This is a hard problem on integration. It involves the use of substitutions.

Consider the following improper integral:

$$\int_a^b \left[(x-a)(b-x)^2\right]^{-\frac{1}{3}} dx\,,\ a < b$$

(a) By using substitution $x = a\cos^2\theta + b\sin^2\theta$, prove that is equal to

$$2\int_0^{\frac{\pi}{2}} \tan^{\frac{1}{3}}\theta d\theta$$

(b) By using the substitution $t = \tan^{\frac{1}{3}}\theta$, prove that the integral is equal to

$$6\int_0^{\infty} \frac{t^3}{1+t^6} dt$$, hence prove that is also equal to $$3\int_0^{\infty} \frac{u}{1+u^3} du$$

(c) Prove that $\int_0^{\infty} \frac{v}{1+v^3} dv = \int_0^{\infty} \frac{1}{1+v^3} dv$

Hence find the integral in part (a).

(d) Use the substitution $x = \frac{a+bu}{1+u}$ to do part (a) again.

Solution

(a) $x = a\cos^2\theta + b\sin^2\theta$; $x = a$, $\theta = 0$; $x = b$, $\theta = \frac{\pi}{2}$.

$x - a = (b-a)\sin^2\theta$; $b - x = (b-a)\cos^2\theta$; $dx = 2(b-a)\sin\theta\cos\theta\, d\theta$

$$\text{I} = \int_0^{\frac{\pi}{2}} \left[(b-a)\sin^2\theta (b-a)^2\cos^4\theta\right]^{-\frac{1}{3}} 2(b-a)\sin\theta\cos\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \tan^{\frac{1}{3}}\theta d\theta$$

(b) $t = \tan^{\frac{1}{3}}\theta$; $\theta = 0$, $t = 0$, $\theta \to \frac{\pi}{2}$, $t \to \infty$

$$dt = \frac{1}{3}\tan^{-\frac{2}{3}}\theta\sec^2\theta\, d\theta \Rightarrow d\theta = \frac{3t^2 dt}{1+t^6}$$

$$\text{I} = 2\int_0^{\infty} t\cdot\frac{3t^2 dt}{1+t^6} = 6\int_0^{\infty}\frac{t^3}{1+t^6} dt$$

Let $u = t^2$; $t = 0$, $u = 0$; $t \to \infty$, $u \to \infty$; $du = 2t\, dt$

$$\text{I} = 3\int_0^{\infty} \frac{t^2}{1+(t^2)^3} 2t dt = 3\int_0^{\infty} \frac{u}{1+u^3} du$$

(*c*) J = $\int_0^{\infty} \frac{v}{1+v^3}\,dv$. Let $w = \frac{1}{v}$; $v \to 0$, $w \to \infty$; $v \to \infty$, $w \to 0$; dv = $-\frac{1}{w^2}\,dw$

$$J = \int_{\infty}^{0} \frac{\frac{1}{w}}{1+\left(\frac{1}{w}\right)^3}\left(-\frac{1}{w^2}\right)dw.$$

$$J = \int_0^{\infty} \frac{1}{1+w^3}\,dw = \int_0^{\infty} \frac{1}{1+v^3}\,dv$$

$$I = 3J = 3\int_0^{\infty} \frac{1}{1+v^3}\,dv$$

$$I + I = 3\left(\int_0^{\infty} \frac{1}{1+v^3}\,dv + \int_0^{\infty} \frac{v}{1+v^3}\,dv\right)$$

$$2I = 3\int_0^{\infty} \frac{1+v}{1+v^3}\,dv$$

$$= 3\int_0^{\infty} \frac{1}{1-v+v^2}\,dv$$

$$= 3\int_0^{\infty} \frac{1}{\left(v-\frac{1}{2}\right)^2+\frac{3}{4}}\,dv$$

$$= 3\int_0^{\infty} \frac{1}{\frac{3}{4}\left[\left(\frac{v-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2+1\right]}\,d\left(v-\frac{1}{2}\right)$$

$$= 2\sqrt{3}\tan^{-1}\left(\frac{2v-1}{\sqrt{3}}\right)\Bigg|_0^{\infty}$$

$$= 2\sqrt{3}\left(\frac{\pi}{2}+\frac{\pi}{6}\right) = \frac{4\sqrt{3}\pi}{3}$$

$$\int_a^b \left[(x-a)(b-x)^2\right]^{-\frac{1}{3}}dx = I = \frac{2\sqrt{3}\pi}{3}$$

(*d*) $x = \frac{a+bu}{1+u}$; $x = a$, $u = 0$; $x \to b$, $u \to \infty$

$x - a = \frac{(b-a)u}{1+u}$, $b - x = \frac{b-a}{1+u}$, dx = $\frac{(b-a)du}{(1+u)^2}$

$$I = \int_0^\infty \left[\frac{(b-a)u}{1+u} \cdot \left(\frac{b-a}{1+u} \right)^2 \right]^{-\frac{1}{3}} \frac{(b-a)du}{(1+u)^2}$$

$$= \int_0^\infty \frac{u^{-\frac{1}{3}}}{1+u} du$$

$$= -\int_\infty^0 \frac{3v^3}{1+v^{-3}} dv \, ; \; v = u^{-\frac{1}{3}}$$

$$= \int_0^\infty \frac{3}{1+v^3} dv$$

= 3J, same as the answer in part (*c*)