

# Taylor's Theorem

Theorem 1 (Lagrange form)

Suppose  $f^{(n-1)}(x)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  then  $\forall x \in (a, b) \exists c \in (a, x)$  s.t.

$$f(x) = f(a) + \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!} (x-a)^r + \frac{f^{(n)}(c)}{n!} (x-a)^n$$

Proof: Let  $g(t) = (t-a)^n$ ; Note:  $g^{(r)}(a) = 0$  for  $0 \leq r < n$  and  $g^{(n)}(t) = n!$

$$\text{Define } F(t) = f(t) + \sum_{r=1}^{n-1} \frac{f^{(r)}(t)}{r!} (x-t)^r ; G(t) = g(t) + \sum_{r=1}^{n-1} \frac{g^{(r)}(t)}{r!} (x-t)^r$$

$F, G$  satisfies the conditions of Cauchy's mean value theorem

then  $\exists c \in (a, x)$  s.t.  $[F(x) - F(a)]G'(c) = [G(x) - G(a)]F'(c)$

$$F(t) = f(t) + f'(t)(x-t) + \frac{f^{(2)}(t)}{2!} (x-t)^2 + \dots + \frac{f^{(n-1)}(t)}{(n-1)!} (x-t)^{n-1}$$

$$F'(t) = f'(t) - f'(t) + f''(t)(x-t) - f''(t)(x-t) + \frac{f^{(3)}(t)}{2!} (x-t)^2 - \frac{f^{(3)}(t)}{2!} (x-t)^2 + \dots + \frac{f^{(n)}(t)}{(n-1)!} (x-t)^{n-1}$$

$$= \frac{f^{(n)}(t)}{(n-1)!} (x-t)^{n-1}$$

$$G'(t) = \frac{g^{(n)}(t)}{(n-1)!} (x-t)^{n-1} = \frac{n!}{(n-1)!} (x-t)^{n-1} = n(x-t)^{n-1}$$

$$\left[ f(x) - f(a) - \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!} (x-a)^r \right] n(x-c)^{n-1} = [(x-a)^n - 0] \frac{f^{(n)}(c)}{(n-1)!} (x-c)^{n-1}$$

$$f(x) = f(a) + \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!} (x-a)^r + \frac{f^{(n)}(c)}{n!} (x-a)^n . \text{ Q.E.D.}$$

$f(a) + \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!} (x-a)^r$  is called the Taylor Polynomial of degree  $n-1$ .

$\frac{f^{(n)}(c)}{n!} (x-a)^n$  is the remainder of Lagrange form

Theorem 2 (Cauchy form)

Suppose  $f^{(n-1)}(x)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  then  $\forall x \in (a, b) \exists c \in (a, x)$  s.t.

$$f(x) = f(a) + \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!} (x-a)^r + \frac{f^{(n)}(c)}{(n-1)!} (x-a)(x-c)^{n-1}$$

Proof: Define  $F(t) = f(x) - f(t) - \sum_{r=1}^{n-1} \frac{f^{(r)}(t)}{r!} (x-t)^r$

$$F'(t) = -\frac{f^{(n)}(t)}{(n-1)!} (x-t)^{n-1}$$

By mean value theorem,  $\exists c \in (a, x)$  s.t.  $\frac{F(x) - F(a)}{x-a} = F'(c)$

$$\frac{0 - f(x) + f(a) + \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!} (x-a)^r}{x-a} = -\frac{f^{(n)}(c)}{(n-1)!} (x-c)^{n-1}$$

$$f(x) = f(a) + \sum_{r=1}^{n-1} \frac{f^{(r)}(a)}{r!} (x-a)^r + \frac{f^{(n)}(c)}{(n-1)!} (x-a)(x-c)^{n-1}. \text{ Q.E.D.}$$

Corollary (Maclaurin's Theorem)

Put  $a = 0$ , then  $\forall x \in (a, b) \exists c \in (a, x)$  s.t.  $f(x) = f(0) + \sum_{r=1}^{n-1} \frac{f^{(r)}(0)}{r!} x^r + \frac{f^{(n)}(c)}{n!} x^n$

Examples

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n-1}}{(2n-1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \dots + \frac{x^n}{n!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^n \frac{x^n}{n} + \dots; |x| < 1 \quad (\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \dots + (-1)^n \frac{1}{n} + \dots)$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots; |x| < 1$$

$$\ln\left(\frac{1-x}{1+x}\right) = 2\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots\right); |x| < 1$$

Theorem 3 (Integral form) 1984 Paper 2 Q2, 2003 Paper 2 Q12

If  $f$  is  $n$  times continuously differentiable on  $[a - h, a + h]$ , then

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x f^{(n)}(t) \frac{(x-t)^{n-1}}{(n-1)!} dt$$

Proof: Let  $I_m = \frac{1}{(m-1)!} \int_0^x (x-t)^{m-1} f^{(m)}(t) dt, m \geq 1$

$$\begin{aligned} I_{m+1} &= \frac{1}{m!} \int_0^x (x-t)^m f^{(m+1)}(t) dt = \frac{1}{m!} \int_0^x (x-t)^m df^{(m)}(t) \\ &= \frac{1}{m!} \left[ (x-t)^m f^{(m)}(t) \right]_0^x - \frac{1}{m!} \int_0^x f^{(m)}(t) d(x-t)^m \quad (\text{integration by parts}) \\ &= -\frac{1}{m!} x^m f^{(m)}(0) + \frac{1}{(m-1)!} \int_0^x (x-t)^{m-1} f^{(m)}(t) dt \\ &= I_m - \frac{f^{(m)}(0)}{m!} x^m \\ I_n &= I_{n-1} - \frac{f^{(n-1)}(0)}{(n-1)!} x^{n-1} \\ &= I_{n-2} - \frac{f^{(n-2)}(0)}{(n-2)!} x^{n-2} - \frac{f^{(n-1)}(0)}{(n-1)!} x^{n-1} \\ &= \dots \\ &= I_1 - \frac{f^{(1)}(0)}{1!} x - \dots - \frac{f^{(n-2)}(0)}{(n-2)!} x^{n-2} - \frac{f^{(n-1)}(0)}{(n-1)!} x^{n-1} \end{aligned}$$

On the other hand,  $I_1 = \frac{1}{1!} \int_0^x (x-t)^{1-1} f^{(1)}(t) dt = \int_0^x f'(t) dt = \int_0^x df(t) = f(x) - f(0)$

$$\begin{aligned} \therefore I_n &= f(x) - f(0) - \frac{f^{(1)}(0)}{1!} x - \dots - \frac{f^{(n-2)}(0)}{(n-2)!} x^{n-2} - \frac{f^{(n-1)}(0)}{(n-1)!} x^{n-1} \\ f(x) &= f(0) + \frac{f^{(1)}(0)}{1!} x + \dots + \frac{f^{(n-2)}(0)}{(n-2)!} x^{n-2} + \frac{f^{(n-1)}(0)}{(n-1)!} x^{n-1} + \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f^{(n)}(t) dt \end{aligned}$$

1984 Paper 2 Q2c Show that  $0 < \ln(1+x) - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 < \frac{1}{5}x^5$ , for  $0 < x < 1$

Putting  $f(x) = \ln(1+x)$  which is infinitely differentiable on  $(-1, 1)$ ,

$$f'(x) = \frac{1}{1+x}; f^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{(1+x)^n}$$

for any  $0 < x < 1$ ,  $\ln(1+x) = \ln 1 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + I_5$

$$\text{Since } I_5 = \frac{1}{4!} \int_0^x (x-t)^4 \cdot \frac{(-1)^4 (4!)}{(1+t)^5} dt$$

> 0 as  $(x-t)^4$ ,  $(1+x)^5 > 0$  for  $t \in (0, x)$

$$\therefore \ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{3} = I_5 > 0$$

Similarly,  $I_6 = \frac{1}{5!} \int_0^x \frac{(x-t)^5 (-1)^5 (5!)}{(1+t)^6} dt < 0$

$$\therefore \ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} = I_6 + \frac{x^5}{5} < \frac{x^5}{5}$$

2003 Paper 2Q12b

Define  $g(x) = \frac{1}{\sqrt{1-x^2}}$  for all  $x \in (-1, 1)$ . Let  $n$  be a positive integer.

(i) Prove that  $(1-x^2)g'(x) - xg(x) = 0$ .

Hence deduce that  $(1-x^2)g^{(n+1)}(x) - (2n+1)xg^{(n)}(x) - n^2g^{(n-1)}(x) = 0$ , where  $g^{(0)} = g$ .

(ii) Prove that  $g^{(2n-1)}(0) = 0$  and  $g^{(2n)}(0) = \left( \frac{(2n)!}{(2^n)(n!)} \right)^2$ .

(iii) Using (a), prove that  $g(x) = \sum_{k=0}^{n-1} \frac{C_k^{2n}}{2^{2k}} x^{2k} + \frac{1}{(2n-1)!} \int_0^x (x-t)^{2n-1} g^{(2n)}(t) dt$

(i)  $(1-x^2)g^2(x) = 1$

Differentiate both sides w.r.t.  $x$

$$2(1-x^2)g(x)g'(x) - 2xg^2(x) = 0$$

$$\because g(x) \neq 0, (1-x^2)g'(x) - xg(x) = 0 \dots\dots\dots(*)$$

Use Leibniz rule to differentiate (\*) w.r.t.  $x$   $n$  times.

$$(1-x^2)g^{(n+1)}(x) - 2nxg^{(n)}(x) - n(n-1)g^{(n-1)}(x) - xg^{(n)}(x) - n^2g^{(n-1)}(x) = 0$$

$$(1-x^2)g^{(n+1)}(x) - (2n+1)xg^{(n)}(x) - n^2g^{(n-1)}(x) = 0 \dots\dots\dots(**)$$

(ii)  $g(0) = 1 = \left( \frac{(2 \times 0)!}{(2^0)(0!)} \right)^2$

Put  $x=0$  in (\*),  $g'(0) = 0$

$$\text{Put } x=0, n=1 \text{ into (**), } g''(0) = 1 = \left( \frac{2!}{(2^1)(1!)} \right)^2$$

$\therefore$  the statement is true for  $n=1$

$$\text{Suppose } g^{(2k-1)}(0) = 0 \text{ and } g^{(2k)}(0) = \left( \frac{(2k)!}{(2^k)(k!)} \right)^2$$

Put  $x = 0, n = 2k$  into (\*\*):  $g^{(2k+1)}(0) = (2k)^2 g^{2k-1}(0) = 0$

Put  $x = 0, n = 2k+1$  into (\*\*):  $g^{(2k+2)}(0) = (2k+1)^2 g^{(2n)}(0) = \left( \frac{(2k+1)!}{(2^k)(k!)} \right)^2 = \left[ \frac{(2k+2)!}{(2^{k+1})(k+1)!} \right]^2$

The statement is also true for  $n = k + 1$ , by M. I., the statement is true for all  $n \in \mathbf{N} \cup \{0\}$

(iii) By Taylor's Theorem (integral form), (replace  $n$  by  $2n$ , put  $a = 0$ )

$$\begin{aligned} g(x) &= \sum_{k=0}^{2n-1} \frac{g^{(k)}(0)}{k!} x^k + \int_0^x g^{(2n)}(t) \frac{(x-t)^{2n-1}}{(2n-1)!} dt \\ &= \sum_{k=0}^{n-1} \frac{\left[ \frac{(2k)!}{(2^k)(k!)} \right]^2}{(2k)!} x^{2k} + \frac{1}{(2n-1)!} \int_0^x (x-t)^{2n-1} g^{(2n)}(t) dt \\ &= \sum_{k=0}^{n-1} \frac{C_k^{2n}}{2^{2k}} x^{2k} + \frac{1}{(2n-1)!} \int_0^x (x-t)^{2n-1} g^{(2n)}(t) dt \end{aligned}$$