

The following two questions guide you to find $\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$.

Q 26 (a) Use the reduction formula for $\int \sin^n x dx$ to show that

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx.$$

(b) Now show that

$$\int_0^{\pi/2} \sin^{2n+1} x dx = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1},$$

$$\int_0^{\pi/2} \sin^{2n} x dx = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n},$$

and conclude that

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx}.$$

(c) Using the fact that $0 < \sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x$ for $0 < x < \frac{\pi}{2}$,

$$\text{Show that } 1 < \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx} < 1 + \frac{1}{2n}$$

$$\text{Hence show that } \lim_{n \rightarrow \infty} \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right) = \frac{\pi}{2}$$

(d) Show also that $\lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{n}} \cdot \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right] = \sqrt{\pi}$

Q27 (a) Show that $\int_0^1 (1-x^2)^n dx = \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n+1}$,

$$\text{and } \int_0^\infty \frac{1}{(1+x^2)^n} dx = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-3}{2n-2}.$$

(b) Prove, using the derivative, that

$$1 - x^2 \leq e^{-x^2} \text{ for } 0 \leq x \leq 1.$$

$$\text{and } e^{-x^2} \leq \frac{1}{1+x^2} \text{ for } 0 \leq x.$$

(c) Integrate the n^{th} powers of these inequalities from 0 to 1 and from 0 to ∞ , respectively.

Then use the substitution $y = \sqrt{n}x$ to show that

$$\sqrt{n} \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n+1} \leq \int_0^{\sqrt{n}} e^{-y^2} dy \leq \int_0^\infty e^{-y^2} dy \leq \frac{\pi}{2} \cdot \sqrt{n} \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-3}{2n-2}.$$

(d) Use Problem 26(c) to show that $\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$.

$$\begin{aligned}
\text{Q26 (a)} \quad \int_0^{\pi/2} \sin^n x dx &= \int_0^{\pi/2} \sin^{n-1} x \sin x dx \\
&= -\int_0^{\pi/2} \sin^{n-1} x d \cos x \\
&= -\sin^{n-1} x \cos x \Big|_0^{\pi/2} + \int_0^{\pi/2} \cos x d \sin^{n-1} x \\
&= (n-1) \int_0^{\pi/2} \cos^2 x \sin^{n-2} x dx \\
&= (n-1) \int_0^{\pi/2} (1 - \sin^2 x) \sin^{n-2} x dx \\
&= (n-1) \left[\int_0^{\pi/2} \sin^{n-2} x dx - \int_0^{\pi/2} \sin^n x dx \right]
\end{aligned}$$

$$\int_0^{\pi/2} \sin^n x dx = (n-1) \int_0^{\pi/2} \sin^{n-2} x dx - (n-1) \int_0^{\pi/2} \sin^n x dx$$

$$n \int_0^{\pi/2} \sin^n x dx = (n-1) \int_0^{\pi/2} \sin^{n-2} x dx$$

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x dx$$

$$\begin{aligned}
\text{(b)} \quad \int_0^{\pi/2} \sin^{2n+1} x dx &= \frac{2n}{2n+1} \int_0^{\pi/2} \sin^{2n-1} x dx \\
&= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \int_0^{\pi/2} \sin^{2n-3} x dx \\
&= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdot \frac{2n-4}{2n-3} \cdots \frac{2}{3} \int_0^{\pi/2} \sin x dx \\
&= \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1}
\end{aligned}$$

$$\begin{aligned}
\int_0^{\pi/2} \sin^{2n} x dx &= \frac{2n-1}{2n} \int_0^{\pi/2} \sin^{2n-2} x dx \\
&= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \int_0^{\pi/2} \sin^{2n-4} x dx \\
&= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \int_0^{\pi/2} \sin^0 x dx \\
&= \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n}
\end{aligned}$$

$$\frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx} = \frac{\frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n}}{\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1}}$$

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx}$$

$$\text{(c)} \quad 0 < \sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x \text{ for } 0 < x < \frac{\pi}{2}$$

$$0 < \int_0^{\pi/2} \sin^{2n+1} x dx < \int_0^{\pi/2} \sin^{2n} x dx < \int_0^{\pi/2} \sin^{2n-1} x dx$$

$$1 < \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx} < \frac{\int_0^{\pi/2} \sin^{2n-1} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx}$$

$$1 < \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx} < \frac{\int_0^{\pi/2} \sin^{2n-1} x dx}{\frac{2n}{2n+1} \int_0^{\pi/2} \sin^{2n-1} x dx}, \text{ by (a)}$$

$$1 < \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx} < 1 + \frac{1}{2n}$$

$$\lim_{n \rightarrow \infty} 1 \leq \lim_{n \rightarrow \infty} \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx} \leq 1 + \lim_{n \rightarrow \infty} \frac{1}{2n} = 1$$

By squeezing principle, $\lim_{n \rightarrow \infty} \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx} = 1$.

From (b), $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx}$

$$\frac{\pi}{2} = \lim_{n \rightarrow \infty} \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right) \lim_{n \rightarrow \infty} \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx}$$

$$\therefore \lim_{n \rightarrow \infty} \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right) = \frac{\pi}{2}$$

(d) By (b), $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx}$

$$\sqrt{\frac{\pi}{2}} = \lim_{n \rightarrow \infty} \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdots \frac{2n}{2n-1} \cdot \frac{1}{\sqrt{2n+1}} \sqrt{\lim_{n \rightarrow \infty} \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx}}$$

$$\sqrt{\pi} = \lim_{n \rightarrow \infty} \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdots \frac{2n}{2n-1} \cdot \frac{\sqrt{2}}{\sqrt{2n+1}} \quad (\because \lim_{n \rightarrow \infty} \frac{\int_0^{\pi/2} \sin^{2n} x dx}{\int_0^{\pi/2} \sin^{2n+1} x dx} = 1)$$

$$= \lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{n}} \cdot \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \cdot \sqrt{\frac{2}{2 + \frac{1}{n}}} \right]$$

$$\lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{n}} \cdot \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right] = \sqrt{\pi}$$

Q27 (a) $\int_0^1 (1-x^2)^n dx$, let $x = \cos \theta$, $dx = -\sin \theta d\theta$; $x = 0$, $\theta = \frac{\pi}{2}$; $x = 1$, $\theta = 0$.

$$\begin{aligned} \int_0^1 (1-x^2)^n dx &= -\int_{\pi/2}^0 (1-\cos^2 \theta)^n \sin \theta d\theta \\ &= \int_0^{\pi/2} \sin^{2n+1} \theta d\theta \\ &= \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n+1} \text{ by Q26(b)} \end{aligned}$$

$\int_0^\infty \frac{1}{(1+x^2)^n} dx$, let $x = \cot \theta$, $dx = -\csc^2 \theta d\theta$; $x \rightarrow 0^+$, $\theta \rightarrow \frac{\pi}{2}^-$; $x \rightarrow \infty$, $\theta \rightarrow 0^+$.

$$\begin{aligned} \int_0^\infty \frac{1}{(1+x^2)^n} dx &= -\int_{\pi/2}^0 \frac{1}{(1+\cot^2 \theta)^n} \csc^2 \theta d\theta \\ &= \int_0^{\pi/2} \frac{1}{\csc^{2n} \theta} \csc^2 \theta d\theta \\ &= \int_0^{\pi/2} \sin^{2n-2} \theta d\theta \\ &= \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-3}{2n-2} \text{ by Q26(b)} \end{aligned}$$

(b) Let $f(x) = e^{-x^2} + x^2 - 1$

$$f'(x) = -2xe^{-x^2} + 2x$$

$$= 2x(1 - e^{-x^2})$$

$$f'(x) = 0 \Rightarrow x = 0$$

For $0 < x$, $e^{x^2} > e^0 = 1$

$$\Rightarrow e^{-x^2} < 1$$

$$\Rightarrow 1 - e^{-x^2} > 0$$

$$\Rightarrow 2x(1 - e^{-x^2}) > 0$$

$$\Rightarrow f'(x) > 0$$

$\therefore f'(x)$ is strictly increasing for $x > 0$

$$f(x) > f(0)$$

$$e^{-x^2} + x^2 - 1 > 0$$

$$1 - x^2 \leq e^{-x^2} \text{ for } 0 \leq x \leq 1.$$

Let $g(x) = 1 + x^2 - e^{x^2}$

$$g'(x) = 2x - 2xe^{x^2}$$

$$= 2x(1 - e^{x^2})$$

$$g'(0) = 0$$

For $x > 0$, $e^{x^2} > e^0 = 1$

$$1 - e^{x^2} < 0 \Rightarrow g'(x) < 0$$

$\therefore g(x)$ is strictly decreasing for $x \geq 0$

$$g(x) < g(0) \text{ for } x > 0$$

$$1 + x^2 < e^{x^2}$$

$$e^{-x^2} < \frac{1}{1+x^2}$$

$$e^{-x^2} \leq \frac{1}{1+x^2} \text{ for } 0 \leq x.$$

(c) By (b), $1 - x^2 \leq e^{-x^2}$ for $0 \leq x \leq 1$ and $e^{-x^2} \leq \frac{1}{1+x^2}$ for $x \geq 0$.

$$\Rightarrow (1 - x^2)^n \leq e^{-nx^2} \text{ for } 0 \leq x \leq 1 \text{ and } e^{-nx^2} \leq \frac{1}{(1+x^2)^n} \text{ for } x \geq 0.$$

$$\Rightarrow \int_0^1 (1 - x^2)^n dx \leq \int_0^1 e^{-nx^2} dx \text{ and } \int_0^\infty e^{-nx^2} dx \leq \int_0^\infty \frac{1}{(1+x^2)^n} dx \dots\dots\dots (*)$$

$$y = \sqrt{n}x, dx = \frac{dy}{\sqrt{n}}; x = 0, y = 0; x = 1, y = \sqrt{n}; x \rightarrow \infty, y \rightarrow \infty.$$

$$\int_0^1 e^{-nx^2} dx = \int_0^{\sqrt{n}} e^{-y^2} \frac{dy}{\sqrt{n}} \text{ and } \int_0^\infty e^{-nx^2} dx = \int_0^\infty e^{-y^2} \frac{dy}{\sqrt{n}}$$

$$= \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-y^2} dy \text{ and } = \frac{1}{\sqrt{n}} \int_0^\infty e^{-y^2} dy$$

$$\text{By } (*) \int_0^1 (1 - x^2)^n dx \leq \int_0^1 e^{-nx^2} dx \leq \int_0^\infty e^{-nx^2} dx \leq \int_0^\infty \frac{1}{(1+x^2)^n} dx$$

$$\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \dots \frac{2n}{2n+1} \leq \frac{1}{\sqrt{n}} \int_0^{\sqrt{n}} e^{-y^2} dy \leq \frac{1}{\sqrt{n}} \int_0^\infty e^{-y^2} dy \leq \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \dots \frac{2n-3}{2n-2}$$

$$\sqrt{n} \frac{2}{3} \cdot \frac{4}{5} \dots \frac{2n}{2n+1} \leq \int_0^{\sqrt{n}} e^{-y^2} dy \leq \int_0^\infty e^{-y^2} dy \leq \frac{\pi}{2} \cdot \sqrt{n} \frac{1}{2} \cdot \frac{3}{4} \dots \frac{2n-3}{2n-2} \dots\dots\dots (**)$$

(d) Use Problem 26(c) to show that $\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$.

(d) Take limit as $n \rightarrow \infty$ in (**)

$$\lim_{n \rightarrow \infty} \left(\sqrt{n} \frac{2}{3} \cdot \frac{4}{5} \dots \frac{2n}{2n+1} \right) \leq \int_0^\infty e^{-y^2} dy \leq \frac{\pi}{2} \cdot \lim_{n \rightarrow \infty} \left(\sqrt{n} \frac{1}{2} \cdot \frac{3}{4} \dots \frac{2n-3}{2n-2} \right)$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n}} \cdot \frac{2 \cdot 4 \cdot 6 \dots 2n}{1 \cdot 3 \cdot 5 \dots (2n-1)} \right) \lim_{n \rightarrow \infty} \frac{n}{2n+1} \leq \int_0^\infty e^{-y^2} dy \leq \frac{\pi}{2} \cdot \lim_{n \rightarrow \infty} \left(\frac{\sqrt{n}}{1} \cdot \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} \right) \lim_{n \rightarrow \infty} \frac{2n}{2n-3}$$

$$\frac{\sqrt{\pi}}{2} \leq \int_0^\infty e^{-y^2} dy \leq \frac{\pi}{2} \cdot \frac{1}{\sqrt{\pi}} = \frac{\sqrt{\pi}}{2}; \text{ by the result of 26(d)}$$

$$\text{By squeezing principle, } \int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}.$$