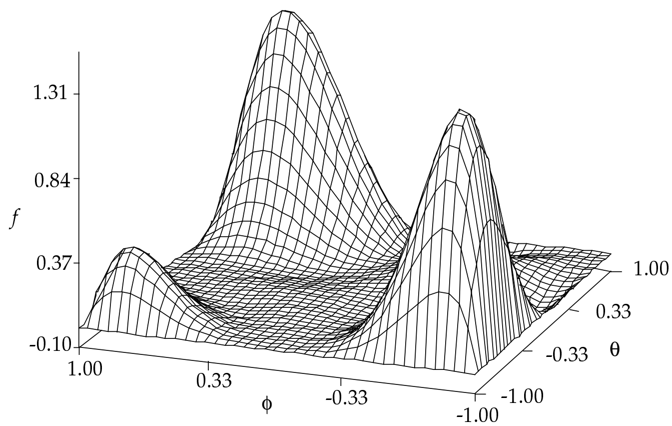

Figure 4.4: Heron-2-median Surface

which however is not an elliptic curve. When $c \neq 1/3$ there are $\binom{7}{2} = 21$ ways in which one linear factor, in ϕ, can be a multiple of some other factor and hence possibly produce an elliptic curve. However one can readily eliminate those pairings which lead to a contradiction e.g. $\phi = k(1-\phi)$ as well as those pairings which lead to $c = 0, \pm 1$ since then γ^2 will be identically zero. This leaves just three cases to consider.
Case (i) : If $\phi = k[\phi + (2c-1)]$ then equating coefficients of ϕ leads to $k = 1$ and $c = 1/2$. Substituting this into (4.8) leads to

$$2^6\gamma^2 = 3\phi^2(1-\phi)(1+\phi)(\phi+3)(4\phi+3)(-2\phi+3)$$

which reduces to a quintic equal to a rational square but is still not an elliptic curve.
Case (ii) : $(\phi - 1) = k[2\phi + (c+1)]$ leads to $k = 1/2$ and $c = -3$. Again substitution into equation (4.8) reveals that

$$\gamma^2 = -2^5 \cdot 3\phi(1-\phi)^2(1+\phi)(5\phi+1)(\phi-7)(\phi+2)$$

which is not an elliptic curve.
Case (iii) : If $(\phi + 1) = k[-\phi + (c+1)]$ then $k = -1$ and $c = -2$. Substitution leads to

$$\gamma^2 = 6\phi(1-\phi)(1+\phi)^2(7\phi+1)(\phi-5)(2\phi-1)$$

another quintic not reducible to an elliptic curve.

So it is not possible for any plane of the form $\theta = c$ to intersect the surface defined by equation (4.7) in an elliptic curve. By symmetry no plane of the form $\phi = c$ can intersect the surface in an elliptic curve either. Now of all the planes of the form $\phi = a\theta + c$, consider those which pass through the origin i.e.

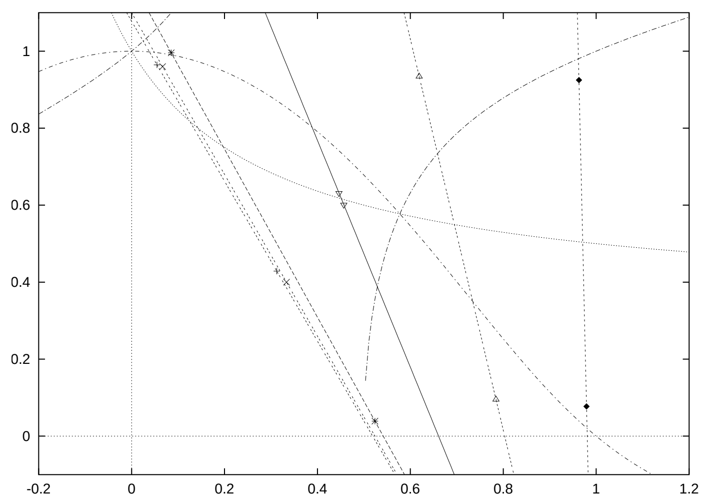

Figure 4.5: Heron-2-median triangles in the $\theta\phi$-plane

$\phi = a\theta$. Then equation (4.3) becomes

$$\begin{aligned}\gamma^2 &= a\theta^2(1-\theta^2)(1-(a\theta)^2)[3a\theta^2+(1-a)\theta+1] \\ &\quad \times [(2+a)\theta-1][(2a+1)\theta+1][(1-a)\theta+1].\end{aligned} \tag{4.9}$$

As before this can lead identically to an elliptic curve in one of two possible ways. Either

$$\gamma^2 = [p_4(\theta)]^2 p_3(\theta) \text{ when } a \neq -2, -\frac{1}{2}, 0, 1 \text{ or}$$
$$\gamma^2 = [p_3(\theta)]^2 p_4(\theta) \text{ when } a = -2, -\frac{1}{2}, 0, 1.$$

First examine the degenerate degree cases. When $a = -2$ equation (4.9) becomes

$$\gamma^2 = 2\theta^2(1-\theta^2)(1-4\theta^2)(6\theta^2-3\theta-1)(5\theta-1)(3\theta+1)$$

and is not an elliptic curve. Similarly

$$2^6\gamma^2 = \theta^2(1-\theta^2)(4-\theta^2)(3\theta^2-3\theta-1)(3\theta-2)(3\theta+2)$$

also not an elliptic curve. If $a = 0$ then γ^2 is identically zero and hence uninteresting. The case $a = 1$ leads to the first occurrence of an elliptic curve since now

$$\gamma^2 = \theta^2(1-\theta^2)^2(3\theta^2+1)(3\theta-1)(3\theta+1).$$

From now on assume that $a \neq -2, -\frac{1}{2}, 0, 1$. Now the quadratic factor $[3a\theta^2 + (1-a)\theta+1]$ from equation (4.9) cannot be expressed in the form $k(d\theta+e)^2$

for rational d and e, since equating coefficients of θ leads to $a^2 - 14a + 1 = 0$ which has no rational solutions. So, at best, the quadratic factor could reduce to a pair of distinct linear factors with rational coefficients. Suppose that the quadratic factor is irreducible over the field of rationals. Then there are $\binom{7}{2} =$ 21 ways in which the remaining linear factors can be matched up to possibly produce a $k(d\theta + e)^2$ term and hence possibly lead to an elliptic curve. Again eliminating those matchings which lead to a contradiction and those which make γ^2 identically zero leads to eleven cases of which just five are distinct.
Case (i) : If $(1-\theta) = k[1 - a\theta]$ then $k = 1$ and $a = -1$ which when substituted back into equation (4.9) gives the elliptic curve

$$\gamma^2 = \theta^2(1-\theta)4(1+\theta)^2[-3\theta^2 + 2\theta + 1][2\theta + 1].$$

Case (ii) : If $(1-\theta) = k[(2+a)\theta - 1]$ then $k = -1$ and $a = -1$ which leads to the same elliptic curve as the first case.
Case (iii) : If $(1-\theta) = k[(2a+1)\theta + 1]$ then $k = 1$ and $a = -1$ which is the same as (i).
Case (iv) : If $(1-\theta) = k[(1-a)\theta + 1]$ then $k = 1$ and $a = 2$. For this value of a, equation (4.9) becomes

$$\gamma^2 = 2\theta^2(1-\theta)^2(1+\theta)(1-2\theta)(1+2\theta)(6\theta^2 - \theta + 1)(4\theta - 1)(5\theta + 1)$$

which is not an elliptic curve.
Case (v) : If $(1+\theta) = k[1 - a\theta]$ then $k = 1$ and $a = -1$ which is the same as (i).
Case (vi) : If $(1+\theta) = k[(2+a)\theta - 1]$ then $k = -1$ and $a = -3$ for which

$$\gamma^2 = 3\theta^2(1-\theta)(1+\theta)^2(1-3\theta)(1+3\theta)(9\theta^2 - 4\theta - 1)(5\theta - 1)(4\theta + 1)$$

and does not lead to an elliptic curve.
Case (vii) : If $(1 - a\theta) = k[(2a+1)\theta + 1]$ then $k = 1$ and $a = -\frac{1}{3}$ which leads to

$$3^7\gamma^2 = -\theta^2(1-\theta)(1+\theta)(3+\theta)^2(3-\theta)(-3\theta^2 + 4\theta + 1)(5\theta - 3)(4\theta + 3)$$

again not an elliptic curve.
Case (viii) : If $(1 + a\theta) = k[(2+a)\theta - 1]$ then $k = -1$ and $a = -1$ which is the same as (i).
Case (ix) : If $(1 + a\theta) = k[(2a+1)\theta + 1]$ then $k = 1$ and $a = -1$ which is the same as (i).
Case (x) : If $(1 + a\theta) = k[(1-a)\theta + 1]$ then $k = 1$ and $a = \frac{1}{2}$ so that

$$2^6\gamma^2 = \theta^2(1-\theta)(1+\theta)(2+\theta)^2(2-\theta)(3\theta^2 + \theta + 2)(5\theta - 2)(2\theta + 1)$$

which is not an elliptic curve.
Case (xi) : If $(2+a)\theta - 1) = k[(2a+1)\theta + 1]$ then $k = -1$ and $a = -1$ which is the same as (i).
The final cases for planes through the origin occur when the quadratic factor $[3a\theta^2 + (1-a)\theta + 1]$ is expressible in the form $k(\theta + b)(\theta + c)$ for some rational

choice of k, b and c. Equating coefficients of θ leads to

$$k = 3a$$
$$b = \frac{1}{3ac}$$
$$c = \frac{(1-a) \pm \sqrt{a^2 - 14a + 1}}{6a}$$

which are rational only when $a^2 - 14a + 1 = m^2$. So without loss of generality let $[3a\theta^2 + (1-a)\theta + 1] = (3a\theta + \frac{1}{c})(\theta + c)$ where $c = \frac{1-a+m}{6a}$, a and m are related as above. Matching these two linear factors to the remaining seven in equation (4.9) leads to fourteen separate cases. Of these one results in a contradiction and the remaining thirteen lead to $a = 0$ or $a = -1$ or both. So the planes $\phi = a\theta$, $\theta = c$ and $\phi = c$ intersect the Heron-2-median surface in an elliptic curve only when $a = \pm 1$. Of course this does not rule out the existence of other elliptic curves on the surface. For example one could consider planes of the form $\phi = a\theta + c$ where both a and c are non-zero. Even if these lead to no new elliptic curves then it is still possible that higher degree relationships between ϕ and θ will.

4.8 Rational Points on the Elliptic Curves

Setting $\phi = \theta$ in equation (4.7) leads to

$$f(\theta, \phi) = \theta^2(1-\theta^2)^2(3\theta^2+1)(9\theta^2-1).$$

The Heron-2-median triangles correspond to points (θ, ϕ) such that $f(\theta, \phi)$ is the square of some rational number. So there exists some $y \in \mathbb{Q}$ such that

$$y^2 = (3\theta^2+1)(9\theta^2-1). \tag{4.10}$$

This quartic can be transformed into a cubic via Mordell's transformation and since $(\theta, y) = (1/3, 0)$ satisfies equation (4.10) the quartic equation represents an elliptic curve. However it turns out that there are no points of infinite order on (4.10) (first pointed out by Andrew Bremner). Let $\theta = \frac{r}{s}$ where r and s are strictly positive integers such that $\gcd(r, s) = 1$. Then $y = \frac{t}{s^2}$ for some integer t. Hence equation (4.10) becomes

$$t^2 = (9r^2 - s^2)(3r^2 + s^2). \tag{4.11}$$

If the $\gcd(9r^2 - s^2, 3r^2 + s^2) = k$ then $\gcd(12r^2, -4s^2) = k$. However the fact that $\gcd(r, s) = 1$ implies that $k \mid 12$. Since the two factors on the right hand side of equation (4.11) are clearly greater than zero then one need only consider k to be greater than zero. This leaves just six cases, namely $k = 1, 2, 3, 4, 6$ or 12.
Case (i) : If $k = 1$ then there exist integers, p and q say, such that $p \neq 0$ and

$$9r^2 - s^2 = p^2$$
$$3r^2 + s^2 = q^2.$$

The first of these equations is of Pythagorean form and so when p is even has solutions $p = 2xy$, $s = x^2 - y^2$, $3r = x^2 + y^2$ where $\gcd(x, y) = 1$ and $2 \mid xy$. Substituting these into the second equation of the pair above gives

$$4x^4 + 4x^2y^2 + 4y^4 = 3q^2.$$

Since $4 \mid q^2$ then q must be even so letting $q = 2Q$ leads to

$$(x^2 - y^2)^2 + x^2y^2 = 3Q^2.$$

This last equation has no solutions in integers since reducing it modulo 3 one obtains $x^2 - y^2 \equiv 0 \pmod 3$ and $xy \equiv 0 \pmod 3$ which in turn implies that $\gcd(x, y) = 3 \ldots$ a contradiction.

Similarly when p is odd the solutions of $9r^2 - s^2 = p^2$ are $p = x^2 - y^2$, $s = 2xy$, $3r = x^2 + y^2$ where $\gcd(x, y) = 1$ and $2 \mid xy$. Substitution into $3r^2 + s^2 = q^2$ leads to

$$(x^2 - y^2)^2 + 16x^2y^2 = 3Q^2$$

which also has no integer solutions by a modulo 3 argument.
Case (ii) : If $k = 2$ then there exist integers, p and q say, such that $p \neq 0$ and

$$\begin{aligned} 9r^2 - s^2 &= 2p^2 \\ 3r^2 + s^2 &= 2q^2. \end{aligned}$$

These equations imply that r and s have the same parity but then $\gcd(r, s) = 1$ means that r and s are both odd. So let $r = m + n$ and $s = m - n$ where $\gcd(m, n) = 1$ and $2 \mid mn$. So the second equation leads to

$$4m^2 + 4mn + 4n^2 = 2q^2.$$

Since 2 divides q^2 means that 2 divides q let $q = 2Q$ to give

$$m^2 + mn + n^2 = 2Q^2.$$

This leads to a contradiction since the left hand side is odd (by the constraints on m and n) while the right hand side is clearly even.
Case (iii) : If $k = 3$ then the the equations become

$$\begin{aligned} 9r^2 - s^2 &= 3p^2 \\ 3r^2 + s^2 &= 3q^2. \end{aligned}$$

First notice that s is divisible by three. Hence setting $s = 3\sigma$ implies that p is divisible by three. So $p = 3\rho$ leads to

$$\begin{aligned} r^2 - \sigma^2 &= 3\rho^2 \\ r^2 + 3\sigma^2 &= q^2. \end{aligned}$$

Solutions to the first equation are either

$$\sigma = u^2 - 3v^2,\ \rho = 2uv,\ r = u^2 + 3v^2 \text{ where } \gcd(u,v) = 1 \text{ and } 2 \mid uv \text{ or}$$

$$\sigma = 3u^2 - v^2,\ \rho = 2uv,\ r = 3u^2 + v^2 \text{ where } \gcd(u,v) = 1 \text{ and } 2 \mid uv.$$

While solutions to the second equation are either

$$q = x^2 + 3y^2,\ \sigma = 2xy,\ r = x^2 - 3y^2 \text{ where } \gcd(x,y) = 1 \text{ and } 2 \mid xy \text{ or}$$

$$q = 3x^2 + y^2,\ \sigma = 2xy,\ r = 3x^2 - y^2 \text{ where } \gcd(x,y) = 1 \text{ and } 2 \mid xy.$$

But in the first pair of solutions σ is always odd while in the second pair of solutions σ is always even. Hence no simultaneous solution is possible.
Case (iv) : If $k = 4$ then the two equations become

$$\begin{aligned} 9r^2 - s^2 &= 4p^2 = (2p)^2 \\ 3r^2 + s^2 &= 4q^2 = (2q)^2. \end{aligned}$$

These are equivalent to the equations of case (i) and so have no solution.
Case (v) : If $k = 6$ then the equations are

$$\begin{aligned} 9r^2 - s^2 &= 6p^2 \\ 3r^2 + s^2 &= 6q^2. \end{aligned}$$

This time the argument used in the second case will also work here and so there are no solutions again.
Case (vi) : If $k = 12$ then the equations are

$$\begin{aligned} 9r^2 - s^2 &= 12p^2 = 3(2p)^2 \\ 3r^2 + s^2 &= 12q^2 = 3(2q)^2. \end{aligned}$$

These are the same form as the equations of Case (iii) and so there are no solutions here either. So finally the elliptic curve on the Heron-2-median surface defined by $\phi = \theta$ has only one rational point. To consider the other elliptic curve Set $\phi = -\theta$ in equation (4.7) to obtain

$$f(\theta, \phi) = \theta^2(1-\theta)^4(1+\theta)^2(-3\theta^2 + 2\theta + 1)(2\theta + 1).$$

As before Heron triangles with two integer medians occur when $f(\theta, \phi) = \gamma^2$ for some rational γ. So letting $y = \frac{\gamma}{\theta(1-\theta)^2(1+\theta)}$ leads to the elliptic curve

$$y^2 = (1-\theta)(3\theta + 1)(2\theta + 1).$$

Replacing y by $\frac{y}{6}$ and letting $\theta = \frac{6-x}{6}$ one obtains the curve

$$\begin{aligned} & y^2 = x(x-8)(x-9) \text{ or} \\ E \quad : \quad & y^2 = x^3 - 17x^2 + 72x. \end{aligned}$$

The rational points of order two are all given by $y = 0$ and so $P_1 = (0, 0)$, $P_2 = (8, 0)$, $P_3 = (9, 0)$ are the only 2-torsion points on $E(\mathbb{Q})$. All other finite

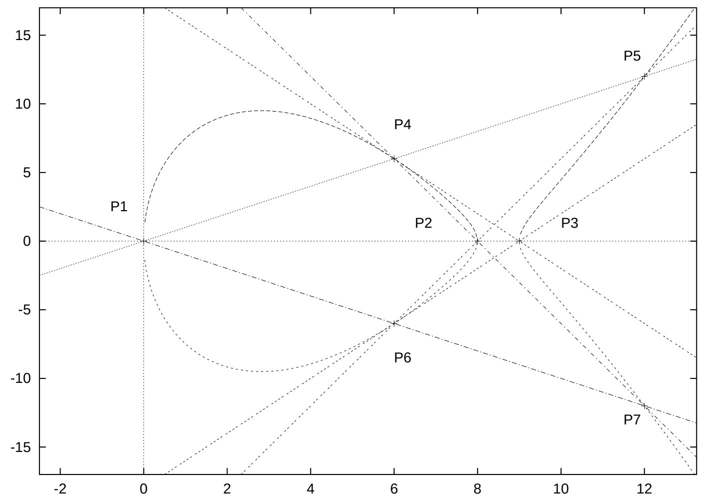

Figure 4.6: Torsion subgroup of $y^2 = x(x-8)(x-9)$

order rational points satisfy $y^2 \mid 2^6 \cdot 3^4$ and $x, y \in \mathbb{Z}$ (see [16, p. 221]). So $y \mid 2^3 \cdot 3^2$ and a finite search of the twelve cubics reveals the only four rational points satisfying the criteria, namely, $P_4 = (6,6)$, $P_5 = (12,12)$, $P_6 = (6,-6)$ and $P_7 = (12,-12)$. Denote the point at infinity by $\mathcal{O}$. Then applying the group laws, which describe the tangent-chord process [16, pp. 58-59], to the rational points of $E(\mathbb{Q})$ leads to $2P_1 = \mathcal{O}$, $2P_2 = \mathcal{O}$ and $2P_3 = \mathcal{O}$ as expected, see Figure 4.6. The remaining points all have order 4 since $2P_4 = P_3$ which, in turn, implies that $4P_4 = \mathcal{O}$ and $3P_4 = -P_4 = P_6$. Similarly $2P_5 = P_3$ which implies that $4P_5 = \mathcal{O}$ and $3P_5 = -P_5 = P_7$. Furthermore $P_1 + P_2 = P_3$, $P_1 + P_4 = P_7$, $P_2 + P_4 = P_5$ and $P_3 + P_4 = P_6$ so that P_2, P_3, P_5, P_6, and P_7 can all be expressed in terms of P_1 and P_4. So the torsion subgroup of rational points on $E(\mathbb{Q})$, denoted by $E_{tors}(\mathbb{Q})$, is given by

$$E_{tors}(\mathbb{Q}) = \{P_1, P_4 : 2P_1 = \mathcal{O}, 4P_4 = \mathcal{O}\} \cong C_2 \times C_4.$$

The aim now is to show that $E(\mathbb{Q})$ has no rational points of infinite order, i.e. the rank, g, of the group of all rational points on E is zero. The following argument was suggested by both A. Bremner and R. Guy.

Consider the isogeny, ϕ, and its dual, $\hat{\phi}$, (homomorphisms between elliptic curves which map $\mathcal{O}$ to $\mathcal{O}$) given by

$$\phi(x,y) = \left(\frac{y^2}{x^2}, \frac{y(72-x^2)}{x^2}\right) \quad \text{and} \quad \hat{\phi}(X,Y) = \left(\frac{Y^2}{4X^2}, \frac{Y(1-X^2)}{8X^2}\right).$$

Then ϕ maps from the elliptic curve E to E' defined by

$$E' \quad : \quad Y^2 = X^3 + 34X^2 + X$$

while $\hat{\phi}$ maps from E' to E. Furthermore $\phi \circ \hat{\phi}$ is duplication on E' and $\hat{\phi} \circ \phi$ is duplication on E. For the elliptic curve E let $x = \frac{dr^2}{es^2}$ and $y = \frac{u}{v}$ where $\gcd(d, e) = 1$, $\gcd(r, s) = 1$, $\gcd(u, v) = 1$ and d and e are squarefree. Then E becomes

$$e^3 s^6 u^2 = dr^2 v^2 (dr^2 - 8es^2)(dr^2 - 9es^2). \tag{4.12}$$

So e^3 can only divide v^2 and since e is squarefree $e^2 \mid v$. Let $v = we^2$ to give

$$s^6 u^2 = dr^2 w^2 e(dr^2 - 8es^2)(dr^2 - 9es^2).$$

Hence either $e \mid u$ or $e \mid s$. Now if $e \mid u$ then $e = 1$ since $\gcd(u, v) = 1$. If $e \mid s$ then suppose $e^p \parallel s$ so that $s = Se^p$ where $\gcd(e, S) = 1$. Substitution yields

$$eS^6 e^{6p-2} u^2 = dr^2 w^2 (dr^2 - 8es^2)(dr^2 - 9es^2).$$

Thus e^{3p-1} divides w so let $w = We^{3p-1}$ to give

$$eS^6 u^2 = dr^2 W^2 (dr^2 - 8es^2)(dr^2 - 9es^2).$$

Now one can see that e must divide W so letting $W = Ve$ gives

$$S^6 u^2 = dr^2 V^2 e(dr^2 - 8es^2)(dr^2 - 9es^2)$$

which also leads to the conclusion that $e = 1$. So (4.12) becomes

$$s^6 u^2 = dr^2 v^2 (dr^2 - 8s^2)(dr^2 - 9s^2).$$

Since $\gcd(r, s) = 1$, $\gcd(u, v) = 1$ and d is squarefree one must have v^2 divides s^6 while dr^2 divides u^2. Hence $v \mid s^3$ while $dr \mid u$ so let $s^3 = kv$ and $u = ldr$ to give

$$dt^2 = (dr^2 - 8s^2)(dr^2 - 9s^2) \tag{4.13}$$

where $t = kl$. Now $d \mid 72s^4$ implies that $d \mid 72 = 2^3 \cdot 3^2$. But d is squarefree which implies that $d \mid 6$. If d is negative then the above equation has no real solutions let alone rational solutions. The remaining values that d can take each admit a solution since when

$d = 1$ then $t^2 = (r^2 - 8s^2)(r^2 - 9s^2)$ has $(r, s, t) = (3, 1, 0)$ as a solution,

$d = 2$ then $t^2 = (r^2 - 4s^2)(2r^2 - 9s^2)$ has $(r, s, t) = (2, 1, 0)$ as a solution,

$d = 3$ then $t^2 = (3r^2 - 8s^2)(r^2 - 3s^2)$ has $(r, s, t) = (2, 1, 2)$ as a solution,

$d = 6$ then $t^2 = (3r^2 - 4s^2)(2r^2 - 3s^2)$ has $(r, s, t) = (1, 1, 1)$ as a solution.

Following a similar argument for the isogenous curve E', without loss of generality, let $X = \frac{DR^2}{S^2}$ and $Y = \frac{U}{V}$ where $\gcd(R, S) = 1$, $\gcd(U, V) = 1$ and D is squarefree to give

$$S^6 U^2 = DR^2 V^2 (D^2 R^4 + 34DR^2 S^2 + S^4).$$

As before $V \mid S^3$ while $DR \mid U$ so let $S^3 = KV$ and $U = LDR$ to give

$$DT^2 = (D^2R^4 + 34DR^2S^2 + S^4). \tag{4.14}$$

Since $\gcd(D, S) = 1$ then $D \mid 1$. When $D = 1$ the equation (4.14) becomes $T^2 = R^4 + 34R^2S^2 + S^4$ which has a solution $(R, S, T) = (1, 1, 6)$. When $D = -1$ there is no solution since now (4.14) taken modulo 3 is

$$-T^2 \equiv R^4 + 2R^2S^2 + S^4 \pmod 3.$$

But $\gcd(R, S) = 1$ implies that $R, S \equiv \pm 1 \pmod 3$ so $-T^2 \equiv 1 \pmod 3$ which is impossible. It turns out [2, p. 95] that g is obtained from the inequality $g \le \lambda + \lambda_1 - 2$ where 2^λ is the number of distinct d for which (4.13) has at least one solution and 2^{λ_1} is the number of distinct D for which (4.14) has at least one solution. Thus $2^\lambda = 4$ and so $\lambda = 2$ while $2^{\lambda_1} = 1$ which means that $\lambda_1 = 0$ hence $g \le 0$. Since g is a non-negative integer g must be zero.

In conclusion, the only two elliptic curves found so far on the Heron-2-median surface turn out to have only a finite number of rational points and so the question of the existence of an infinite number of Heron triangles with two integer medians remains open.

Chapter 5

General Rational Concurrent Cevians

Recall that a line from the vertex of a triangle to any point on the opposite side is called a cevian after Giovanni Ceva who proved a theorem concerning these lines in 1678. This chapter begins with two results on the cevians of triangles - one by Ceva himself and one by M. Stewart. These theorems turn out to be useful when restricting the lengths of the sides and cevians to rational numbers.

5.1 Renaissance Results

The first result, namely Theorem 15, is actually stronger than Ceva's original theorem since he only proved the sufficiency condition. However the converse was readily shown to be true. In Figure 5.1 let $\overline{AD} = p$, $\overline{BE} = q$ and $\overline{CF} = r$.

Theorem 15 *The three cevians p, q, r say, of any triangle are concurrent if and only if*

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1.$$

Proof: $\Longrightarrow$ If the three cevians are concurrent then consider the areas in Figure 5.1 which lead to

$$\frac{a_1}{a_2} = \frac{\triangle(ADB)}{\triangle(ADC)} = \frac{\triangle(PDB)}{\triangle(PDC)}.$$

However, $x = \frac{a}{b} = \frac{c}{d}$ implies that $x = \frac{a \pm c}{b \pm d}$ so that

$$\frac{a_1}{a_2} = \frac{\triangle(ADB) - \triangle(PDB)}{\triangle(ADC) - \triangle(PDC)} = \frac{\triangle(APB)}{\triangle(APC)}.$$

Similarly

$$\frac{b_1}{b_2} = \frac{\triangle(BPC)}{\triangle(BPA)} \quad \text{and} \quad \frac{c_1}{c_2} = \frac{\triangle(CPA)}{\triangle(CPB)}.$$

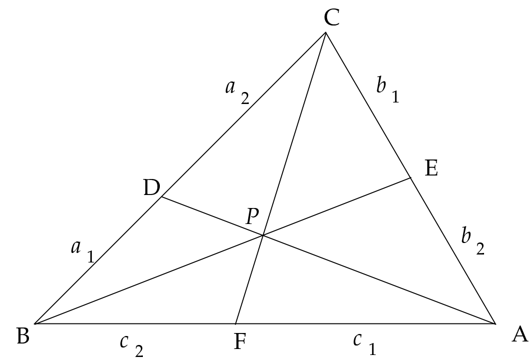

Figure 5.1: Cevians of a triangle

Combining all this produces

$$\frac{a_1}{a_2}\frac{b_1}{b_2}\frac{c_1}{c_2} = \frac{\triangle(APB)}{\triangle(APC)}\frac{\triangle(BPC)}{\triangle(BPA)}\frac{\triangle(CPA)}{\triangle(CPB)} = 1$$

which proves Ceva's theorem.
$\Longleftarrow$ Now to prove the converse assume that the feet of the cevians p, q and r satisfy $\frac{a_1}{a_2}\frac{b_1}{b_2}\frac{c_1}{c_2} = 1$ and that p and q intersect at the point P. Let r' be the cevian from vertex C which passes through the point P and subdivides $\overline{AB}$ into base segments c_1' and c_2'. Since p, q and r' are concurrent then by Ceva's theorem $\frac{a_1}{a_2}\frac{b_1}{b_2}\frac{c_1'}{c_2'} = 1$ so that $\frac{c_1}{c_2} = \frac{c_1'}{c_2'}$. But using the fact that $c = c_1 + c_2 = c_1' + c_2'$ leads to $c_1 = c_1'$ and hence $r = r'$. ■

In 1746 M. Stewart first stated the following theorem (since known as Stewart's Theorem) which was subsequently proved by R. Simpson in 1751.

Theorem 16 *If p is any cevian of a triangle (a, b, c) which divides side a into the base segments a_1 and a_2 then p satisfies*

$$a(p^2 + a_1 a_2) = b^2 a_1 + c^2 a_2.$$

Proof: Considering the cosine of angle B in Figure 5.1 in two different ways leads to

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{a_1^2 + c^2 - p^2}{2a_1 c}$$

which gives upon rearrangement

$$\begin{aligned} a^2 a_1 + c^2 a_1 - b^2 a_1 &= a a_1^2 + ac^2 - ap^2 \\ a(p^2 + a a_1 - a_1^2) &= b^2 a_1 + c^2 a - c^2 a_1 \\ a(p^2 + a_1(a - a_1)) &= b^2 a_1 + c^2(a - a_1). \end{aligned}$$

But since $a_1 + a_2 = a$ the desired result is obvious. ■

5.2 Rational Cevians

Restricting attention to firstly one rational cevian via Stewart's theorem and then to three concurrent rational cevians by Ceva's theorem leads to the following interesting results.

Theorem 17 *If the sides a, b, c and one cevian p of a triangle are rational then both the base segments corresponding to the cevian p are elements of the same extension field $\mathbb{Q}(\sqrt{s})$ for some nonnegative rational s.*

Proof: Using Stewart's theorem for the cevian p of Figure 5.1 leads to

$$a(p^2 + a_1a_2) = b^2a_1 + c^2a_2 \quad \text{and}$$
$$a_1 + a_2 = a.$$

Solving these two equations for a_1 and a_2 results in

$$a_1 = \frac{(c^2 + a^2 - b^2) + \sqrt{4a^2p^2 - 16\triangle^2}}{2a}$$
$$a_2 = \frac{(a^2 + b^2 - c^2) - \sqrt{4a^2p^2 - 16\triangle^2}}{2a}.$$

In other words a_1 and a_2 can be written as

$$a_1 = \alpha_1 + \sqrt{A} \quad \text{and} \quad a_2 = \alpha_2 - \sqrt{A}$$

where a_1, a_2 and A are all rationals. Note that $a_1 + a_2 = a$ and that

$$A = \frac{4a^2p^2 - 16\triangle^2}{4a^2} > 0 \quad \text{since} \quad \frac{ap}{2} > \triangle.$$

∎

For example if the triangle $(a, b, c) = (13, 14, 15)$ has a cevian of length 14 to the side $b = 14$ then the base segments are $b_1 = 5 + \sqrt{52}$ and $b_2 = 9 - \sqrt{52}$. In fact it is possible to find all rational solutions to the equation in Stewart's Theorem.

Theorem 18 *For any integer-sided triangle (a, b, c) any rational cevian (and its base segments) to one side is given by*

$$p = \frac{\alpha u^2 + \beta v^2}{4auv}$$
$$a_1 = \frac{\alpha u^2 - \beta v^2 + 2uv(a^2 - b^2 + c^2)}{4auv}$$
$$a_2 = a - a_1$$

where α, β, u, v are all integers and $\alpha\beta = 16\triangle^2$.

Proof: Replacing a_2 by $a - a_1$ in Stewart's Theorem leads to

$$a(p^2 + a_1(a - a_1)) = b^2 a_1 + c^2(a - a_1).$$

Now multiply by a and then complete the square to obtain

$$(2ap)^2 - (2aa_1 - (a^2 - b^2 + c^2))^2 = (2ac)^2 - (a^2 - b^2 + c^2)^2.$$

Expanding the right-hand-side and comparing with Heron's formula for the area of a triangle results in $16\triangle^2$. Since p and a_1 can be rational let $p = \frac{P}{Y}$ and $a_1 = \frac{A_1}{Y}$ where $P, A_1, Y \in \mathbb{N}$, to give

$$(2aP)^2 - (2aA_1 - Y(a^2 - b^2 + c^2))^2 = 16\triangle^2 Y^2.$$

This can be reduced to

$$Z^2 - X^2 = nY^2$$

where $X = 2aA_1 - Y(a^2 - b^2 + c^2)$, $Z = 2aP$ and $n = 16\triangle^2 \in \mathbb{N}$. If $n = \alpha\beta$ then the parametric solution to this last equation is

$$\begin{aligned} X &= \alpha u^2 - \beta v^2 \\ Y &= 2uv \\ Z &= \alpha u^2 + \beta v^2. \end{aligned}$$

So it is possible to express P and A_1 in terms of the sides and arbitrary parameters as follows.

$$\begin{aligned} P &= \frac{\alpha u^2 + \beta v^2}{2a} \\ A_1 &= \frac{\alpha u^2 - \beta v^2 + 2uv(a^2 - b^2 + c^2)}{2a}. \end{aligned}$$

The desired result follows upon division by Y. ■

5.3 Generating New Ceva points from Old

Definition : A ceva point of a rational-sided triangle is any internal or external point such that the lengths of the three cevians through that point are rational.

With the aid of Theorem 18 it is possible to systematically construct rational-sided triangles with two rational cevians p and q say. This is achieved simply by using two sets of parameters such as

$$\begin{aligned} p &= \frac{\alpha u_1^2 + \beta v_1^2}{4au_1v_1} \\ a_1 &= \frac{\alpha u_1^2 - \beta v_1^2 + 2u_1v_1(a^2 - b^2 + c^2)}{4au_1v_1} \\ a_2 &= a - a_1. \end{aligned}$$

$$\begin{aligned}
q &= \frac{\gamma u_2^2 + \delta v_2^2}{4bu_2v_2} \\
b_1 &= \frac{\gamma u_2^2 - \delta v_2^2 + 2u_2v_2(a^2+b^2-c^2)}{4bu_2v_2} \\
b_2 &= b - b_1.
\end{aligned}$$

To ensure that the third cevian, r say, through the intersection of p and q is also rational note that Ceva's Theorem together with the identity $c_1 + c_2 = c$ leads to

$$c_1 = \frac{ca_2b_2}{a_1b_1 + a_2b_2} \quad \text{and} \quad c_2 = \frac{ca_1b_1}{a_1b_1 + a_2b_2}. \tag{5.1}$$

Substituting this into Stewart's Theorem applied to the cevian r, namely

$$c(r^2 + c_1c_2) = a^2c_1 + b^2c_2$$

gives

$$r^2 = \frac{a_1b_1a_2b_2(a^2+b^2-c^2) + a^2a_2^2b_2^2 + b^2a_1^2b_1^2}{(a_1b_1 + a_2b_2)^2}. \tag{5.2}$$

Finally r is rational if and only if the numerator of equation (5.2) is the square of some rational number. For any specific triangle (a, b, c) fixing one cevian, q say, fixes the two corresponding base segments and so equation (5.2) becomes an elliptic curve in the parameters u_1 and v_1. Hence, beginning with one ceva point of a triangle it is possible to construct an infinite number of ceva points. For example, applying this process to the Heron triangle $(a, b, c) = (13, 14, 15)$ with an area given by $\triangle = 84$, equation (5.2) becomes

$$y^2 = 140a_1b_1a_2b_2 + 169a_2^2b_2^2 + 196a_1^2b_1^2.$$

One ceva point is given by the orthocentre at which $p = \frac{168}{13}$, $q = 12$ and $r = \frac{56}{5}$ are the three rational cevians. Fixing q and hence the base segments to $b_1 = 5$ and $b_2 = 9$ leads to

$$y^2 = 6300a_1a_2 + 13689a_2^2 + 4900a_1^2.$$

Without loss of generality let $\alpha = \beta = 4\triangle = 336$. Then in terms of the parameters of Theorem 18 one obtains

$$\begin{aligned}
a_1 &= \frac{84u^2 - 84v^2 + 99uv}{13uv} \\
a_2 &= \frac{-84u^2 + 84v^2 + 70uv}{13uv}.
\end{aligned}$$

Substituting these into the equation above leads to

$$C \quad : \quad Y^2 = 86711184X^4 - 94832640X^3 - 14662368X^2 + 94832640X + 86711184$$

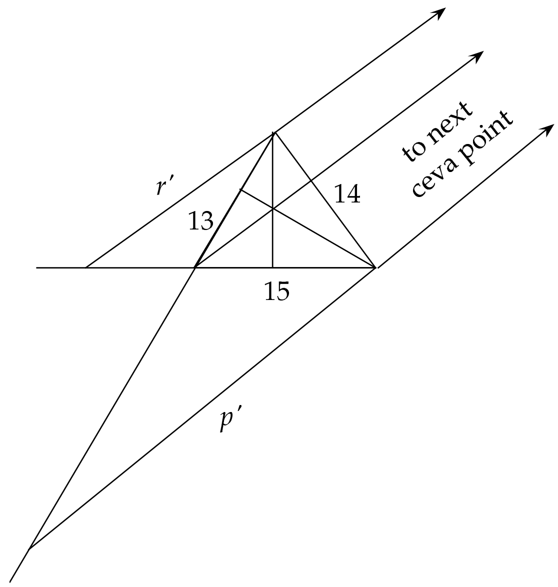

Figure 5.2: New Ceva Point

where $X = \frac{u}{v}$ and $Y = \frac{13uvy}{v^2}$. Now $p = \frac{168}{13}$ corresponds to the parameters $(u, v) = (1, 1)$. So a rational point on the quartic curve $C(\mathbb{Q})$ is given by $(X, Y) = (1, 150)$. As in Chapter 3 one finds that the quadratic equation

$$625Y = 57683X^2 - 77616X + 113683$$

touches the quartic curve $C(\mathbb{Q})$ at $(X, Y) = (1, 150)$ with multiplicity three. Intersecting the two curves results in a new rational point on $C(\mathbb{Q})$, namely $X = -\frac{1693}{307}$. Hence $(u, v) = (-1693, 307)$ leads to the corresponding new ceva point where $p' = -\frac{248681832}{6756763}$, $q' = 12$ and $r' = -\frac{129256568}{6738275}$ (see Figure 5.2). Notice that this new ceva point is outside the triangle which is why the p' and r' are negative. The previous argument serves to prove the following theorem.

Theorem 19 *Any rational sided triangle with at least one ceva point has an infinite number of ceva points.*

This means that all Heron triangles have an infinite number of ceva points since the orthocentre of any Heron triangle is a ceva point. Using the search procedure outlined at the beginning of this section it is possible to find many ceva points for a specific triangle. For an equilateral triangle it is sufficient to consider just the $(1, 1, 1)$ triangle. Concentrating on the interior of the triangle, one finds six ceva points corresponding to the reflections and rotations of the set of three rational concurrent cevians $(p, q, r) = (\frac{511}{589}, \frac{7}{8}, \frac{133}{153})$, as well as many others (see Figure 5.3).

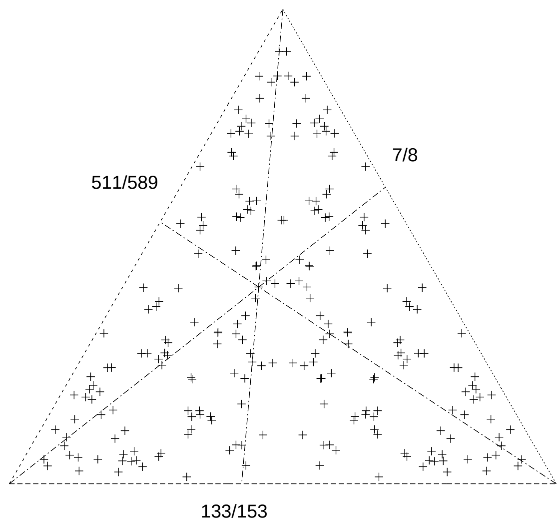

Figure 5.3: Ceva Points of an equilateral triangle

5.4 Allowable Base Segments

At this stage (or perhaps even earlier) several questions arise. Given a triangle with six rational base segments which correspond to three concurrent cevians - are those cevians also rational? No! A counterexample is clearly provided by the altitudes of an equilateral triangle of edge length one. The base segments are all $1/2$ while the altitudes are all $\sqrt{3}/2$.

What about the converse? If the sides and the three concurrent cevians are rational are all the base segments rational? This led to several new theorems. To determine the rationality or otherwise of the base segments of a triangle with three rational cevians requires the following preparatory result.

Theorem 20 *If the three sides and at least two base segments (on different sides) of a triangle are rational and the three cevians are concurrent then all of the base segments are rational.*

Proof: Without loss of generality let a_1 and b_1 be the rational base segments. Then a_2 and b_2 are rational since $a_2 = a - a_1$ and $b_2 = b - b_1$. Recall from equations (5.1) that the remaining two base segments can be expressed as

$$c_1 = \frac{ca_2b_2}{a_1b_1 + a_2b_2} \quad \text{and} \quad c_2 = \frac{ca_1b_1}{a_1b_1 + a_2b_2}$$

which are clearly also rational. ■

Theorem 21 *Assume that the sides and three concurrent cevians of a triangle are rational. If one base segment is irrational then all base segments are irrational.*

Proof: Suppose that a_1 is irrational. Then a_2 must also be irrational since $a_2 = a - a_1$ and a is rational. Now if b_1 and c_1 are both rational then a_1 would have to be rational by Theorem 20. But this contradicts our initial assumption so let b_1 be irrational. Then as before b_2 is irrational since $b_2 = b - b_1$ and b is rational. Since $c_1 + c_2 = c \in \mathbb{Q}$ then c_1 and c_2 are either both rational or both irrational. The aim now is to show that assuming $c_1, c_2 \in \mathbb{Q}$ leads to a contradiction. Ceva's theorem leads to

$$\frac{a_2}{a_1}\frac{b_2}{b_1} = \frac{c_1}{c_2} \in \mathbb{Q}$$

while Theorem 17 led to

$$a_1 = \alpha_1 + \sqrt{A}, a_2 = \alpha_2 - \sqrt{A}, b_1 = \beta_1 + \sqrt{B} \text{ and } b_2 = \beta_2 - \sqrt{B}.$$

Substituting these into the above requires that

$$\left(\frac{\alpha_2 - \sqrt{A}}{\alpha_1 + \sqrt{A}}\right)\left(\frac{\beta_2 - \sqrt{B}}{\beta_1 + \sqrt{B}}\right) \in \mathbb{Q}.$$

Rationalising the denominator and multiplying out the resulting numerator leads to the equation

$$a(\beta_1\beta_2 + B)\sqrt{A} + b(\alpha_1\alpha_2 + A)\sqrt{B} - ab\sqrt{AB} = 0. \tag{5.3}$$

Let $u = \sqrt{A} + \sqrt{B}$ which must be irrational since if A and B were squares of rational numbers then a_1 and b_1 would be rational contradicting the initial assumption again. Since $\mathbb{Q}(\sqrt{A}, \sqrt{B}) \cong \mathbb{Q}(\sqrt{A} + \sqrt{B})$ it is possible to rewrite the above equation in terms of powers of u. In fact, squaring and cubing u provides the desired relationships, namely

$$\begin{aligned}
\sqrt{A} &= \left(\frac{3A + B}{2A - 2B}\right)u - \left(\frac{1}{2A - 2B}\right)u^3 \\
\sqrt{B} &= \left(\frac{3B + A}{2B - 2A}\right)u - \left(\frac{1}{2B - 2A}\right)u^3 \\
\sqrt{AB} &= -\left(\frac{A + B}{2}\right) - \frac{u^2}{2}.
\end{aligned}$$

Substituting these into equation (5.3) and equating the coefficients of powers of

u on both sides leads to the four equations

$$ab\left(\frac{A+B}{2}\right)=0$$
$$a(\beta_1\beta_2+B)\left(\frac{3A+B}{2A-2B}\right)+b(\alpha_1\alpha_2+A)\left(\frac{3B+A}{2B-2A}\right)=0$$
$$-\frac{ab}{2}=0$$
$$a(\beta_1\beta_2+B)\left(\frac{-1}{2A-2B}\right)+b(\alpha_1\alpha_2+A)\left(\frac{-1}{2B-2A}\right)=0.$$

Since A and B are both positive by Theorem 17 the first equation implies that the product of the sides must be zero which leads to a degenerate triangle. Hence c_1 and c_2 are not both rational and so all the base segments must be irrational. ∎

Theorem 22 *If the sides and three concurrent cevians of a triangle are rational then all of the base segments are rational.*

Proof: By Theorem 21 it is enough to show that it is impossible for all the base segments to be simultaneously irrational. From Theorem 17 the base segments take the form

$$a_1=\alpha_1+\sqrt{A},\quad b_1=\beta_1+\sqrt{B},\quad c_1=\gamma_1+\sqrt{C},$$
$$a_2=\alpha_2-\sqrt{A},\quad b_2=\beta_2-\sqrt{B},\quad c_2=\gamma_2-\sqrt{C}.$$

Substituting these into Ceva's Theorem requires that

$$\left(\frac{\alpha_2-\sqrt{A}}{\alpha_1+\sqrt{A}}\right)\left(\frac{\beta_2-\sqrt{B}}{\beta_1+\sqrt{B}}\right)\left(\frac{\gamma_2-\sqrt{C}}{\gamma_1+\sqrt{C}}\right)\in\mathbb{Q}.$$

Rationalising the denominator leads to

$$(\alpha_1\alpha_2+A+a\sqrt{A})(\beta_1\beta_2+B+b\sqrt{B})(\gamma_1\gamma_2+C+c\sqrt{C})\in\mathbb{Q}.$$

For brevity let $A'=(\alpha_1\alpha_2+A)$, $B'=(\beta_1\beta_2+B)$, $C'=(\gamma_1\gamma_2+C)$. Then the rationality constraint becomes

$$0=B'C'a\sqrt{A}+A'C'b\sqrt{B}+A'B'c\sqrt{C}+C'ab\sqrt{AB}$$
$$+A'bc\sqrt{BC}+B'ac\sqrt{AC}+abc\sqrt{ABC}.$$

Since $\mathbb{Q}(\sqrt{A},\sqrt{B},\sqrt{C})\cong\mathbb{Q}(\sqrt{A}+\sqrt{B}+\sqrt{C})$ it is possible to rewrite the above equation in terms of powers of $u=\sqrt{A}+\sqrt{B}+\sqrt{C}$. It turns out that the odd and even powers of u split the above rationality constraint into two independent equations. All the odd powers of u occur in

$$B'C'a\sqrt{A}+A'C'b\sqrt{B}+A'B'c\sqrt{C}+abc\sqrt{ABC}=0 \tag{5.4}$$

while all the even powers of u occur in

$$C'ab\sqrt{AB} + A'bc\sqrt{BC} + B'ac\sqrt{AC} = 0.$$

Concentrating on the odd powers of u one obtains

$$\begin{aligned}
\sqrt{A} &= \frac{a_1u + a_3u^3 + a_5u^5 + a_7u^7}{16(A-B)(A-C)(A^2+B^2+C^2-2AB-2BC-2AC)} \\
\sqrt{B} &= \frac{b_1u + b_3u^3 + b_5u^5 + b_7u^7}{16(A-B)(B-C)(A^2+B^2+C^2-2AB-2BC-2AC)} \\
\sqrt{C} &= \frac{c_1u + c_3u^3 + c_5u^5 + c_7u^7}{16(B-C)(A-C)(A^2+B^2+C^2-2AB-2BC-2AC)} \\
\sqrt{ABC} &= \frac{d_1u + d_3u^3 + d_5u^5 + d_7u^7}{8(A^2+B^2+C^2-2AB-2BC-2AC)}
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= 35A^4 + (B+C)(-56A^3 - 80ABC) + (B^2+BC+C^2)(6A^2 - 3BC) \\
&\quad + 182A^2BC + 16A(B+C)(B^2+C^2) \\
&\quad - (B^4 + B^3C + 13B^2C^2 + AB^3 + B^4) \\
a_3 &= -35A^3 + (B+C)(-7A^2 + 4BC) + (B+C)^2(-25A + 3B + 3C) \\
&\quad + 60ABC \\
a_5 &= 21A^2 + 14A(B+C) - 7BC - 3(B^2+BC+C^2) \\
a_7 &= B + C - 5A
\end{aligned}$$

and similarly for the b_i and c_i, while

$$\begin{aligned}
d_1 &= 2(A^3 - A^2B - A^2C - AB^2 + 26ABC - AC^2 - B^3 - B^2C - BC^2 + C^3) \\
d_3 &= -5(A^2+B^2+C^2) - 6(BC + AC + AB) \\
d_5 &= 4(A+B+C) \\
d_7 &= -1.
\end{aligned}$$

Equating coefficients of u in equation (5.4) leads to four equations whose only solution is

$$B'C'a = 0, A'C'b = 0, A'B'c = 0, abc = 0.$$

This again leads to a degenerate triangle and so the result follows. ■

5.5 Rational Cevians through the Circumcentre

From the results of Chapters 1, 2 and 3 the orthocentre of any Heron triangle, the incentre of any bis triangle and the centroid of any med triangle are all ceva points of those respective types of triangles.

Are there any other common ceva points?

This section considers the circumcentre of any Heron triangle.

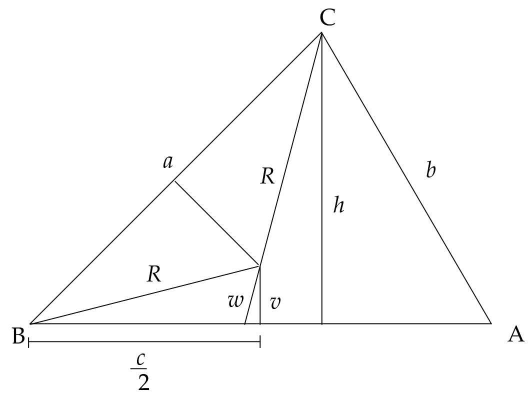

Figure 5.4: Cevian through the circumcentre

Theorem 23 *The circumcentre of any rational sided triangle is a ceva point if and only if the triangle has rational area.*

Proof: Consider the triangle in Figure 5.4. Let the cevian through the circumcentre onto side c be R_c i.e. $R_c = R + w$ where the circumradius is given by $R = \frac{abc}{4\triangle}$. Now since $\frac{w}{v} = \frac{R+w}{h}$ then $w = \frac{Rv}{h-v}$. Adding R to both sides leads to $R_c = \frac{Rh}{h-v}$. But $h = \frac{2\triangle}{c}$ and $v = \sqrt{R^2 - c^2/4}$ so that replacing R, h and v in the expression for R_c leads to

$$R_c = \frac{4abc\triangle}{16\triangle^2 - c^2\sqrt{4a^2b^2 - 16\triangle^2}}.$$

Using Heron's formula for the area of a triangle one obtains

$$R_c = \frac{4abc\triangle}{c^2(a^2+b^2) - (a^2-b^2)^2}.$$

Clearly R_c is rational if and only if the area is rational. Since this is also true for the other two cevians through the circumcentre the result follows. ■

5.6 Vincents of a Triangle

The final type of cevian to be considered in this thesis is a more obscure type, rarely mentioned elsewhere.
Definition : A vincent of a triangle is a cevian from any vertex to the opposite incircle tangent. A vin triangle is an integer-sided triangle with three integer vincents. The set of vin triangles will be denoted by **Vin**.

Notice that the relationship of the vincents to the incentre is the same as the relationship of the medians to the circumcentre. The cevians through the

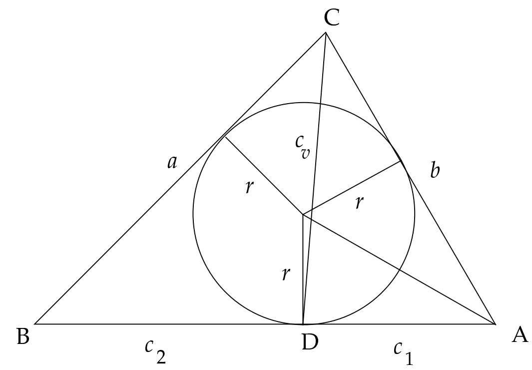

Figure 5.5: Vincent of a triangle

circumcentre are relatively easy to analyse (cf. previous section) while the medians are more difficult. Similarly the angle bisectors (which pass through the incentre) are easy to parametrize (cf. Chapter 2) while the vincents seem at least as difficult as the medians. Referring to Figure 5.5 one obtains

$$c_v^2 = c_1^2 + b^2 - 2c_1 b \cos A$$

where $c_1 = \frac{r}{\tan(A/2)}$ and $r = \frac{\triangle}{s}$. Since $\tan(A/2) = \frac{1-\cos A}{\sin A} = \sqrt{\frac{1-\cos A}{1+\cos A}}$ then we have

$$\begin{aligned} c_1 &= \frac{\triangle}{s}\sqrt{\frac{2bc + b^2 + c^2 - a^2}{2bc - (b^2 + c^2 - a^2)}} \\ &= \frac{b + c - a}{2}. \end{aligned}$$

Substituting into the equation above leads to

$$c_v^2 = \left(\frac{b+c-a}{2}\right)^2 + b^2 - 2b\left(\frac{b+c-a}{2}\right)\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

which upon rearrangement becomes

$$4cc_v^2 = (a+b-c)\left[c(a+b+c) - 2(a-b)^2\right].$$

By symmetry one obtains the defining equations for the three vincents, namely

$$\begin{aligned} 4aa_v^2 &= (b+c-a)\left[a(a+b+c) - 2(b-c)^2\right] \\ 4bb_v^2 &= (a+c-b)\left[b(a+b+c) - 2(a-c)^2\right] \\ 4cc_v^2 &= (a+b-c)\left[c(a+b+c) - 2(a-b)^2\right]. \end{aligned} \tag{5.5}$$

Now $c_1 = \frac{b+c-a}{2}$ implies that $c_2 = \frac{a-b+c}{2}$. Similarly $a_1 = \frac{a-b+c}{2}$, $a_2 = \frac{a+b-c}{2}$, $b_1 = \frac{a+b-c}{2}$ and $b_2 = \frac{b-a+c}{2}$. Hence

$$\frac{a_1}{a_2}\frac{b_1}{b_2}\frac{c_1}{c_2} = \left(\frac{a-b+c}{a+b-c}\right)\left(\frac{a+b-c}{b-a+c}\right)\left(\frac{b+c-a}{a-b+c}\right) = 1.$$

So by Ceva's Theorem the three vincents of a triangle are concurrent. In fact this common point of intersection is called the Gergonne point [4, p.13]. Requiring all three vincents to be integers forces the perimeter of the triangle to be even. Consider equations (5.5) modulo 2

$$\begin{aligned} a(b+c) &\equiv a \pmod 2 \\ b(a+c) &\equiv b \pmod 2 \\ c(a+b) &\equiv c \pmod 2. \end{aligned}$$

Adding these three equations together leads to $a + b + c \equiv 0 \pmod 2$ and so the semiperimeter is always an integer in a vin triangle. As a result equations (5.5) can be simplified to

$$\begin{aligned} aa_v^2 &= (s-a)\left[sa - (b-c)^2\right] \\ bb_v^2 &= (s-b)\left[sb - (a-c)^2\right] \\ cc_v^2 &= (s-c)\left[sc - (a-b)^2\right]. \end{aligned}$$

At this stage the only positive result concerning vincents is a parametrization of all isosceles triangles with two integer vincents. This will be followed by a demonstration that no vin triangle can be isosceles. If $a = b$ then $a_v = b_v$ and the first two equations of set (5.5) reduce to

$$4aa_v^2 = c^2(5a - 2c).$$

This equation implies that $\frac{5a-2c}{a}$ must be the square of a rational number. So letting $\gcd(5a-2c, a) = k$ leads to $5a-2c = kq^2$ and $a = kp^2$ where $\gcd(p,q) = 1$. Hence $c = \frac{5kp^2-5kq^2}{2}$ and $a_v = \frac{qc}{2p}$. Scaling these equations by a factor of $4p$ leads to the parametrization

$$\begin{aligned} a = b &= 4kp^3 \\ c &= 2kp(5p^2 - q^2) \\ a_v = b_v &= kq(5p^2 - q^2). \end{aligned}$$

The triangle inequality restricts the values of p and q to $\frac{1}{\sqrt{5}} < \frac{p}{q} < 1$ and so such triangles can be listed systematically (see Table 5.1).

Theorem 24 *No integer-sided isosceles triangle can have three integer vincents.*

Proof: Letting $a = b$ in equations (5.5) one obtains the pair

$$\begin{aligned} 4aa_v^2 &= c^2(5a - 2c) \\ 4c_v^2 &= (2a)^2 - c^2. \end{aligned}$$

p	q	$a(=b)$	c	$a_v(=b_v)$
1	2	$2\cdot 2$	2	2
2	3	32	44	33
3	4	$4\cdot 27$	$6\cdot 29$	$4\cdot 34$
3	5	$4\cdot 27$	$4\cdot 30$	$4\cdot 25$
4	5	256	440	275
5	6	250	445	267
4	7	256	248	217
5	7	$4\cdot 125$	$4\cdot 190$	$4\cdot 133$
6	7	864	1572	917
5	8	$2\cdot 250$	$2\cdot 305$	$2\cdot 244$

Table 5.1: Isosceles triangles with two integer vincents

The latter of these two equations has pythagorean triples as its solution. It is easiest to consider c_v as odd or even separately.
Case (i) : If c_v is even then the pythagorean triples take the form $c = 2u^2 - 2v^2$, $c_v = 2uv$, $a = u^2 + v^2$ where $\gcd(u, v) = 1$ and $2 \mid uv$. Substituting this into the first equation gives

$$\left(\frac{2av}{c}\right)^2 = \frac{u^2 + 9v^2}{u^2 + v^2}.$$

If $\gcd(u^2 + 9v^2, u^2 + v^2) = k$ then there exist integers p and q, say such that

$$u^2 + 9v^2 = kp^2$$
$$u^2 + v^2 = kq^2.$$

Now by the transformation $a = u^2q^2 - v^2p^2$ and $b = 2uvpq$ one obtains

$$a^2 + b^2 = (u^2q^2 + v^2p^2)^2$$
$$a^2 + 9b^2 = \left(\frac{u^4 + 18u^2 + 9v^4}{k}\right)^2.$$

But the quadratic forms $a^2 + b^2$ and $a^2 + 9b^2$ are discordant (i.e. cannot be simultaneously squares of integers) by Lemma 4. Hence this case has no solutions.
Case (ii) : If c_v is odd then the pythagorean triples take the form $c = 4uv$, $c_v = u^2 - v^2$, $a = u^2 + v^2$ where $\gcd(u, v) = 1$ and $2 \mid uv$. Substituting this into the equation defining a_v gives

$$\left(\frac{2av}{c}\right)^2 = \frac{5(u-v)^2 + 2uv}{u^2 + v^2}.$$

Using the transformation $U = u - v$ and $V = u + v$ then

$$\left(\frac{2av}{c}\right)^2 = \frac{U^2 + 9V^2}{U^2 + V^2}.$$

By the same argument used in case (i) this has no solutions and the theorem is proved. ∎

To date, a search of all dissimilar triangles with semiperimeter less than 4000 has not turned up a single example of a rational-sided triangle with three rational vincents.

5.7 Tiling a Triangle with Rational Triangles

Recalling the result of Theorem 18 it is possible to tile any rational sided triangle with any number of rational triangles by simply subdividing one side. More generally one could begin by subdividing the original triangle into two rational triangles (by Theorem 18). Then subdivide one or both of the two new triangles by their own rational cevians (with associated rational base segments). This process can obviously be continued indefinitely.

A tiling which requires more than just Theorem 18 is one in which the common vertex of any two triangles does not lie on the boundary of any triangle of the configuration (e.g. Figure 5.6). It turns out that the distances to any ceva point are rational and hence they provide a rational tiling of this non-trivial type. The triangle in Figure 5.6 was obtained by appropriately scaling up the one shown in Figure 5.3. (In fact the example of Figure 5.6 had been discovered previously by Arnfried Kemnitz but using a different approach.) Referring back to Figure 5.1, let $AP = p^*$, $BP = q^*$ and $CP = r^*$. Then the following preliminary result is useful.

Theorem 25 *For any triangle*

$$\frac{p^*}{p} + \frac{q^*}{q} + \frac{r^*}{r} = 2.$$

Proof: From Figure 5.1 one obtains $\frac{p-p^*}{p} = \frac{\triangle(BPC)}{\triangle(ABC)}$ since the triangles have the same base. Similarly $\frac{q-q^*}{q} = \frac{\triangle(APC)}{\triangle(ABC)}$ and $\frac{r-r^*}{r} = \frac{\triangle(APB)}{\triangle(ABC)}$. Adding these together leads to

$$\frac{p-p^*}{p} + \frac{q-q^*}{q} + \frac{r-r^*}{r} = \frac{\triangle(BPC)+\triangle(APC)+\triangle(APB)}{\triangle(ABC)} = 1.$$

Rearranging this leads to the form as stated in the theorem. ∎

Theorem 26 *Any ceva point of a triangle corresponds to a rational tiling of that triangle.*

Proof: Firstly the three cevians through a ceva point are rational by definition and by Theorem 22 so are all the base segments. Hence the cosines of all the angles shown in Figure 5.1 are also rational. In Figure 5.1

$$p^* \cos PAB + q^* \cos PBA = c$$
$$r^* \cos PCA + p^* \cos PAC = b.$$

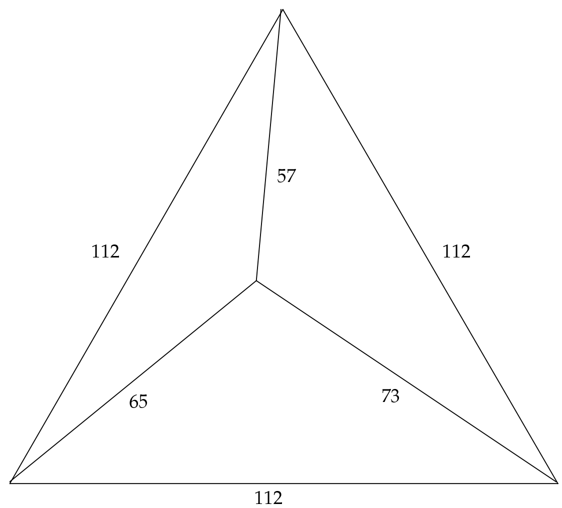

Figure 5.6: An integer-tiled equilateral triangle

So q^* and r^* can be expressed in terms of p^* as follows

$$q^* = \frac{c - p^* \cos PAB}{\cos PBA}$$
$$r^* = \frac{b - p^* \cos PAC}{\cos PCA}.$$

Using Theorem 25 leads to

$$p^* \left[\frac{1}{p} - \frac{\cos PAB}{q \cos PBA} - \frac{\cos PAC}{r \cos PCA}\right] = \left[2 - \frac{c}{q \cos PBA} - \frac{b}{r \cos PCA}\right].$$

So rearranging this gives

$$p^* = \frac{p(2qr \cos PBA \cos PCA - cr \cos PCA - bq \cos PBA)}{qr \cos PBA \cos PCA - pr \cos PAB \cos PCA - pq \cos PBA \cos PAC}.$$

Since

$$\cos PBA = \frac{q^2 + c^2 - b_2^2}{2qc}, \quad \cos PAB = \frac{p^2 + c^2 - a_1^2}{2pc},$$
$$\cos PAC = \frac{p^2 + b^2 - a_2^2}{2pb}, \quad \cos PCA = \frac{r^2 + b^2 - c_1^2}{2rb}$$

p^* is expressible rationally in terms of p, q, r, a_1, a_2, b_1, b_2, c_1 and c_2. At any ceva point these nine parameters are all rational and so p^* is rational. Similarly q^* and r^* are expressible as rational functions of the cevians and base segments and hence are rational. ■

As a consequence of Theorem 26 the orthocentre and circumcentre of alt triangles, the centroid of med triangles and the incentre of bis triangles all correspond to rational tilings of their respective types of triangle. Since the unit equilateral triangle can tile the infinite plane and each equilateral triangle itself can be rationally tiled at any ceva point (e.g Figure 5.3) this leads to a rational tiling of the infinite plane by dissimilar rational triangles.

Part III

Addenda

Epilogue

As is usual for most research, this thesis has raised many more questions than it has answered. Unfortunately the motivating problem for this thesis, namely D21, remains unsolved. However the appearance of Heron triangles with two integer medians does make it somewhat more probable that D21 can be solved in the positive sense rather than in the negative. A few of the more important unanswered questions which may be more amenable to attack are listed below as conjectures.

- Is it possible to invert the defining equations (2.1) for angle bisectors to express the sides in terms of the angle bisectors? I suspect it is. The analogous has been achieved for the defining equations for altitudes and medians. Even Carmichael's parametrization of heron triangles and the two median parametrization have been inverted in terms of their respective parameters. Yet the angle bisector equations remain stubbornly irreversible.

- Do there exist med, bis triangles? That is, are there integer-sided triangles with three integer medians and three integer angle bisectors? This seems very doubtful. In fact I suspect that med triangles cannot even have one integer angle bisector. There are alt triangles with one integer median e.g. the isosceles triangles given in parametrizations (2.2) and (2.2). There are also triangles with two integer medians and two integer angle bisectors e.g. $(238, 529, 529)$. This triangle corresponds to the rational point $(x, y) = \left(\frac{17}{7}, \frac{828}{49}\right)$ on the elliptic curve

  $$y^2 = 8x^4 + 4x^3 - 8x^2 - 2x + 2$$

  and so there are an infinite number of such triangles. However the transition from isosceles to scalene removes all of the degeneracy of these cases. This seems to make the occurrence of a non-isosceles triangles with two integer medians and two integer angle bisectors very rare.

- Do there exist an infinite number of dissimilar heron triangles with two integer medians? Only six such triangles have appeared in the course of the computer searches and both the elliptic curves found on the appropriate surface have rank zero. Despite this, it still seems likely that there are an infinite number of such triangles. There is the suggestive pattern connecting four of the six triangles of Table 4.2 which may lead to an infinite family. Also, restricting many other sets of six parameters of a triangle to integers (e.g. alt, bis, med) still lead to infinite solution sets.

- Do all rational tilings of rational-sided triangles correspond only to ceva points? I think so. The proof of this probably requires a theorem analogous to Theorem 22 however the corresponding equations to Stewart's Theorem for p^*, q^* and r^* seem much more difficult to deal with.

Bibliography

1 J. Baines and J. Malek, Atlas of Ancient Egypt, Phaidon, 1985.

2 B.J. Birch and H.P.F. Swinnerton-Dyer, Notes on elliptic curves, J. Reine und Angewandte Mathematik, vol.218, pp. 79-108, Walter de Gruyter and Co., 1965

3 R.D. Carmichael, The Theory of Numbers and Diophantine Analysis, Dover, 1952.

4 H.S.M. Coxeter and S.L. Greitzer, Geometry Revisited, Random House, 1967.

5 L.E. Dickson, History of the Theory of Numbers, vol. 2, Chelsea, 1952.

6 L. Euler, Opera Omnia, Commentationes Arithmeticae, vol. 2, paper 451, Teubner, 1941.

7 L. Euler, Solutio facilior problematis Diophantei circa triangulum in quo rectae ex angulis latera opposita bisecantes rationaliter exprimantur, Mémoires Acad. Sci. St-Pétersbourg, 2 (1807/8), 1810, 10-16. See L. Euler, Opera Omnia, Commentationes Arithmeticae, vol. 3, paper 732, Teubner, 1911.

8 L. Euler, Opera Omnia, Commentationes Arithmeticae, vol. 4, Teubner, 1911.

9 R.K. Guy, Unsolved Problems in Number Theory, Springer, 1981.

10 T.L. Heath, Diophantus of Alexandria, Dover, 1964.

11 K. Kendig, Elementary Algebraic Geometry, Springer, 1977.

12 L.J. Mordell, Diophantine Equations, Academic Press, 1969.

13 J.R. Newmann, The World of Mathematics, vol. 1, Simon and Schuster, 1956

14 T.E. Peet, The Rhind Mathematical Papyrus, Hodder and Stoughton, 1923, Kraus Reprint, 1970.

15 H. Schubert, Die Ganzzahligkeit in der algebraischen Geometrie, Leipzig, 1905, 1-16. See [1, p. 199].

16 J. H. Silverman, The Arithmetic of Elliptic Curves, Springer, 1986.

Appendix A

Search Program

This appendix contains a listing of the main program used by the author in the course of the research, namely that used to search for heron triangles with two integer medians. It also shows a general triangle together with the coordinates of all of the relevant points mentioned in this thesis. This turned out to be a useful computational aid as well as a conjecture testing device (especially when implemented on a computer as an interactive, animated graphic).

```
C     Program Heron_2_Median_triangles
C     ********************************************************
C     This program produces integer sided triangles with two
C     integer medians with the aid of the parametrization (3.15).
C     It then checks the rationality (or otherwise) of the area
C     of each triangle and prints out those with integer area
C     ********************************************************
      common/b1/ p(50000),q(50000),nmax
      common/b2/ m,n
      common/b3/ a,b,c
      common/b4/ prime(10000),pmax
      double precision a,b,c,m1,m2,m3,r,s,t,u,v,w,g
      double precision sr,sa,sb,sc,p,q
      real*16 area,area2
      integer i,sum,m,n,nmax,pmax
      Call Pairs
      write (*,*) '# Pairs  = ',nmax
      Call Primes
      write (*,*) '# Primes = ',pmax
      do 10 sum=2,2*nmax
      if (sum.eq.200*int(sum/200)) write (*,*) sum
      do 10 n=1,sum-1
      m=sum-n
      r=2*p(n)*q(n)+p(n)*p(n)-q(n)*q(n)
```

```
      s=p(n)*p(n)+q(n)*q(n)
      t=2*p(n)*q(n)-p(n)*p(n)+q(n)*q(n)
      u=2*p(m)*q(m)+p(m)*p(m)-q(m)*q(m)
      v=p(m)*p(m)+q(m)*q(m)
      w=2*p(m)*q(m)-p(m)*p(m)+q(m)*q(m)
      a=abs(u*r+v*t)
      b=abs(w*r+v*s)
      c=abs(w*t-s*u)
      if (a+b.le.c .or. a+c.le.b .or. b+c.le.a) goto 10
      call fng(b,a,gcd)
      call fng(c,gcd,g)
      g=abs(g)
      a=a/g
      b=b/g
      c=c/g
      call checkarea
10    continue
      stop
      end



      subroutine fng(a1,b1,gcd)
C     *******************************************************
C     This subroutine calculates the greatest common divisor of
C     a1 and b1 and puts the result into gcd.
C     *******************************************************
      common/b1/ p(50000),q(50000),nmax
      common/b2/ m,n
      double precision a1,b1,perm1,perm2,tst,q1,r1,gcd
      double precision sr,sa,sb,sc,p,q
      a1=dint(a1)
      b1=dint(b1)
      perm1=a1
      perm2=b1
      if (a1.gt.b1) then
            tst=a1
            a1=b1
            b1=tst
      end if
10    q1=dint(b1/a1)
      r1=dint(b1-q1*a1)
      if (r1.lt.0.01) goto 20
      b1=a1
      a1=r1
      goto 10
```

```
20    gcd=dint(a1)
      a1=perm1
      b1=perm2
      return
      end

      subroutine pairs
C     ********************************************************
C     This subroutine puts all relatively prime pairs of integers
C     less thanfour hundred into two arrays p(n) and q(n) for use
C     in the mainprogram when generating triangles.
C     ********************************************************
      common/b1/ p(50000),q(50000),nmax
      double precision sr,sa,sb,sc,p,q,gcd,u,v
      n=0
      do 99 u=2,400
      do 99 v=1,u-1
      call fng(u,v,gcd)
      if (gcd.gt.1) goto 99
      n=n+1
      p(n)=v
      q(n)=u
99    continue
      nmax=n
      return
      end

      subroutine primes
C     ********************************************************
C     This subroutine puts all primes numbers less than 99999
C     into an array called prime(n)
C     ********************************************************
      common/b4/ prime(10000),pmax
      integer pmax
      n=1
      prime(1)=2
      do 10 i=3,99999,2
            do 5 j=1,aint(sqrt(i+0.5))
              if (i.eq.prime(j)*aint(i/prime(j))) goto 10
5           continue
            n=n+1
            prime(n)=i
10    continue
      pmax=n
      return
      end
```

```
      subroutine checkarea
C     *********************************************************
C     This subroutine checks the number of prime divisors of each
C     factor in Herons formula for the square of the area. If any
C     prime occurs an odd number of times processing is returned
C     to the main program since the area cannot be the square of
C     an integer.
C     *********************************************************
      common/b1/ p(50000),q(50000),nmax
      common/b2/ m,n
      common/b3/ a,b,c
      common/b4/ prime(10000)
      double precision a,b,c,m1,m2,m3,area,r,s,t,u,v,w,g
      double precision sr,sa,sb,sc,p,q,tmp
      integer power(10000)
      sr=a+b+c
      sa=sr-(2*a)
      sb=sr-(2*b)
      sc=sr-(2*c)
      i=1
5     power(i)=0
10    tmp=sr/prime(i)
      if (tmp-dint(tmp).lt.0.00001) then
            sr=tmp
            power(i)=power(i)+1
            goto 10
      end if
20    tmp=sa/prime(i)
      if (tmp-dint(tmp).lt.0.00001) then
            sa=tmp
            power(i)=power(i)+1
            goto 20
      end if
30    tmp=sb/prime(i)
      if (tmp-dint(tmp).lt.0.00001) then
            sb=tmp
            power(i)=power(i)+1
            goto 30
      end if
40    tmp=sc/prime(i)
      if (tmp-dint(tmp).lt.0.00001) then
            sc=tmp
            power(i)=power(i)+1
            goto 40
      end if
```

```
      if (power(i).ne.2*int(power(i)/2)) goto 100
      sup_f=max(sr,sa,sb,sc)
      i=i+1
      if (prime(i).le.int(sqrt(sup_f+0.5))) goto 5
      if (sa.eq.sb .and. sc.eq.sr) then
            area=sa*sc
      else if (sa.eq.sc .and. sb.eq.sr) then
            area=sa*sr
      else if (sa.eq.sr .and. sb.eq.sc) then
            area=sa*sb
      else
            goto 100
      end if
      max_i=i
      do 50 i=1,max_i-1
         if (power(i).gt.0) area=area*(prime(i)**(power(i)/2))
50    continue
      write (*,60) m+n,a+b+c,2*a,2*b,2*c,area
60    format( '**',i6,4f8.0,f16.0 )
100   return
      end
```

Appendix B

Coordinatised Triangle

Let $\overline{BC} = a$, $\overline{AC} = b$, $\overline{AB} = c$, $\overline{AE} := x$ and $\overline{BE} := y$. Place the origin at vertex A with $\overline{AC}$ lying along the x-axis. Then the coordinates of the points in Figure B.1 are given by the following expressions of a, b, c, x and y.

$$A : (0, 0) \qquad B : (x, y) \qquad C : (b, 0)$$

$$D : \left(\frac{by^2}{a^2}, \frac{yb^2 - xyb}{a^2}\right) \qquad E : (x, 0) \qquad F : \left(\frac{x^2 b}{c^2}, \frac{xyb}{c^2}\right)$$

$$G : \left(\frac{x+b}{3}, \frac{y}{3}\right) \qquad H : \left(x, \frac{xb - x^2}{y}\right) \qquad I : \left(\frac{b+c-a}{2}, \frac{yb}{2s}\right)$$

$$J : \left(\frac{x+b}{2}, \frac{y}{2}\right) \qquad K : \left(\frac{b}{2}, 0\right) \qquad L : \left(\frac{x}{2}, \frac{y}{2}\right)$$

$$O : \left(\frac{b}{2}, \frac{x^2 + y^2 - xb}{2y}\right)$$

$$R : \left(\frac{x}{2}, \frac{xb - x^2}{2y}\right) \qquad S : \left(x, \frac{xb + y^2 - x^2}{2y}\right) \qquad T : \left(\frac{x+b}{2}, \frac{xb - x^2}{2y}\right)$$

$$X : \left(\frac{2ab + (x-b)(a+b-c)}{2a}, \frac{y(a+b-c)}{2a}\right)$$

$$Y : \left(\frac{b+c-a}{2}, 0\right)$$

$$Z : \left(\frac{x(b+c-a)}{2c}, \frac{y(b+c-a)}{2c}\right)$$

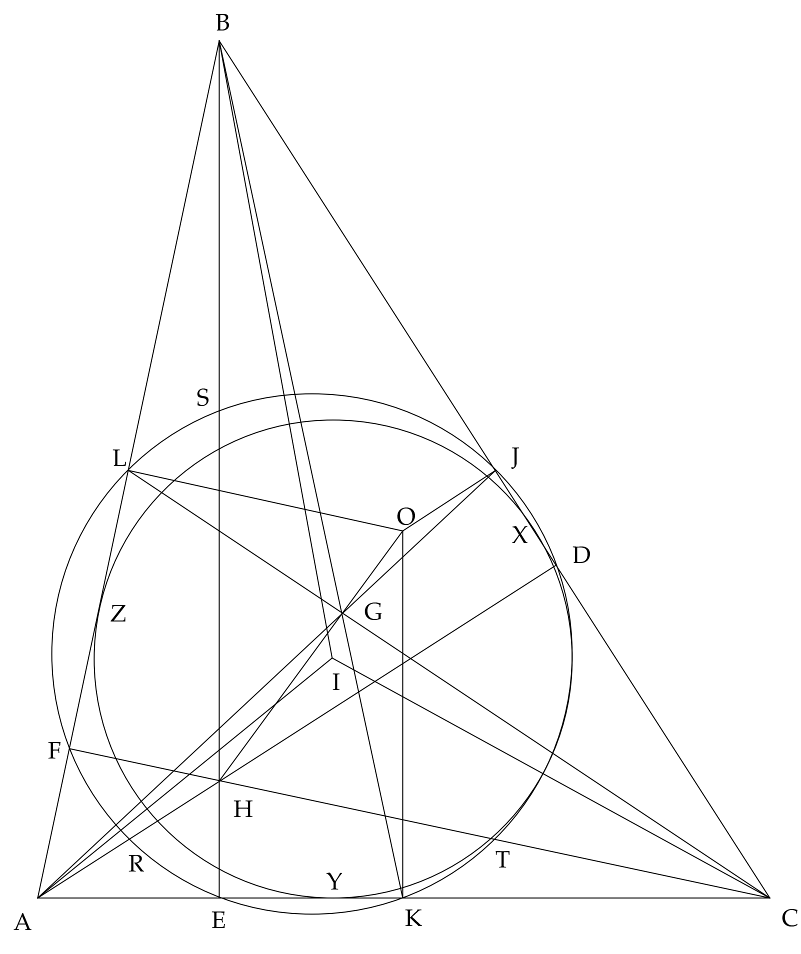

Figure B.1: The coordinatised triangle

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