

Tangent Inflexion

Last updated on 20090702 by Mr. Francis Hung

Breakthrough Calculus and Coordinate Geometry by Y.L. Ng & K.M. Pang p.211 Q8

Given a curve $y = e^{-\frac{(x-a)^2}{2}}$, ($a > 0$).

If two distinct tangents can be drawn from the origin to the curve,

- (i) show that $a > 2$.
 (ii) show that, between the two points of contact P and Q , there is only one point of inflexion.

$$(i) \quad \frac{dy}{dx} = -(x-a)e^{-\frac{(x-a)^2}{2}} = (a-x)e^{-\frac{(x-a)^2}{2}}$$

Suppose the point of contact is (x_0, y_0) ,

$$\left. \frac{dy}{dx} \right|_{(x_0, y_0)} = (a-x_0)e^{-\frac{(x_0-a)^2}{2}}$$

$$\text{Equation of tangent: } \frac{y-y_0}{x-x_0} = (a-x_0)e^{-\frac{(x_0-a)^2}{2}}$$

$\therefore$ It passes through the origin,

$$y_0 = x_0(a-x_0)e^{-\frac{(x_0-a)^2}{2}}$$

$$e^{-\frac{(x_0-a)^2}{2}} = x_0(a-x_0)e^{-\frac{(x_0-a)^2}{2}}$$

$$1 = x_0(a-x_0)$$

$$x_0^2 - ax_0 + 1 = 0 \quad (*)$$

For distinct real roots, $\Delta > 0$.

$$a^2 - 4 > 0$$

$$(a+2)(a-2) > 0$$

$$a > 2 \quad (\because \text{given that } a > 0)$$

$$(ii) \quad \frac{d^2y}{dx^2} = -e^{-\frac{(x-a)^2}{2}} - (a-x)(x-a)e^{-\frac{(x-a)^2}{2}}$$

$$= [(x-a)^2 - 1]e^{-\frac{(x-a)^2}{2}}$$

$$\frac{d^2y}{dx^2} = (x-a+1)(x-a-1)e^{-\frac{(x-a)^2}{2}} \quad (**)$$

$$\frac{d^2y}{dx^2} = 0 \Rightarrow x = a-1 \text{ or } a+1$$

$$\text{Solve } (*): x = \frac{a + \sqrt{a^2 - 4}}{2} \quad \text{or} \quad x = \frac{a - \sqrt{a^2 - 4}}{2}$$

$$a+1 - \frac{a + \sqrt{a^2 - 4}}{2} = \frac{a+2 - \sqrt{a^2 - 4}}{2} > 0$$

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$$\begin{aligned}a+1 &> \frac{a+\sqrt{a^2-4}}{2} \\a-1 - \frac{a-\sqrt{a^2-4}}{2} &= \frac{a-2+\sqrt{a^2-4}}{2} > 0 \\a-1 &> \frac{a-\sqrt{a^2-4}}{2} \\ \frac{a+\sqrt{a^2-4}}{2} - (a-1) &= \frac{\sqrt{a^2-4}-a+2}{2} \\ &= \frac{\sqrt{a^2-4}^2 - (a-2)^2}{2(\sqrt{a^2-4}+a-2)} \\ &= \frac{4a-8}{2(\sqrt{a^2-4}+a-2)} > 0 \\ \therefore \frac{a-\sqrt{a^2-4}}{2} &< a-1 < \frac{a+\sqrt{a^2-4}}{2} < a+1\end{aligned}$$

$\therefore$ There is at most one point ($x = a - 1$) between the two points of contact P ($x = \frac{a-\sqrt{a^2-4}}{2}$) and Q

($x = \frac{a+\sqrt{a^2-4}}{2}$) at which $\frac{d^2y}{dx^2} = 0$

Consider (**), when $x < a - 1$, $\frac{d^2y}{dx^2} < 0$; when $x > a - 1$, $\frac{d^2y}{dx^2} > 0$.

$f(x)$ changes from concave to convex, $f(a - 1)$ is a point of inflexion.