Indefinite Integral Examples
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1. $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

Let $t=\sqrt[6]{x}$, then $x = t^6$

$dx = 6t^5\,dt$, $\sqrt{x}=t^3$, $\sqrt[3]{x}=t^2$

$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = \int \frac{6t^5 dt}{t^3+t^2} = 6\int \frac{t^3}{1+t}dt$$

$$=6\int\left(t^2-t+1-\frac{1}{1+t}\right)dt$$

$$=6\left[\frac{t^3}{3}-\frac{t^2}{2}+t-\ln|1+t|\right]+c$$

$$=2t^3-3t^2+6t-6\ln|1+t|+c$$

$$=2\sqrt{x}-3\cdot\sqrt[3]{x}+6\sqrt[6]{x}-6\ln\left(1+\sqrt[6]{x}\right)+c$$, where c is a constant.

2. Calculus Exercise 3.8 p.103 Question 10

I = $\int \frac{dx}{(x^2+b^2)\sqrt{x^2+a^2}}$ $(a > b > 0)$

Let $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$, $x^2 + b^2 = a^2\tan^2\theta + b^2$

I = $\int \frac{a\sec^2\theta d\theta}{(a^2\tan^2\theta + b^2)a\sec\theta}$

$= \int \frac{\sec\theta d\theta}{(a^2\tan^2\theta + b^2)}$

$= \int \frac{\cos\theta d\theta}{(a^2\sin^2\theta + b^2\cos^2\theta)}$

$= \int \frac{d\sin\theta}{(a^2 - b^2)\sin^2\theta + b^2}$

$= \frac{1}{\sqrt{a^2-b^2}}\int \frac{d\sqrt{a^2-b^2}\sin\theta}{(a^2 - b^2)\sin^2\theta + b^2}$

$= \frac{1}{\sqrt{a^2-b^2}}\int \frac{du}{u^2+b^2}$, $u = \sqrt{a^2-b^2}\sin\theta$

$= \frac{1}{b\sqrt{a^2-b^2}}\tan^{-1}\frac{u}{b}$ + C, by formula 6.4 p.82

$= \frac{1}{b\sqrt{a^2-b^2}}\cos^{-1}\frac{b}{\sqrt{u^2+b^2}}$ + C

$= \frac{1}{b\sqrt{a^2-b^2}}\left(\frac{\pi}{2} - \sin^{-1}\frac{b}{\sqrt{(a^2-b^2)\sin^2\theta + b^2}}\right)$ + C, since $\sin^{-1}\alpha + \cos^{-1}\alpha = \frac{\pi}{2}$

$= -\frac{1}{b\sqrt{a^2-b^2}}\sin^{-1}\frac{b}{\sqrt{\frac{(a^2-b^2)x^2 + b^2(x^2+a^2)}{x^2+a^2}}}$ + C'

$= -\frac{1}{b\sqrt{a^2-b^2}}\sin^{-1}\frac{b}{a}\sqrt{\frac{x^2+a^2}{x^2+b^2}}$ + C'

3. Solution to Exercise 3.5 Question 16 p.95

$$\int \frac{x^3 + 8x - 2}{(x^2 + 4x + 9)^2} dx$$

Let $\dfrac{x^3 + 8x - 2}{(x^2 + 4x + 9)^2} \equiv \dfrac{Ax + B}{x^2 + 4x + 9} + \dfrac{Cx + D}{(x^2 + 4x + 9)^2}$

$x^3 + 8x - 2 \equiv (Ax + B)(x^2 + 4x + 9) + Cx + D$

Compare coefficients,

x^3: $A = 1$

x^2: $B + 4A = 0 \Rightarrow B = -4$

x: $9A + 4B + C = 8 \Rightarrow C = 15$

1: $9B + D = -2 \Rightarrow D = 34$

$$\int \frac{x^3 + 8x - 2}{(x^2 + 4x + 9)^2} dx$$

$$= \int \frac{x - 4}{x^2 + 4x + 9} dx + \int \frac{15x + 34}{(x^2 + 4x + 9)^2} dx$$

$$= \frac{1}{2}\int \frac{2x + 4 - 12}{x^2 + 4x + 9} dx + \frac{1}{2}\int \frac{30x + 60 + 8}{(x^2 + 4x + 9)^2} dx$$

$$= \frac{1}{2}\int \frac{d(x^2 + 4x + 9)}{x^2 + 4x + 9} - 6\int \frac{1}{x^2 + 4x + 9} dx + \frac{15}{2}\int \frac{d(x^2 + 4x + 9)}{(x^2 + 4x + 9)^2} + 4\int \frac{dx}{(x^2 + 4x + 9)^2}$$

$$= \frac{1}{2}\ln\left|x^2 + 4x + 9\right| - 6\int \frac{dx}{(x + 2)^2 + 5} - \frac{15}{2(x^2 + 4x + 9)} + 4\int \frac{dx}{\left[(x + 2)^2 + 5\right]^2}$$

$$= \frac{1}{2}\ln\left|x^2 + 4x + 9\right| - \frac{6}{\sqrt{5}}\tan^{-1}\frac{x + 2}{\sqrt{5}} - \frac{15}{2(x^2 + 4x + 9)} + 4\int \frac{dx}{\left[(x + 2)^2 + 5\right]^2}$$

(Let $x + 2 = \sqrt{5}\tan\theta$)

$$= \frac{1}{2}\ln\left|x^2 + 4x + 9\right| - \frac{6}{\sqrt{5}}\tan^{-1}\frac{x + 2}{\sqrt{5}} - \frac{15}{2(x^2 + 4x + 9)} + 4\int \frac{\sqrt{5}\sec^2\theta d\theta}{\left[\sqrt{5}\sec\theta\right]^4}$$

$$= \frac{1}{2}\ln\left|x^2 + 4x + 9\right| - \frac{6}{\sqrt{5}}\tan^{-1}\frac{x + 2}{\sqrt{5}} - \frac{15}{2(x^2 + 4x + 9)} + \frac{4\sqrt{5}}{25}\int \cos^2\theta d\theta$$

$$= \frac{1}{2}\ln\left|x^2 + 4x + 9\right| - \frac{6}{\sqrt{5}}\tan^{-1}\frac{x + 2}{\sqrt{5}} - \frac{15}{2(x^2 + 4x + 9)} + \frac{2\sqrt{5}}{25}\int 1 + \cos 2\theta d\theta$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{6}{\sqrt{5}}\tan^{-1}\frac{x+2}{\sqrt{5}} - \frac{15}{2\left(x^2+4x+9\right)} + \frac{2\sqrt{5}}{25}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{6}{\sqrt{5}}\tan^{-1}\frac{x+2}{\sqrt{5}} - \frac{15}{2\left(x^2+4x+9\right)} + \frac{2\sqrt{5}}{25}\left(\tan^{-1}\frac{x+2}{\sqrt{5}} + \sin\theta\cos\theta\right) + C$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{28\sqrt{5}}{25}\tan^{-1}\frac{x+2}{\sqrt{5}} - \frac{15}{2\left(x^2+4x+9\right)} + \frac{2\sqrt{5}}{25}\cdot\frac{\sqrt{5}(x+2)}{(x+2)^2+5} + C$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{28\sqrt{5}}{25}\tan^{-1}\frac{x+2}{\sqrt{5}} - \frac{15}{2\left(x^2+4x+9\right)} + \frac{2}{5}\frac{(x+2)}{\left(x^2+4x+9\right)} + C$$

$$= \frac{1}{2}\ln\left|x^2+4x+9\right| - \frac{28\sqrt{5}}{25}\tan^{-1}\frac{x+2}{\sqrt{5}} + \frac{4x-67}{10\left(x^2+4x+9\right)} + C$$