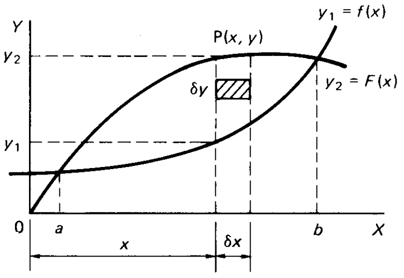
To evaluate $I=\int_{-\infty}^{\infty} e^{-x^2}\,dx$

$$I\times I=\int_{-\infty}^{\infty} e^{-x^2}\,dx\times\int_{-\infty}^{\infty} e^{-x^2}\,dx \;=\int_{-\infty}^{\infty} e^{-x^2}\,dx\times\int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$==\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dxdy \;=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)}dxdy$$

(a) *Areas of plane figures*



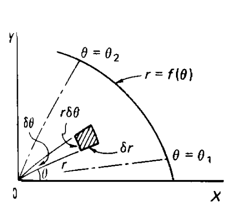
Area of element $\delta A = \delta x \delta y$

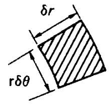
Area of strip $\simeq \sum_{y=y_1}^{y=y_2} \delta x\,\delta y$

Area of all such strips $\simeq \sum_{x=a}^{x=b}\left\{\sum_{y=y_1}^{y=y_2} \delta x\,\delta y\right\}$

If $\delta x \to 0$ and $\delta y \to 0$, $\quad A=\int_a^b\int_{y_1}^{y_2} \mathrm{d}y\,\mathrm{d}x$

(b) *Areas of plane figures bounded by a polar curve $r=f(\theta)$ and radius vectors at $\theta=\theta_1$ and $\theta=\theta_2$.*





Small arc of circle of radius r, subtending angle $\delta\theta$ at centre.

$\therefore$ Arc $= r\delta\theta$

Area of element $\delta A \simeq r\,\delta\theta\,\delta r$

Area of thin sector $\simeq \sum_{r=0}^{r=f(\theta)} r\,\delta\theta\,\delta r$

$\therefore$ Total area of all such sectors $\simeq \sum_{\theta=\theta_1}^{\theta=\theta_2}\left\{\sum_{r=0}^{r=f(\theta)} r\,\delta r\,\delta\theta\right\}$

$\therefore$ If $\delta r \to 0$ and $\delta\theta \to 0$ $\quad A=\int_{\theta_1}^{\theta_2}\int_0^{r=f(\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta$

Using the transformation $x = r\cos\theta$, $y = r\sin\theta$

then $dx\,dy = r\,dr\,d\theta$

when $x \to \infty$, $y \to \infty$ is equivalent to $r = 0$ to $r \to \infty$, $\theta = 0$ to $\theta = 2\pi$

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\left(r^2\cos^2\theta+r^2\sin^2\theta\right)} r\,dr\,d\theta = \int_0^{2\pi} d\theta\int_0^{\infty} e^{-r^2}\,r\,dr$$

$$= 2\pi\left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_0^{\infty} = \pi$$

$\therefore I = \sqrt{\pi}$

Exercise

Show that $\int_0^{\infty}\frac{x}{1+x^3}\,dx \;=\int_0^{\infty}\frac{1}{1+x^3}\,dx$, and hence find the common integral.

Numerical answer: $\dfrac{4\sqrt{3}\pi}{9}$