The following notes try to show that for $0 < p < 1$, $\lim_{n\to\infty}\frac{1}{n^p} = 0$.

Step 1: $\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)=\infty$

First, we prove by mathematical induction that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^m}\geq 1+\frac{m}{2}\quad \forall m\in\mathbf{N}$$

$m = 1$, LHS $=1+\frac{1}{2}$; RHS $=1+\frac{1}{2}$. $\therefore$ LHS $\geq$ RHS

It is true for $m = 1$.

Suppose that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^k}\geq 1+\frac{k}{2}$ for some $k\in\mathbf{N}$,

$m = k + 1$, LHS $=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^k}+\underbrace{\frac{1}{2^k+1}+\cdots+\frac{1}{2^{k+1}}}_{2^k\text{ terms}}$

$$\geq 1+\frac{k}{2}+\underbrace{\frac{1}{2^{k+1}}+\frac{1}{2^{k+1}}+\cdots+\frac{1}{2^{k+1}}}_{2^k\text{ equal terms}}$$

$$\geq 1+\frac{k}{2}+\frac{2^k}{2^{k+1}}$$

$$=1+\frac{k}{2}+\frac{1}{2}$$

$$=1+\frac{k+1}{2}=\text{RHS}$$

It is also true for $m = k + 1$ if it is true for $m = k$.

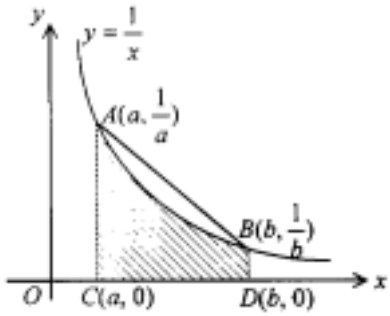
By the principle of mathematical induction, $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^m}\geq 1+\frac{m}{2}\quad \forall m\in\mathbf{N}$

Now for $n > 1$, $\exists m\in\mathbf{N}$ such that $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\geq 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^m}\geq 1+\frac{m}{2}$

$\because \lim_{n\to\infty}\left(1+\frac{m}{2}\right)=\infty$

$\therefore \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)=\infty$

Step 2: $\lim_{n\to\infty}\log n=\infty$



Proof: Consider the curve $y=\frac{1}{x}$ for $1\leq x\leq n$.

Area under the curve $\geq$ sum of areas of rectangles width = 1 under the curve

$$\int_1^n\frac{1}{x}dx\geq\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

$$\ln x\Big|_1^n\geq\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

$$\ln n\geq\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

$$\therefore \lim_{n\to\infty}\log n\geq\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-1=\infty$$

Step 3: For $0 < p < 1$, $1 \le x \le n$

$$x^p \ge 1$$

$$\frac{x^p}{x} \ge \frac{1}{x}$$

$$\frac{1}{x^{1-p}} \ge \frac{1}{x} \text{ for } 1 \le x \le n$$

$$\int_1^n \frac{1}{x^{1-p}}\,dx \ge \int_1^n \frac{1}{x}\,dx$$

$$\left.\frac{x^p}{p}\right|_1^n \ge \ln x\Big|_1^n$$

$$\frac{n^p - 1}{p} \ge \ln n$$

$$n^p \ge p \ln n + 1$$

$$\because \lim_{n\to\infty}\left(p \ln n + 1\right) = \infty$$

$$\therefore \lim_{n\to\infty} n^p = \infty$$

$$\therefore \lim_{n\to\infty} \frac{1}{n^p} = \frac{1}{\infty} = 0$$