Symmetries of triangles with two rational medians*

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Abstract

We study the symmetry group of the solutions to equations which define rational triangles with two rational medians. The group action is used to discover three new elliptic curves parametrizing such triangles which also have rational area. We prove that only finitely many of the rational triangles with two rational medians and rational area, which correspond to rational points on eight elliptic curves, can also have a third rational median. Finally, we present a new analysis of sporadic examples of such triangles along with the discovery of a new sporadic triangle.

Keywords : rational area triangle, fundamental domain, elliptic curve, rational medians.

1 Introduction

A perfect triangle, as defined by Richard Guy (see [6] D21), is a Heron triangle, namely one with three rational sides and rational area, which also has three rational medians. To date, no-one has found such an object—nor has anyone proven that such a triangle cannot exist.

There are partial results which show that triangles do exist in which six of the seven parameters are rational. In fact we now know of infinite families of triangles with

(a) three rational sides and three rational medians ([4] p. 399), or
(b) three rational sides, two rational medians and rational area ([1]), or

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(c) two rational sides, three rational medians and rational area ([6]).

The main subject of this paper is triangles with two rational medians and so we focus on family (b) above. Furthermore, there is a one-to-one correspondence between triangles in family (b) and those in family (c) so that much of what we find applies to the latter case as well.

There are an infinite number of triangles in family (b) which correspond to rational points on elliptic curves isomorphic to the curve

$$E : y^2 + xy = x^3 + x^2 - 2x$$

of conductor 102. The symmetries of these triangles have been studied before but we show that the previous results were incomplete—in fact we show that the complete symmetry group is isomorphic to the wreath product of the Klein four group and the finite simple group of order two[1], namely

$$(C_2 \times C_2) \wr C_2.$$

We use Faltings' theorem [5] to prove that there are only finitely many perfect triangles corresponding to rational points on any of the 8 isomorphic copies of E generated by this symmetry group.

Finally we report on the status of the search for sporadic versions of Heron triangles with two rational medians, namely those which do not correspond to rational points on the elliptic curves mentioned above. We actually find a new one,

$$(a, b, c) = (22816608, 20565641, 19227017),$$

which is only the fourth known sporadic triangle.

2 Defining equations

The search for perfect triangles requires one to find rational solutions to the equations defining the area and the medians in terms of the sides. These æquations have been handed down to us from Hero of Alexandria (in his Metrica circa the first century A.D.) and Apollonius of Perga (who lived ca 262 BC - 190 BC). Given a triangle with sides a, b, c, medians k, l, m, and area Δ we have

$$\begin{aligned} 16\Delta^2 &= (a+b+c)(-a+b+c)(a-b+c)(a+b-c), \\ 4k^2 &= 2b^2 + 2c^2 - a^2, \\ 4l^2 &= 2c^2 + 2a^2 - b^2, \\ 4m^2 &= 2a^2 + 2b^2 - c^2. \end{aligned} \tag{1}$$

It was Brahmagupta [3] who showed us how to parametrize all Heron triangles, while it was Euler [4] who provided an infinite (though incomplete) family of

[1] With apologies to The Klein Four

Apollonian triangles, namely rational sided triangles with three rational medians.

If we consider various subsets of equations (1) from a slightly more modern perspective we see some interesting structure (summarised in Table 1).

median equations	surface type	ambient space
1	homogeneous quadratic surface	$\mathbb{P}^3(\mathbb{C})$
2	degree 4 del Pezzo surface	$\mathbb{P}^4(\mathbb{C})$
3	K3 surface	$\mathbb{P}^5(\mathbb{C})$

Table 1: Surface classification

First, the equation for any single median, say that defining k, is a homogeneous quadratic with a trivial rational point. As such one can use the chord method to produce a complete parameterisation, for example

$$(a : b : c : k) = (4p(q+r) : p^2+2q^2-2r^2+4qr : p^2-2q^2+2r^2+4qr : p^2-2q^2-2r^2)$$

where p, q, r are arbitrary rational parameters.

Next, the equations defining any pair of medians, say the two k and l quadratic equations in 5 variables, represent a so-called degree 4 del Pezzo surface [**?**]. A simple check of the zeros of the appropriate partial derivatives reveals that it is in fact non-singular. Furthermore, it is well known that such surfaces contain 16 lines, which in this particular case are given by triples of homogeneous linear equations, namely

$$L_{(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)} = \{c + \epsilon_1(a + \epsilon_0 b), 2k + \epsilon_2(2b + \epsilon_0 a), 2l + \epsilon_3(2a + \epsilon_0 b)\}$$

where $\epsilon_i \in \{-1, +1\}$ for $i \in \{0, 1, 2, 3\}$. When these 16 lines are defined over $\mathbb{Q}$ it turns out that general theory predicts that the surface is birationally equivalent to the rational projective plane. This parametrisation of rational sided triangles with two rational medians (see [1]) is given by

$$\begin{aligned} a &= (-2\phi\theta^2 - \phi^2\theta) + (2\theta\phi - \phi^2) + \theta + 1 \\ b &= (\phi\theta^2 + 2\phi^2\theta) + (2\theta\phi - \theta^2) - \phi + 1 \\ c &= (\phi\theta^2 - \phi^2\theta) + (\theta^2 + 2\theta\phi + \phi^2) + \theta - \phi \end{aligned} \tag{2}$$

where θ and ϕ are arbitrary rational parameters. We can of course substitute these into equations (1) to show that the medians are given by

$$\begin{aligned} k &= (3\phi^2\theta + 2\theta^2 + 2\theta\phi - \phi^2 + \theta + 2\phi - 1)/2 \\ l &= (3\phi\theta^2 + \theta^2 - 2\theta\phi - 2\phi^2 + 2\theta + \phi + 1)/2 \\ 4m^2 &= 4 + 9\phi^2\theta^4 - 4\phi + 18\theta\phi + 4\theta + 6\phi\theta^2 - 6\phi^2\theta - 6\phi\theta^4 - 22\theta\phi^3 \\ &\quad + 6\phi^2\theta^2 + 6\phi^4\theta + 9\phi^4\theta^2 - 22\phi\theta^3 + 18\phi^3\theta^2 - 18\phi^2\theta^3 \\ &\quad + 18\phi^3\theta^3 - 3\phi^2 - 3\theta^2 + \phi^4 + \theta^4 - 2\theta^3 + 2\phi^3 = f_4(\theta, \phi). \end{aligned} \tag{3}$$

Notice that the medians k and l, bisecting sides a and b respectively, are automatically rational, while the median m, to side c, is not necessarily rational since the multiquartic polynomial, $f_4(\theta, \phi)$, is irreducible over the integers.

Now since the equations (1) allow for an ambiguity in the resulting signs of the expressions for a, b, c, k, l in equations (2) and (3) the signs have been chosen so that a proper triangle appears in the positive $\theta\phi$-quadrant.

We can invert the set of equations (2) to obtain

$$\theta_{\pm} = \frac{c - a \pm \sqrt{2a^2 + 2c^2 - b^2}}{a + b + c}$$
$$\phi_{\pm} = \frac{b - c \pm \sqrt{2b^2 + 2c^2 - a^2}}{a + b + c}$$

where re-substitution of equations (2) into these force the sign of the discriminant to be positive in each case. In the sequel we will always use

$$\begin{aligned} \theta &= \frac{c - a + 2l}{a + b + c} \\ \phi &= \frac{b - c + 2k}{a + b + c} \end{aligned} \tag{4}$$

as the defining inverse equations.

Finally, the equations for all three medians define a non-singular K3 surface. In fact, in [?] it is shown that this can be viewed as a one parameter elliptic curve which generically has rank 2 and so the rational points are dense in this surface.

3 The Group of Symmetries

We now restrict our attention to just the case of rational-sided triangles with two rational medians and show how to compute their group of symmetries. We restate the defining equations for convenience:

$$\begin{aligned} 4k^2 &= 2b^2 + 2c^2 - a^2 \\ 4l^2 &= 2c^2 + 2a^2 - b^2. \end{aligned} \tag{5}$$

Now we are interested in all the rational (θ, ϕ)-pairs which correspond, via equations (2) and (3) to rational quintuples (a, b, c, k, l) which are solutions to equations (5). Rather than work directly in the $\theta\phi$-plane we find that the problem is greatly simplified by consideration of the $\mathbb{R}^5$-space defined by the a, b, c, k, l coordinates. Once the symmetries are understood here they can be easily mapped to $\mathbb{R}^2$ via equations (4).

Clearly we can change the signs of a, b, c, k, and l independently and the new 5-tuple will still form a solution to equations (5). Furthermore, we can interchange sides a and b while simultaneously swapping medians k and l to also obtain a solution to equations (5).

The only other symmetries, of solutions to equations (5), are compositions of these operations. Accordingly, we define the following maps

$$\begin{aligned} A &: (a,b,c,k,l) \mapsto (-a,b,c,k,l) \\ B &: (a,b,c,k,l) \mapsto (a,-b,c,k,l) \\ C &: (a,b,c,k,l) \mapsto (a,b,-c,k,l) \\ K &: (a,b,c,k,l) \mapsto (a,b,c,-k,l) \\ L &: (a,b,c,k,l) \mapsto (a,b,c,k,-l) \\ T &: (a,b,c,k,l) \mapsto (b,a,c,l,k), \end{aligned} \tag{6}$$

and call the group generated by all these symmetries $\overline{G}$ say. First we observe that each of these maps is an involution, since they are just reflections. Furthermore, the first five generators, A, B, C, K, L, all commute with each other and so generate a subgroup, H say, of $\overline{G}$ of order 32. Meanwhile, it is trivial to verify that T satisfies the relations

$$TA = BT, \quad TB = AT, \quad TK = LT, \quad TL = KT, \quad TC = CT,$$

so in principle we could replace B and L by A^T and K^T respectively.[2] However, for convenience, we will continue to include B and L in any description of an arbitrary element of $\overline{G}$. The relations above imply that T permutes the elements of H under conjugation, from which we can infer that $\overline{G}$ contains 64 elements distributed in two cosets, namely H and TH.

Since we only care about distinct (θ, ϕ)-pairs which correspond to a given triangle, and changing the sign of all of a, b, c, k, l leaves θ and ϕ unchanged, we factor out the action of $ABCKL$ from $\overline{G}$. Hence from now on we consider the modified group

$$G = \overline{G}/\langle BLACK \rangle$$

containing 32 elements and basically ignore the generator C. So if we let an arbitrary element of G be denoted by g then without loss of generality we can represent it as

$$g = A^\alpha B^\beta K^\kappa L^\lambda T^\tau$$

where α, β, κ, λ, τ are zero or one. Since there is a one-to-one correspondence between group elements and binary 5-tuples (via the exponents) we can identify each element with an integer from 0 to 31 as shown in Table 2. First we compute the centre of the group G, denoted by $Z(G)$, which is the set of all elements in G which commute with everything in G. The non-commuting equations above

[2] We use the standard notation $A^T := T^{-1}AT$ for conjugation of one element by another. The difference between conjugation, exponentiation and action on a point should be clear by context.

$(\alpha,\beta,\kappa,\lambda,\tau)_2$	element	order	$(\alpha,\beta,\kappa,\lambda,\tau)_2$	element	order
0	1	1	16	A	2
1	T	2	17	AT	4
2	L	2	18	AL	2
3	LT	4	19	ALT	4
4	K	2	20	AK	2
5	KT	4	21	AKT	4
6	KL	2	22	AKL	2
7	KLT	2	23	$AKLT$	4
8	B	2	24	AB	2
9	BT	4	25	ABT	2
10	BL	2	26	ABL	2
11	BLT	4	27	$ABLT$	4
12	BK	2	28	ABK	2
13	BKT	4	29	$ABKT$	4
14	BKL	2	30	$ABKL$	2
15	$BKLT$	4	31	$ABKLT$	2

Table 2: Symmetries of equivalence classes of triangles with two rational medians

show us that

$$\begin{aligned}
g^A &= \begin{cases} A^\alpha B^\beta K^\kappa L^\lambda T^\tau & \text{if } \tau = 0 \\ A^{\alpha+1} B^{\beta+1} K^\kappa L^\lambda T^\tau & \text{if } \tau = 1 \end{cases} \\
g^B &= \begin{cases} A^\alpha B^\beta K^\kappa L^\lambda T^\tau & \text{if } \tau = 0 \\ A^{\alpha+1} B^{\beta+1} K^\kappa L^\lambda T^\tau & \text{if } \tau = 1 \end{cases} \\
g^K &= \begin{cases} A^\alpha B^\beta K^\kappa L^\lambda T^\tau & \text{if } \tau = 0 \\ A^\alpha B^\beta K^{\kappa+1} L^{\lambda+1} T^\tau & \text{if } \tau = 1 \end{cases} \\
g^L &= \begin{cases} A^\alpha B^\beta K^\kappa L^\lambda T^\tau & \text{if } \tau = 0 \\ A^\alpha B^\beta K^{\kappa+1} L^{\lambda+1} T^\tau & \text{if } \tau = 1 \end{cases} \\
g^T &= A^\beta B^\alpha K^\lambda L^\kappa T^\tau
\end{aligned} \tag{7}$$

from which we deduce that

$$\begin{aligned}
&Ag = gA, Bg = gB, Kg = gK, Lg = gL \text{ iff } \tau = 0 \text{ and} \\
&Tg = gT \text{ iff } \alpha = \beta, \kappa = \lambda.
\end{aligned}$$

So the centre of G contains only elements of the form $(AB)^\alpha (KL)^\kappa$, hence $Z(G) = \langle 1, AB, KL, ABKL \rangle$ which is isomorphic to $C_2 \times C_2$.

Next we calculate the derived group of G, denoted by G', which is the subgroup generated by all the commutators in G. Recall that the derived group is the unique smallest normal subgroup such that G/G' is Abelian. If we consider the commutators of the generators with an arbitrary element of G then we have

$$\begin{aligned}
[A,g] &= AgAg^{-1} = \begin{cases} 1 & \text{if } \tau = 0 \\ AB & \text{if } \tau = 1 \end{cases} \\
[B,g] &= BgBg^{-1} = \begin{cases} 1 & \text{if } \tau = 0 \\ AB & \text{if } \tau = 1 \end{cases} \\
[K,g] &= KgKg^{-1} = \begin{cases} 1 & \text{if } \tau = 0 \\ KL & \text{if } \tau = 1 \end{cases} \\
[L,g] &= LgLg^{-1} = \begin{cases} 1 & \text{if } \tau = 0 \\ KL & \text{if } \tau = 1 \end{cases} \\
[T,g] &= TgTg^{-1} = 1.
\end{aligned}$$

Thus $G' = \langle AB, KL\rangle$ which leads to the somewhat surprising fact that $G' = Z(G)$. We now determine the group structure of G.

Theorem 1 $G \cong (C_2 \times C_2) \wr C_2$.

Proof : First observe that the transformation group, G, can be defined abstractly by

$$G \cong \langle A, B, K, L, T | A^2, B^2, K^2, L^2, T^2, A^T = B, K^T = L\rangle.$$

It is then trivial to construct a permutation group, P_1, a subgroup of the symmetric group on 10 elements, defined by

$$P_1 = \langle (1,2), (3,4), (5,6), (7,8), (1,4)(2,3)(6,7)(5,8)(9,10)\rangle,$$

which is isomorphic to G. In fact the isomorphism, $\phi : G \to P_1$ is clearly given by

$$\begin{aligned}
A &\mapsto (1,2) \\
B &\mapsto (3,4) \\
K &\mapsto (5,6) \\
L &\mapsto (7,8) \\
T &\mapsto (1,4)(2,3)(6,7)(5,8)(9,10).
\end{aligned}$$

By rewriting the wreath product as a semidirect product we find that

$$(C_2 \times C_2) \wr C_2 = (C_2 \times C_2)^2 \rtimes C_2$$

which in turn is isomorphic to the permutation group, P_2, defined by

$$P_2 = \langle (\overline{1}, \overline{2}), (\overline{3}, \overline{4}), (\overline{5}, \overline{6}), (\overline{7}, \overline{8}), (\overline{1}, \overline{5})(\overline{2}, \overline{6})(\overline{3}, \overline{7})(\overline{4}, \overline{8})\rangle < S_8.$$

Now since $(\overline{5},\overline{6}) = (\overline{1},\overline{2})^{(\overline{1},\overline{5})(\overline{2},\overline{6})(\overline{3},\overline{7})(\overline{4},\overline{8})}$ and $(\overline{7},\overline{8}) = (\overline{3},\overline{4})^{(\overline{1},\overline{5})(\overline{2},\overline{6})(\overline{3},\overline{7})(\overline{4},\overline{8})}$ one observes that P_2 can be equivalently described in terms of just 3 generators, namely

$$P_2 = \langle (\overline{1},\overline{2}), (\overline{3},\overline{4}), (\overline{1},\overline{5})(\overline{2},\overline{6})(\overline{3},\overline{7})(\overline{4},\overline{8})\rangle.$$

Finally, one easily checks that the mapping $\theta : P_2 \to P_1$ given by

$$\begin{aligned}
(\overline{1},\overline{2}) &\mapsto (3,4) \\
(\overline{3},\overline{4}) &\mapsto (5,6) \\
(\overline{1},\overline{5})(\overline{2},\overline{6})(\overline{3},\overline{7})(\overline{4},\overline{8}) &\mapsto (1,4)(2,3)(5,8)(6,7)(9,10).
\end{aligned}$$

is an isomorphism. Thus we have $G \cong P_1 \cong P_2 \cong (C_2 \times C_2) \wr C_2$ to give us the result we want. $\square$

4 The Fundamental Domain

We would like to determine the fundamental domain of $\mathbb{R}^2/G$ and the easiest path is to simply determine the effect of each generator in turn. The action of the generators of G expressed in terms of the parameters θ and ϕ is easily obtained by simple substitution. For example, to determine the action of A on θ one simply substitutes the definition of A from equation (6) into the definition of θ from equation (4) and then replace a, b, c, k, l using equations (2) to get

$$\begin{aligned}
\theta(-a,b,c,k,l) &= \frac{c+a+2l}{-a+b+c} \\
&= \frac{\theta\phi+\theta-\phi^2+1}{\phi(2\theta+\phi-1)}.
\end{aligned}$$

Using the same trick on the ϕ coordinate leads to

$$\phi(-a,b,c,k,l) = \frac{3\theta\phi+\theta-\phi+1}{2\theta^2+\theta\phi+\theta+\phi-1}.$$

Extending this process to all the other generators leads to

$$
\begin{aligned}
(\theta,\phi)^A &= \left(\frac{\theta\phi+\theta-\phi^2+1}{\phi(2\theta+\phi-1)}, \frac{3\theta\phi+\theta-\phi+1}{2\theta^2+\theta\phi+\theta+\phi-1}\right)\\
(\theta,\phi)^B &= \left(\frac{-3\theta\phi-\theta+\phi-1}{\theta\phi+2\phi^2-\phi-\theta-1}, \frac{\theta^2-\theta\phi+\phi-1}{\theta(2\phi+\theta+1)}\right)\\
(\theta,\phi)^C &= \left(\frac{\phi(2\theta+\phi-1)}{\phi^2-\theta\phi-\theta-1}, \frac{\theta(2\phi+\theta+1)}{\theta\phi-\theta^2-\phi+1}\right)\\
(\theta,\phi)^K &= \left(\theta, \frac{1-\theta-\phi-\theta\phi-2\theta^2}{3\theta\phi+\theta-\phi+1}\right)\\
(\theta,\phi)^L &= \left(\frac{2\phi^2+\theta\phi-\theta-\phi-1}{3\theta\phi+\theta-\phi+1}, \phi\right)\\
(\theta,\phi)^T &= \left(\frac{2\theta^2+\theta\phi+\theta+\phi-1}{3\theta\phi+\theta-\phi+1}, \frac{1+\theta+\phi-\theta\phi-2\phi^2}{3\theta\phi+\theta-\phi+1}\right).
\end{aligned}
\tag{8}
$$

The boundaries of each of these reflections are shown in Figure 1 where a fundamental domain corresponding to the positive a, b, c, k, l hyper-quadrant is la-

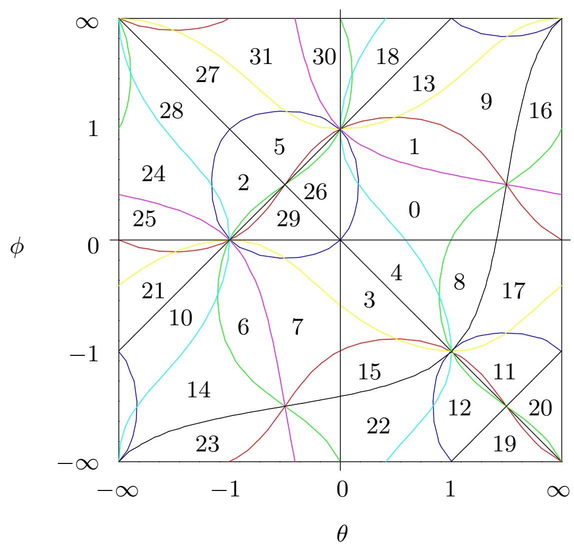

Figure 1: Symmetry boundaries in the $\theta\phi$-plane

belled as region 0. There are three extra boundaries corresponding to the curves

$$
\begin{aligned}
\phi+\theta &= 0\\
\phi &= \left(\frac{1+\theta}{1-\theta}\right)\\
\phi-\theta+2 &= 0
\end{aligned}
$$

which together with the boundaries obtained by setting $(\theta, \phi)^g = (\theta, \phi)$ for each of the 6 generators carves up the $\theta\phi$-plane[3] into 32 regions. These are labelled in Figure 1 with numbers corresponding to each of the elements from Table 2.

When all of the transformations are applied to any point, (θ, ϕ) say, then the resulting orbit of 32 points are distributed, 4 per line, amongst 8 lines which all pass through the point $(\theta, \phi) = (1, -1)$. This is always true since the transformations T and $ABKL$ move points on a line through $(1, -1)$ back to the same line. Furthermore, the group generated by these two transformations has order 4 and so the orbit contains 4 points. For example, the points

$$
\begin{aligned}
(5/2, -1/2)^1 &= (5/2, -1/2) \\
(5/2, -1/2)^T &= (49, 15) \\
(5/2, -1/2)^{ABKL} &= (7/8, -25/24) \\
(5/2, -1/2)^{ABKLT} &= (-3, -7/3)
\end{aligned}
$$

all lie on a line through $(1, -1)$.

The full G-orbit of the point $(1/3, 2/5)$ includes those in the set

{(1/3, 2/5),	(-24/25, 2/5),	(1/15, 24/25),	(-2/5, 24/25),
(1/3, -1/15),	(-24/25, -1/15),	(1/15, -1/3)	(-2/5, -1/3),
(25/24, -7/8),	(-3, -7/8),	(7/3, 3),	(7/8, 3),
(25/24, -7/3),	(-3, -7/3),	(7/3, -25/24)	(7/8, -25/24),
(49, 15),	(1/2, 15),	(5/2, -1/2),	(-15, -1/2),
(49, -5/2),	(1/2, -5/2),	(5/2, -49),	(-15, -49),
(-2, 3/7),	(-1/49, 3/7),	(-8/7, 1/49),	(-3/7, 1/49),
(-2, 8/7),	(-1/49, 8/7),	(-8/7, 2),	(-3/7, 2)}.

Notice that these points are arranged into 2×2 blocks which form rectangles in the $\theta\phi$-plane (two of which are shown in Figure 2). The rectangles are paired by reflection about the line $\theta + \phi = 0$ by the operation of the symmetry TLK, since $(\theta, \phi)^{TLK} = (-\phi, -\theta)$. The rectangles themselves are formed from a single point and the repeated application of K and L symmetries since K fixes θ, L fixes ϕ and $(\theta, \phi)^{KLKL} = (\theta, \phi)$.

5 Heron triangles with three rational medians

If we restrict our attention to Heron triangles which also have two rational medians then previous work [1] [2] had shown that there is a correspondence between infinite families of such triangles and rational points on various curves birationally equivalent to an elliptic curve defined over $\mathbb{Q}$. This work had turned up five genus 1 curves in the $\theta\phi$-plane (the first five shown in Figure 3) with the

[3]We have used a tan^{-1} transformation to squeeze the entire plane onto the page.

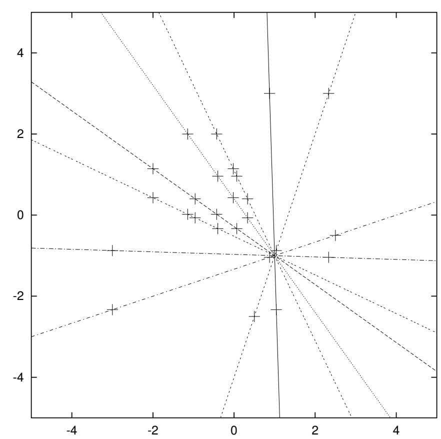

Figure 2: Orbit of $(\theta, \phi) = (1/3, 2/5)$

property that their rational points corresponded to such triangles. The question of the rationality or otherwise of the third median for these infinite families was left unresolved.

When we apply the work of the previous section we find that the action of the group G on these 5 curves produced 3 new curves (the last three in Figure 3). Notice that the first four curves are all symmetric about the line $\phi + \theta = 0$

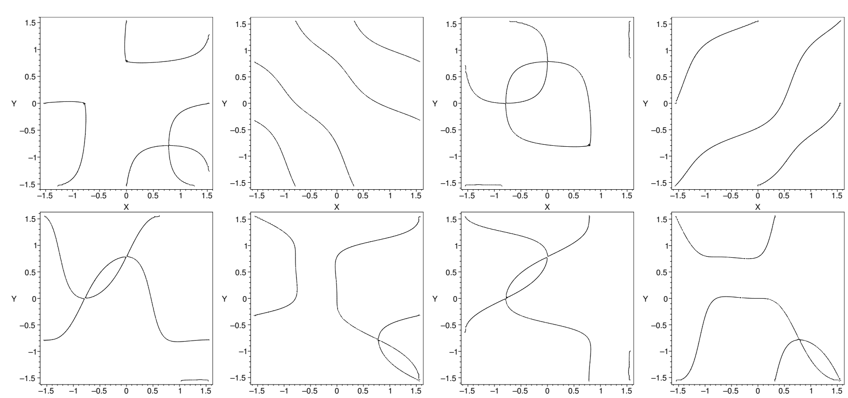

Figure 3: Elliptic curves corresponding to Heron triangles with 2 rational medians $(x = \tan^{-1}(\theta), y = \tan^{-1}(\phi))$

while the last four curves are pairwise symmetric about that line. Also, the last three curves, C_6, C_7, and C_8 do not pass through the unit square in the positive quadrant $(0 \leq \theta, \phi \leq 1)$ which is why previous authors missed them.

The complete action of G on these curves is provided by considering the action of the generators (as shown in Table 3). Due to the closure of Table 3 no new

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A	C_4	C_3	C_2	C_1	C_6	C_5	C_8	C_7
B	C_4	C_3	C_2	C_1	C_6	C_5	C_8	C_7
C	C_2	C_1	C_4	C_3	C_7	C_8	C_5	C_6
K	C_6	C_8	C_7	C_5	C_4	C_1	C_3	C_2
L	C_8	C_6	C_5	C_7	C_3	C_2	C_4	C_1
T	C_2	C_1	C_4	C_3	C_5	C_6	C_7	C_8

Table 3: Action of G on elliptic curves

curves can be produced by the action of G on this collection.

Of particular interest is the question of the rationality of the third median for any of the triangles obtained from these curves. It turns out that there are at most a finite number of such triangles.

Theorem 2 *There are a finite number of rational points on each of the curves $C_1, \ldots, C_8$ with the property that they correspond to a Heron triangle with three rational medians.*

Proof: We sketch the proof for C_4. Recall that equations (3) show that we have an expression for the third median as a function of θ and ϕ, namely,

$$4m^2 = f_4(\theta, \phi).$$

We want to find all rational points on this surface that are also rational points on the curve C_4. The equation for C_4 is given, [1], by the cubic equation

$$C_4 : \theta\phi(\theta - \phi) + \theta\phi + 2(\theta - \phi) - 1 = 0.$$

If we take the resultant of the curve C_4 with the surface defining m and eliminate ϕ say, then we obtain a new curve

$$D_4 : 16\theta^4 m^4 + p_{10}(\theta)m^2 + p_{14}(\theta) = 0$$

relating m and θ. Now the common rational solutions to C_4 and the surface are given by the rational points on the curve D_4. But the genus of D_4 is 7 and so by Faltings' Theorem can have only finitely many rational points. This finite, possibly empty, list of points would produce, when substituted into C_4 only a finite number of corresponding values of ϕ.

A similar result holds for each of the other seven curves, in fact the corresponding resultants, D_i, are genus 7 in every case. $\square$

A good deal more work needs to be done to actually find this finite list of triangles or show that there are none at all coming from these eight curves. Of course this does not rule out the possibility of finding perfect triangles amongst the sporadic triangles, namely those which are not obtained from any of the infinite families we have so far.

One can now ask the question, how is the G-orbit of a point distributed amongst the eight elliptic curves?

6 The sporadic triangles

We had hoped that a better knowledge of the fundamental domain would help speed the computational search. However, in practice, this failed to materialise since any restriction of a search to a fundamental region might miss otherwise easily spotted low height solutions in the orbit outside the region.

One way to search for Heron triangles with two rational medians is to substitute the parametrizations for the sides of triangles with two rational medians from equations (2) into Heron's formula to obtain

$$C : \Delta^2 = 16\theta\phi(\theta^2-1)(\phi^2-1)(3\theta\phi+\theta-\phi+1)(2\theta+\phi-1)(\theta+2\phi+1)(\theta-\phi+1).$$

Fixing ϕ say forces this to become a hyperelliptic curve in θ which, by Faltings' Theorem, has only finitely many rational points.

The basis of the current computational search was Michael Stoll's `ratpoints.c` code which efficiently searches for rational points on hyperelliptic curves. We modified this by adding an outer loop over ϕ and then searching the region $H(\theta) < 20000$, $H(\phi) < 2000$ where $H(m/n) = \max\{|m|, |n|\}$ is the naïve height function. The results of this search revealed a fourth sporadic triangle (see Table 4) which does not lie on any of the eight elliptic curves found so far. Of

(θ, ϕ)	(a, b, c)	sporadic
$(\frac{5}{2}, -\frac{1}{2})$	$(51, 73, 26)$	
$(-\frac{16}{5}, -\frac{5}{2})$	$(875, 626, 291)$	
$(-\frac{37}{40}, \frac{16}{5})$	$(13816, 28779, 15155)$	
$(-\frac{51}{40}, -\frac{4}{17})$	$(4368, 1241, 3673)$	$*$
$(\frac{25}{56}, \frac{12}{19})$	$(14384, 14791, 11257)$	$*$
$(\frac{285}{296}, \frac{37}{40})$	$(185629, 1823675, 1930456)$	
$(-\frac{560}{1089}, \frac{47}{72})$	$(1976471, 2288232, 2025361)$	$*$
$(\frac{2192}{2109}, -\frac{285}{296})$	$(2396426547, 2442655864, 46263061)$	
$(-\frac{2665}{816}, \frac{121}{408})$	$(22816608, 20565641, 19227017)$	$*$

Table 4: Small Heron triangles with 2 rational medians

the 32 different (θ, ϕ) representatives corresponding to each triangle, the one chosen for the table is that which minimizes the product $H(\theta) \cdot H(\phi)$. Since there are always two closely related solutions ((θ, ϕ) and $(-\phi, -\theta)$) with the same value for this product we arbitrarily select the one with the smaller $H(\phi)$. Notice that the non-sporadic triangles are related since one can alternately fix the θ and ϕ values and find the partners which produce a rational point on the curve C above. Starting with $\theta = \frac{5}{2}$ gives:

$$\left(\frac{5}{2}, -\frac{1}{2}\right) \mapsto \left(\frac{5}{2}, \frac{16}{5}\right) \mapsto \left(-\frac{37}{40}, \frac{16}{5}\right) \mapsto \left(-\frac{37}{40}, -\frac{285}{296}\right) \mapsto \left(\frac{2192}{2109}, -\frac{285}{296}\right).$$

If we continue this sequence we simply fix $\theta = \frac{2192}{2109}$ and find the corresponding rational points on the curve C. This leads to

$$\phi = -\frac{285}{296}, -\frac{2275}{2109}, \frac{4301}{2109}, -\frac{4301}{4218}, -\frac{4301}{4467}, \frac{16835}{15618}, -\frac{8408455}{57}, -\frac{21526505}{21156119},$$

of which only $\phi = -\frac{285}{296}$, $\phi = \frac{16835}{15618}$, $\phi = -\frac{8408455}{57}$ and $\phi = -\frac{21526505}{21156119}$ lead to non-degenerate triangles. The first and third ϕ values correspond to the known triangle in Table 4 while the second and fourth values correspond to a new (non-sporadic) triangle represented by

$$(\theta, \phi) = \left(\frac{2192}{2109}, \frac{16835}{15618}\right).$$

It is conceivable that a sporadic triangle does lie between the end of the table at $H(\theta) \cdot H(\phi) = 1087320$ and this new triangle for which $H(\theta) \cdot H(\phi) = 36902320$ but we have not yet completed that portion of the search.

The sporadic triangles do not seem to be related in the same way. Notice that the sets of distinct fractions (up to negation and inversion) used in the representations of the four sporadic triangles (see Table 5) are all disjoint. Furthermore,

(θ, ϕ)	fractions
$(-\frac{51}{40}, -\frac{4}{17})$	$\frac{3}{88}, \frac{21}{221}, \frac{11}{91}, \frac{4}{17}, \frac{13}{21}, \frac{40}{51}, \frac{100}{121}, \frac{85}{91}$
$(\frac{25}{56}, \frac{12}{19})$	$\frac{7}{31}, \frac{72}{289}, \frac{289}{775}, \frac{31}{81}, \frac{25}{56}, \frac{243}{532}, \frac{217}{361}, \frac{12}{19}$
$(-\frac{560}{1089}, \frac{47}{72})$	$\frac{25}{119}, \frac{9409}{34272}, \frac{679}{2209}, \frac{529}{1649}, \frac{560}{1089}, \frac{765}{1444}, \frac{24863}{43681}, \frac{47}{72}$
$(-\frac{2665}{816}, \frac{121}{408})$	$\frac{3185}{14801}, \frac{121}{408}, \frac{816}{2665}, \frac{14801}{44376}, \frac{29575}{59177}, \frac{1849}{3481}, \frac{287}{529}, \frac{5808}{8993}$

Table 5: Fractions used in sporadic representatives

if we attempt to generate a new sporadic triangle, from the first one say, using the same technique as that for the non-sporadics, we fail.

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