

Aids to Advanced Level Pure Mathematics Part 1 p.189 Q29 (Problem on Integration)

(a) Two sequences $\{a_n\}$ and $\{b_n\}$ of positive numbers are related as follows:

$$b_1 > a_1, a_{n+1} = \sqrt{a_n b_n}, b_{n+1} = \frac{a_n + b_n}{2} \quad (n \geq 1)$$

Prove that both sequences converge to the same limit ℓ , say.

(b) If $b > a > 0$, show that the function f given by $u = f(t) = \frac{ab+t^2}{2t}$, $t \in (0, \infty)$ is strictly decreasing on the interval $(0, \sqrt{ab}]$ and strictly increasing on the interval $[\sqrt{ab}, \infty)$. Hence find an explicit expression for each of the inverse function of f .

(c) Let $I(a, b) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}$, $0 < a < b$.

By making the substitution $t = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}$ and $u = \frac{ab+t^2}{2t}$, show that

$$I(a, b) = I\left(\sqrt{ab}, \frac{a+b}{2}\right).$$

(d) Let $I(a_n, b_n) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a_n^2 \cos^2 \theta + b_n^2 \sin^2 \theta}}$ where the sequences $\{a_n\}$ and $\{b_n\}$ are given by

$$a_1 = \sqrt{ab}, b_1 = \frac{a_1 + b_1}{2}, a_{n+1} = \sqrt{a_n b_n}, b_{n+1} = \frac{a_n + b_n}{2} \quad \text{for all } n \geq 1.$$

Using (a), show that $I(a, b) = I(a_n, b_n)$.

(e) Using (a) and (d), show that $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} = \frac{\pi}{2\ell}$.

Solution

$$\begin{aligned} (a) \quad b_{n+1} - a_{n+1} &= \frac{a_n + b_n}{2} - \sqrt{a_n b_n} \\ &\geq \sqrt{a_n b_n} - \sqrt{a_n b_n} = 0 \end{aligned}$$

$$\Rightarrow b_{n+1} \geq a_{n+1}$$

$$\Rightarrow b_n \geq a_n \quad \forall n \geq 1$$

$$b_n - b_{n+1} = b_n - \frac{a_n + b_n}{2}$$

$$= \frac{b_n - a_n}{2} \geq 0$$

$$\Rightarrow b_n \geq b_{n+1}$$

$$a_{n+1} - a_n = \sqrt{a_n b_n} - a_n$$

$$= \sqrt{a_n} (\sqrt{b_n} - \sqrt{a_n})$$

$$= \sqrt{a_n} \frac{b_n - a_n}{\sqrt{b_n} + \sqrt{a_n}} \geq 0$$

$$\Rightarrow a_{n+1} \geq a_n$$

$$\therefore b_1 > \dots > b_n > b_{n+1} > a_{n+1} > a_n > \dots > a_1.$$

The sequences $\{a_n\}$ is monotonic increasing and bounded above by b_1 and $\{b_n\}$ is monotonic decreasing and bounded below by a_1 .

\therefore Both sequences converge.

$$\text{Let } \lim_{n \rightarrow \infty} a_n = k, \quad \lim_{n \rightarrow \infty} b_n = m$$

$$b_{n+1} = \frac{a_n + b_n}{2} \Rightarrow \lim_{n \rightarrow \infty} b_{n+1} = \lim_{n \rightarrow \infty} \frac{a_n + b_n}{2}$$

$$m = \frac{k+m}{2} \Rightarrow k = m, \text{ let the common limit be } \ell.$$

$$(b) \quad f(t) = \frac{ab+t^2}{2t} = \frac{ab}{2t} + \frac{t}{2}$$

$$f'(t) = -\frac{ab}{2t^2} + \frac{1}{2}; \text{ let } f'(t) = 0, t^2 = ab, t = \sqrt{ab}$$

$$f'(t) = \frac{1}{2t^2} (t + \sqrt{ab})(t - \sqrt{ab})$$

If $t \in (0, \sqrt{ab}]$, $f'(t) < 0 \Rightarrow f(t)$ is strictly decreasing.

If $t \in [\sqrt{ab}, \infty)$, $f'(t) > 0 \Rightarrow f(t)$ is strictly increasing.

$$u = \frac{ab+t^2}{2t}$$

$$t^2 - 2ut + ab = 0$$

$$t = u \pm \sqrt{u^2 - ab}$$

$$(c) \quad I(a, b) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}, \quad 0 < a < b.$$

$$t = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \Rightarrow t^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

$$2t dt = (-2a^2 \cos \theta \sin \theta + 2b^2 \sin \theta \cos \theta) d\theta \Rightarrow d\theta = \frac{t dt}{(b^2 - a^2) \sin \theta \cos \theta}$$

$$\theta = 0, t = a; \theta = \frac{\pi}{2}, t = b.$$

$$t^2 - a^2 = b^2 \sin^2 \theta - a^2 \sin^2 \theta = (b^2 - a^2) \sin^2 \theta \Rightarrow \sin \theta = \frac{\sqrt{t^2 - a^2}}{\sqrt{b^2 - a^2}}$$

$$b^2 - t^2 = b^2 \cos^2 \theta - a^2 \cos^2 \theta = (b^2 - a^2) \cos^2 \theta \Rightarrow \cos \theta = \frac{\sqrt{b^2 - t^2}}{\sqrt{b^2 - a^2}}$$

$$I(a, b) = \int_a^b \frac{t dt}{t(b^2 - a^2) \sqrt{\frac{t^2 - a^2}{b^2 - a^2}} \sqrt{\frac{b^2 - t^2}{b^2 - a^2}}} = \int_a^b \frac{dt}{\sqrt{(b^2 - t^2)(t^2 - a^2)}} \dots\dots\dots (*)$$

$$I(a, b) = \int_a^{\sqrt{ab}} \frac{dt}{\sqrt{(b^2 - t^2)(t^2 - a^2)}} + \int_{\sqrt{ab}}^b \frac{dt}{\sqrt{(b^2 - t^2)(t^2 - a^2)}}$$

$$u = \frac{ab+t^2}{2t}, t = u - \sqrt{u^2 - ab} \text{ for } a \leq t \leq \sqrt{ab}; t = u + \sqrt{u^2 - ab} \text{ for } \sqrt{ab} \leq t \leq b.$$

$$t = a, u = \frac{a+b}{2}; t = \sqrt{ab}, u = \sqrt{ab}; t = b, u = \frac{a+b}{2}.$$

$$\text{When } t = u - \sqrt{u^2 - ab}, dt = du - \frac{udu}{\sqrt{u^2 - ab}} = -\frac{(u - \sqrt{u^2 - ab}) du}{\sqrt{u^2 - ab}} = -\frac{t du}{\sqrt{u^2 - ab}};$$

$$\text{When } t = u + \sqrt{u^2 - ab}, dt = du + \frac{udu}{\sqrt{u^2 - ab}} = \frac{(u + \sqrt{u^2 - ab}) du}{\sqrt{u^2 - ab}} = \frac{t du}{\sqrt{u^2 - ab}}.$$

$$(b^2 - t^2)(t^2 - a^2) = b^2 t^2 + a^2 t^2 - a^2 b^2 - t^4$$

$$\begin{aligned}
&= t^2 \left[a^2 + b^2 - 4 \left(\frac{a^2 b^2 + t^4}{4t^2} \right) \right] \\
&= t^2 \left[a^2 + b^2 + 2ab - 4 \left(\frac{ab + t^2}{2t} \right)^2 \right] \\
&= t^2 [(a+b)^2 - 4u^2] \\
I(a, b) &= \int_a^{\sqrt{ab}} \frac{dt}{\sqrt{(b^2 - t^2)(t^2 - a^2)}} + \int_{\sqrt{ab}}^b \frac{dt}{\sqrt{(b^2 - t^2)(t^2 - a^2)}} \\
&= \int_{\frac{a+b}{2}}^{\sqrt{ab}} \frac{tdu}{\sqrt{u^2 - ab} \sqrt{t^2 [(a+b)^2 - 4u^2]}} + \int_{\sqrt{ab}}^{\frac{a+b}{2}} \frac{tdu}{\sqrt{u^2 - ab} \sqrt{t^2 [(a+b)^2 - 4u^2]}} \\
&= \int_{\frac{a+b}{2}}^{\sqrt{ab}} \frac{du}{\sqrt{u^2 - ab} \sqrt{[(a+b)^2 - 4u^2]}} + \int_{\sqrt{ab}}^{\frac{a+b}{2}} \frac{du}{\sqrt{u^2 - ab} \sqrt{[(a+b)^2 - 4u^2]}} \\
&= 2 \int_{\frac{a+b}{2}}^{\sqrt{ab}} \frac{du}{\sqrt{u^2 - ab} \sqrt{4 \left[\left(\frac{a+b}{2} \right)^2 - u^2 \right]}} \\
&= \int_{\frac{a+b}{2}}^{\sqrt{ab}} \frac{du}{\sqrt{\left(\frac{a+b}{2} \right)^2 - u^2} \sqrt{u^2 - (\sqrt{ab})^2}} \\
&= I \left(\sqrt{ab}, \frac{a+b}{2} \right) \text{ by the formula (*)}.
\end{aligned}$$

$$(d) \quad I(a_n, b_n) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a_n^2 \cos^2 \theta + b_n^2 \sin^2 \theta}}; \quad a_1 = \sqrt{ab}, \quad b_1 = \frac{a_1 + b_1}{2}, \quad a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{a_n + b_n}{2}, \quad n \geq 1.$$

To show $I(a, b) = I(a_n, b_n)$ by mathematical induction on n .

$$\begin{aligned}
\text{By (c), } I(a, b) &= I \left(\sqrt{ab}, \frac{a+b}{2} \right) \\
&= I(a_1, b_1)
\end{aligned}$$

The formula is true for $n = 1$.

Suppose $I(a, b) = I(a_k, b_k)$ for some positive integer k .

Use the result of (c) and replace a by a_k , b by b_k .

$$\begin{aligned}
I(a_k, b_k) &= I \left(\sqrt{a_k b_k}, \frac{a_k + b_k}{2} \right) \\
&= I(a_{k+1}, b_{k+1}) \text{ by the definition.}
\end{aligned}$$

$\therefore I(a, b) = I(a_{k+1}, b_{k+1})$ by induction assumption.

The formula is also true for $n = k + 1$ if it is true for $n = k$.

By the principle of mathematical induction, $I(a, b) = I(a_n, b_n)$ for all positive integer n .

$$\begin{aligned}
(e) \quad \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} &= I(a, b) = I(a_n, b_n) = I \left(\lim_{n \rightarrow \infty} a_n, \lim_{n \rightarrow \infty} b_n \right) = I(\ell, \ell) \\
&= \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\ell^2 \cos^2 \theta + \ell^2 \sin^2 \theta}} = \frac{1}{\ell} \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2\ell}.
\end{aligned}$$