Mean Value Theorem

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Let f:$[a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then $\exists c \in (a, b)$ s.t. $f'(c)=\frac{f(b)-f(a)}{b-a}$

Proof: Consider the auxiliary function $g(x) = f(a) - f(x) + \frac{x-a}{b-a}[f(b)-f(a)]$

Then $g(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

$g(a) = g(b) = 0$.

By Rolle's Theorem, $\exists c \in (a, b)$ s.t. $g'(c) = 0$ and hence $f'(c)=\frac{f(b)-f(a)}{b-a}$.

Cauchy Mean Value Theorem

Let f and g be differentiable on (a, b) and continuous an $[a, b]$, then $\exists c \in (a, b)$ such that

$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$

Proof: Define $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$

Then h is continuous on $[a, b]$ and differentiable on (a, b)

We have $h(a) = f(a)\,g(b) - g(a)f(b)$

$h(b) = f(a)g(b) - g(a)f(b)$

$\therefore h(a) = h(b)$

By Rolle's Theorem, $\exists c \in (a, b)$ s.t. $h'(c) = 0$ and result follows.

Example 1

If $\frac{a_0}{n+1}+\frac{a_1}{n}+\cdots+\frac{a_{n-1}}{2}+a_n=0$, prove that the equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ has at least one root between 0 and 1.

Let $f(x)=\frac{a_0x^{n+1}}{n+1}+\frac{a_1x^n}{n}+\cdots+\frac{a_{n-1}x^2}{2}+a_nx$

Then $f(x)$ is a polynomial which is continuously differentiable everywhere.

$f(0) = 0$ and $f(1)=\frac{a_0}{n+1}+\frac{a_1}{n}+\cdots+\frac{a_{n-1}}{2}+a_n=0$ (given)

By mean valued theorem, $\exists c \in (0, 1)$ such that $f'(c) = 0$

$f'(c) = a_0c^n + a_1c^{n-1} + \ldots + a_{n-1}c + a_n = 0$

i.e. the equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ has at least one root (c) between 0 and 1.

Example 2

By using mean value theorem on $f(x) = \cos x$ (in radians) with $a=\frac{\pi}{3}+\frac{\pi}{180}$, $b=\frac{\pi}{3}$,

prove that $\frac{1}{2}-\frac{\pi}{360}>\cos 61^\circ>\frac{1}{2}-\frac{\pi}{180}$.

$f(x) = \cos x$, $a=\frac{\pi}{3}+\frac{\pi}{180}$, $b=\frac{\pi}{3}$

$$\exists\, c \in (b, a) \text{ such that } \frac{f(a)-f(b)}{a-b}=f'(c) \Rightarrow \frac{\cos\left(\frac{\pi}{3}+\frac{\pi}{180}\right)-\cos\left(\frac{\pi}{3}\right)}{\frac{\pi}{180}}=-\sin c$$

$$\frac{\pi}{180}\sin 30^\circ<\frac{\pi}{180}\sin c=\frac{1}{2}-\cos 61^\circ<\frac{\pi}{180}\sin 90^\circ$$

$$\frac{\pi}{360}<\frac{1}{2}-\cos 61^\circ<\frac{\pi}{180} \Rightarrow \frac{1}{2}-\frac{\pi}{360}>\cos 61^\circ>\frac{1}{2}-\frac{\pi}{180}$$

Example 3

(*a*) If $f'(x)$ exists in $0 < a \le x \le b$, show that there exists $c \in (a, b)$ such that

$$f(b) - f(a) = c f'(c) \ln\frac{b}{a}.$$

(Hint: Let $g(x) = [f(b) - f(a)] \ln\frac{x}{a} - f(x)\ln\frac{b}{a}$)

(*b*) By taking $f(x) = x^{\frac{1}{n}}$, deduce that $\lim_{n\to\infty} n\left(a^{\frac{1}{n}} - 1\right) = \ln a$ for $a > 0$.

(*a*) Let $g(x) = [f(b) - f(a)] \ln\frac{x}{a} - f(x)\ln\frac{b}{a}$

$g(a) = -f(a)\ln\frac{b}{a}$; $g(b) = -f(a)\ln\frac{b}{a}$

By Rolle's Theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$

$$[f(b) - f(a)]\frac{1}{c} - f'(c)\ln\frac{b}{a} = 0$$

$$f(b) - f(a) = c f'(c) \ln\frac{b}{a}.$$

(*b*) When $a = 1$, LHS = RHS = 0

When $a > 1$, $n\left(a^{\frac{1}{n}} - 1\right) = n c f'(c) \ln a = c^{\frac{1}{n}} \ln a$, for some $c \in (1, a)$

Let $c^{\frac{1}{n}} = 1 + h_n$, $h_n > 0$

Then $c = (1 + h_n)^n = 1 + nh_n + \ldots. > 1 + nh_n$

$$\frac{c-1}{n} > h_n > 0$$

$$\lim_{n\to\infty}\frac{c-1}{n} \ge \lim_{n\to\infty} h_n \ge \lim_{n\to\infty} 0$$

By squeezing principle, $\lim_{n\to\infty} h_n = 0$

$$\therefore \lim_{n\to\infty} c^{\frac{1}{n}} = 1$$

$$\lim_{n\to\infty} n\left(a^{\frac{1}{n}} - 1\right) = \ln a$$

When $0 < a < 1$, $n\left(a^{\frac{1}{n}} - 1\right) = c^{\frac{1}{n}} \ln a$, for some $c \in (a, 1)$

It can be easily proved that $\lim_{n\to\infty} n\left(a^{\frac{1}{n}} - 1\right) = \ln a$

Mean Value Theorem for Integrals

Calculus Volume 2 Second Edition by Tom M.APOSTOL p.154, 219

Theorem 1 If f is continuous on $[a, b]$, then for some c in $[a,b]$ we have $\int_a^b f(x)dx = f(c)(b-a)$

Proof: let Max($f(x)$) = M, Min($f(x)$) = m.

$$m \le f(x) \le M$$

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

$$m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$$

By the intermediate-value theorem, there exists a constant c: $a \le c \le b$, such that

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c)$$

$$\int_a^b f(x)dx = f(c)(b-a)$$

Theorem 2 Weighted Mean-Valued Theorem for Integrals

Assume f and g are continuous on $[a,b]$. If g is never changes sign in $[a,b]$, then there exists $c \in [a,b]$ such that $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$

Proof: If $g(x) \ge 0$ for all $x \in [a,b]$, then $\int_a^b g(x)dx \ge 0$

let Max($f(x)$) = M, Min($f(x)$) = m.

$$mg(x) \le f(x)\, g(x) \le Mg(x)$$

$$m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (*)$$

If $\int_a^b g(x)dx = 0$, then $g(x) = 0$ for all $x \in [a,b]$

(otherwise $\exists x_0 \in (a,b)$ such that $g(x_0) > 0$,

since g is continuous, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon$

$-\varepsilon < g(x) - g(x_0) < \varepsilon$

$-\varepsilon + g(x_0) < g(x) < \varepsilon + g(x_0)$

let $\varepsilon = \frac{1}{2}g(x_0) > 0$, then $\frac{1}{2}g(x_0) < g(x) < \frac{3}{2}g(x_0)$, for all x: $|x - x_0| < \delta$

$\int_a^b g(x)dx \ge \int_{x_0-\delta}^{x_0+\delta} g(x)dx > 0$, which leads to a contradiction.)

In this case, $\int_a^b f(x)g(x)dx = \int_a^b 0dx = 0 = f\left(\frac{a}{2}\right)\int_a^b g(x)dx, c = \frac{a}{2}$

If $\int_a^b g(x)dx \ne 0$, then $\int_a^b g(x)dx > 0$; divide (*) by $\int_a^b g(x)dx$.

$$m \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le M$$

By the intermediate-value theorem, there exists a constant c: $a < c < b$, such that

$$\frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} = f(c)$$

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

If $g(x) \le 0$, we can arrive at the same result if we apply on $-g(x) \ge 0$.

Theorem 3 Second Mean Value Theorem for Integrals

Assume g is continuous on $[a, b]$, and assume f has a derivative which is continuous and never change sign in $[a,b]$. Then there exists $c\in[a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx$$

Proof: Let $G(x) = \int_a^x g(t)dt$, since g is continuous, we have $G'(x) = g(x)$

Therefore integrating by parts gives

$$\int_a^b f(x)g(x)dx = \int_a^b f(x)G'(x)dx = \int_a^b f(x)dG(x)$$

$$= f(b)G(b) - \int_a^b f'(x)G(x)dx \quad (\because G(a) = 0)$$

By Theorem 2, we have $\int_a^b f'(x)G(x)dx = G(c)\int_a^b f'(x)dx$ for some $c\in[a,b]$

$$= G(c)[f(b) - f(a)]$$

$$\int_a^b f(x)g(x)dx = f(b)G(b) - G(c)[f(b) - f(a)]$$

$$= f(b)\int_a^b g(t)dt - [f(b) - f(a)]\int_a^c g(t)dt$$

$$= f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx$$

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(*b*) Let $F(x)$ be a function with a continuous second derivative such that $F''(x) \ge 0$ and $F'(x) \ge m > 0$ for $a \le x \le b$. Using Theorem 3 with $f(x) = -\frac{1}{F'(x)}$ and $g(x) = -F'(x)\cos F(x)$, show that

$$\left|\int_a^b \cos F(x)dx\right| \le \frac{4}{m}.$$

$f'(x) = \frac{F''(x)}{[F'(x)]^2} > 0$ for all $x \in [a,b]$, so $f(x)$ satisfies the conditions in Theorem 3.

$$\int_a^b \cos F(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx\text{, for some } c\in[a,b]$$

$$= -\frac{1}{F'(a)}\int_a^c -F'(x)\cos F(x)dx - \frac{1}{F'(b)}\int_c^b -F'(x)\cos F(x)dx$$

$$= \frac{1}{F'(a)}\int_a^c F'(x)\cos F(x)dx + \frac{1}{F'(b)}\int_c^b F'(x)\cos F(x)dx$$

$$= \frac{1}{F'(a)}\int_a^c \cos F(x)dF(x) + \frac{1}{F'(b)}\int_c^b \cos F(x)dF(x)$$

$$= \frac{1}{F'(a)}\sin F(x)\Big|_a^c + \frac{1}{F'(b)}\sin F(x)\Big|_c^b$$

$$\left|\int_a^b \cos F(x)dx\right| = \left|\frac{1}{F'(a)}[\sin F(c) - \sin F(a)] + \frac{1}{F'(b)}[\sin F(b) - \sin F(c)]\right|$$

$$\le \frac{|\sin F(c)|}{|F'(a)|} + \frac{|\sin F(a)|}{|F'(a)|} + \frac{|\sin F(b)|}{|F'(b)|} + \frac{|\sin F(c)|}{|F'(b)|}$$

$$\le \frac{4}{m}$$

(*c*) (*i*) Show that $\int_0^1 \cos(x^n)dx \le \int_0^1 \cos(x^{n+1})dx$

Hence show that $\lim_{n\to\infty}\int_0^1 \cos(x^n)dx$ exists.

(*ii*) Using (*b*), or otherwise, show that $\lim_{n\to\infty}\int_0^{2\pi} \cos(x^n)dx$ exists.

(*i*) For $0 \le x \le 1$, $x^n \ge x^{n+1}$,

$0 \le \cos(x^n) \le \cos(x^{n+1})$

$\int_0^1 \cos(x^n)dx \le \int_0^1 \cos(x^{n+1})dx$

The sequence $\left\{\int_0^1 \cos(x^n)dx\right\}$ is monotonic increasing which is bounded above by 1.

By Monotonic convergent Theorem, $\lim_{n\to\infty}\int_0^1 \cos(x^n)dx$ exists.

(*ii*) Let $F(x) = x^n$; $1 \le x \le 2\pi$.

For $n \ge 2$, $F'(x) = nx^{n-1} \ge n > 0$ and $F''(x) = n(n-1)x^{n-1} > 0$

$\therefore$ By (*b*), $\left|\int_1^{2\pi} \cos x^n dx\right| \le \frac{4}{n}$

$$\Rightarrow \lim_{n\to\infty}\left|\int_1^{2\pi} \cos x^n dx\right| = 0$$

$$\Rightarrow \lim_{n\to\infty}\int_1^{2\pi} \cos x^n dx = 0$$

$$\lim_{n\to\infty}\int_0^{2\pi} \cos\left(x^n\right)dx = \lim_{n\to\infty}\int_0^{1} \cos\left(x^n\right)dx + \lim_{n\to\infty}\int_1^{2\pi} \cos\left(x^n\right)dx$$

$$= \lim_{n\to\infty}\int_0^{1} \cos\left(x^n\right)dx$$

$\therefore$ The limit exists.