Rolle's Theorem

Techniques of Mathematical Analysis by C.J. Tranter p.109 5.3

If a and b are consecutive roots of the equation $f(x) = 0$, then the equation $f\,'(x) = 0$ has a real root between a and b.

Proof: Let a, b be consecutive r-multiple and s-multiple roots respectively so that

$f(x) \equiv (x - a)^r (x - b)^s\, g(x)$,

where $g(x)$ has the same sign throughout the interval a to b. Hence

$\log f(x) = r \log (x - a) + s \log(x - b) + \log g(x)$,

and, differentiating with respect to x,

$$\frac{f\,'(x)}{f(x)} = \frac{r}{x-a} + \frac{s}{x-b} + \frac{g'(x)}{g(x)}.$$

Cross multiplying the substituting for $f(x)$,

$f\,'(x) \equiv (x - a)^{r-1}(x - b)^{s-1}\, h(x)$,

where $h(x) \equiv \{r(x - b)+s(x - a)\}\, g(x) + (x - a)(x - b)g'(x)$

The values taken by $x = a$, $x = b$ respectively are

$h(a) = r(a - b)g(a)$, $h(b) = s(b - a)g(b)$.

Since g is one-signed, these expressions are of opposite signs. It follows that $h(x)$, and therefore also $f\,'(x)$, vanishes for some value of x between a and b.