

L' Hôpital's Rule

Created by Francis Hung on 26 June 2008 Last updated: May 20, 2009

Let f and g be differentiable on $(a - \delta, a + \delta) \setminus \{a\}$ with $g'(x) \neq 0$ on this set. Then

(i) If $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$ as $x \rightarrow a$ and if $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \ell$ exists,

then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ exists and equals to ℓ

(ii) If $f(x) \rightarrow \infty$ and $g(x) \rightarrow \infty$ as $x \rightarrow a$ and if $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \ell$ exists,

then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ exists and equals to ℓ .

Proof: (i) Take x s.t. $a < x < a + \delta$

Now f and g continuous on $[a, x]$ and differentiable on (a, x)

$[f(x) \text{ and } g(x) \rightarrow 0 \text{ as } x \rightarrow a]$

By Cauchy's mean value theorem on $[a, x]$

$\exists c_x \in (a, x)$ s.t. $[f(x) - f(a)]g'(c_x) = [g(x) - g(a)]f'(c_x)$

Note: $g(x) \neq 0$; otherwise $\exists \alpha \in (a, x)$ s.t. $g'(\alpha) = 0$ contradiction

\therefore we get $\frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)} \rightarrow \ell$ as $x \rightarrow a^+$ ($c_x \rightarrow a$)

Similarly $\lim_{x \rightarrow a^-} \frac{f(x)}{g(x)} = \ell$

$\therefore \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \ell$

(ii) Given $\varepsilon > 0$, first choose $x_2 \in (a, a + \delta)$ s.t. whenever $a < x \leq x_2$,

We have $\left| \frac{f'(x)}{g'(x)} - \ell \right| < \varepsilon$ and $g(x) > 0$

Now choose $x_1 \in (a, x_2)$ s.t. $g(x) > \max \left\{ \frac{g(x_2)}{\varepsilon}, \frac{f(x_2)}{\varepsilon} \right\}$ whenever $a < x \leq x_1$

Apply Cauchy's mean value theorem on $[x, x_2]$

$\exists c_x \in (x, x_2)$ s.t. $[f(x_2) - f(x)]g'(c_x) = [g(x_2) - g(x)]f'(c_x)$

$\frac{f(x)}{g(x)} = \frac{f(x_2)}{g(x)} + \frac{f'(c_x)}{g'(c_x)} - \frac{g(x_2)f'(c_x)}{g(x)g'(c_x)}$

$\left| \frac{f(x)}{g(x)} - \ell \right| \leq \left| \frac{f(x_2)}{g(x)} \right| + \left| \frac{f'(c_x)}{g'(c_x)} - \ell \right| + \left| \frac{g(x_2)f'(c_x)}{g(x)g'(c_x)} \right|$

$\leq \varepsilon + \varepsilon + \varepsilon(|\ell| + \varepsilon) < (|\ell| + 3)\varepsilon$ assuming $\varepsilon < 1$

$\therefore \frac{f(x)}{g(x)} \rightarrow \ell$ as $x \rightarrow a^+$

Similarly $\lim_{x \rightarrow a^-} \frac{f(x)}{g(x)} = \ell$

(I) Rules on infinity (∞)

∞ is not a number, for any real number x , $-\infty < x < \infty$

$$(a) \quad \infty + \infty = \infty$$

$$(b) \quad \infty \times \infty = \infty$$

$$(c) \quad \frac{1}{\infty} = 0$$

$$(d) \quad \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty; \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

$$(e) \quad \infty^\infty = \infty$$

$$(f) \quad a^\infty = \infty \text{ for } a > 1$$

$$(g) \quad a^\infty = 0 \text{ for } 0 < a < 1$$

$$(h) \quad 0^a = 0 \text{ for } a > 0, a \text{ can be } \infty$$

$$(i) \quad 1^a = 1 \text{ for any number } a, a \neq \infty$$

(II) Indeterminate Forms

$$(a) \quad \frac{0}{0}$$

$$(b) \quad \frac{\infty}{\infty}$$

$$(c) \quad 0 \times \infty$$

$$(d) \quad \infty - \infty$$

$$(e) \quad 0^0$$

$$(f) \quad \infty^0$$

$$(g) \quad 1^\infty$$

(III) If $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$; (a can be any number or ∞ or $-\infty$)

$$\text{then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

(IV) Examples and exercises:

$$\begin{aligned}
 (1) \quad & \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^3} && \text{(type } \frac{0}{0} \text{)} \\
 & = \lim_{x \rightarrow 0} \frac{-x \sin x + \cos x - \cos x}{3x^2} && \text{(L' Hôpital's rule)} \\
 & = \lim_{x \rightarrow 0} \frac{-\sin x}{3x} && \text{(type } \frac{0}{0} \text{)} \\
 & = \lim_{x \rightarrow 0} \frac{-\cos x}{3} && \text{(L' Hôpital's rule)} \\
 & = -\frac{1}{3}
 \end{aligned}$$

Exercise 1.1 $\lim_{x \rightarrow 0} \frac{\cos\left(\frac{\pi}{2}\right)\cos x}{\sin^2 x}$

Exercise 1.2 $\lim_{x \rightarrow 0} \frac{e - (1+x)^{1/x}}{x}$

Exercise 1.3 If $\lim_{x \rightarrow 0} \frac{x(1 + a \cos x) - b \sin x}{x^3} = 1$, find a and b .

(2) $\lim_{x \rightarrow a^+} \frac{\log(x-a)}{\log(e^x - e^a)}$ (type $\frac{\infty}{\infty}$)

$$= \lim_{x \rightarrow a^+} \frac{1}{\frac{x-a}{e^x}} \quad (\text{L' Hôpital's rule})$$

$$= \lim_{x \rightarrow a^+} \frac{e^x - e^a}{e^x(x-a)} \quad (\text{type } \frac{0}{0})$$

$$= \lim_{x \rightarrow a^+} \frac{e^x}{e^x(x-a) + e^x} \quad (\text{L' Hôpital's rule})$$

$$= \frac{e^a}{e^a} = 1$$

Exercise 2.1 $\lim_{x \rightarrow \infty} \frac{\log(1+e^x)}{x}$

Exercise 2.2 $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\log \tan^2 x}{\log\left(x - \frac{\pi}{2}\right)^2}$

(3) $\lim_{x \rightarrow 0^+} x \log x$ (type $0 \times -\infty$)

$$= \lim_{x \rightarrow 0^+} \frac{\log x}{\frac{1}{x}} \quad (\text{type } \frac{\infty}{\infty})$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{-x} \quad (\text{L' Hôpital's rule})$$

$$= \lim_{x \rightarrow 0^+} (-x) = 0$$

Exercise 3.1 $\lim_{x \rightarrow a} (a-x) \tan\left(\frac{\pi x}{2a}\right)$

Exercise 3.2 $\lim_{x \rightarrow 0} \sin(\sin x) \tan\left(x - \frac{\pi}{2}\right)$

(4) $\lim_{x \rightarrow 2} \left[\frac{1}{x-2} - \frac{1}{\log(x-1)} \right]$ (type $\infty - \infty$)

$$= \lim_{x \rightarrow 2} \frac{\log(x-1) - (x-2)}{(x-2)\log(x-1)} \quad (\text{type } \frac{0}{0})$$

$$= \lim_{x \rightarrow 2} \frac{\frac{1}{x-1} - 1}{\frac{x-2}{x-1} + \log(x-1)} \quad (\text{L' Hôpital's rule})$$

$$= \lim_{x \rightarrow 2} \frac{2-x}{x-2 + (x-1)\log(x-1)} \quad (\text{type } \frac{0}{0})$$

$$= \lim_{x \rightarrow 2} \frac{-1}{1+1+\log(x-1)} \quad (\text{L' Hôpital's rule})$$

$$= -\frac{1}{2}$$

Exercise 4.1 $\lim_{x \rightarrow 0} \left(\frac{a}{x} - \cot \frac{x}{a} \right)$

Exercise 4.2 $\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{2x}{\pi} \sec x - \tan x \right)$

Exercise 4.3 $\lim_{x \rightarrow 0} \left(\frac{x-1}{2x^2} + \frac{e^{-x}}{2x \sinh x} \right)$, where $\sinh z = \frac{1}{2} (e^z - e^{-z})$.

(5) Evaluate $\lim_{x \rightarrow 0} (\cos x)^{1/x^2}$.

Let $y = (\cos x)^{1/x^2}$

$$\log y = \frac{\log(\cos x)}{x^2}$$

$$\lim_{x \rightarrow 0} \log y = \lim_{x \rightarrow 0} \frac{\log \cos x}{x^2} \quad (\text{type } \frac{0}{0})$$

$$= \lim_{x \rightarrow 0} \frac{-\tan x}{2x} \quad (\text{L' Hôpital's rule), (type } \frac{0}{0})$$

$$= \lim_{x \rightarrow 0} \left(-\frac{\sec^2 x}{2} \right) \quad (\text{L' Hôpital's rule})$$

$$\log \lim_{x \rightarrow 0} y = -\frac{1}{2}$$

$$\lim_{x \rightarrow 0} y = e^{-\frac{1}{2}}$$

Exercise 5.1 $\lim_{x \rightarrow 0} (1 + \sin x)^{\cot x}$ (type 1^∞)

Exercise 5.2 $\lim_{x \rightarrow \infty} x^{1/x}$ (type ∞^0)

Exercise 5.3 $\lim_{x \rightarrow a^+} (x-a)^{x-a}$ (type 0^0)

$$\begin{aligned}
 \text{Exercise 1.1} \quad & \lim_{x \rightarrow 0} \frac{\cos\left[\left(\frac{\pi}{2}\right)\cos x\right]}{\sin^2 x} \quad (\text{type } \frac{0}{0}) \\
 & = \lim_{x \rightarrow 0} \frac{-\sin\left[\left(\frac{\pi}{2}\right)\cos x\right]\frac{\pi}{2}(-\sin x)}{2 \sin x \cos x} \quad (\text{L' Hôpital's rule}) \\
 & = \frac{\pi}{4} \lim_{x \rightarrow 0} \frac{\sin\left[\left(\frac{\pi}{2}\right)\cos x\right]}{\cos x} \quad (\text{L' Hôpital's rule}) \\
 & = \frac{\pi}{4} \cdot \frac{\sin \frac{\pi}{2}}{\cos 0} = \frac{\pi}{4}
 \end{aligned}$$

$$\text{Exercise 1.2} \quad \lim_{x \rightarrow 0} \frac{e - (1+x)^{1/x}}{x}$$

$$\begin{aligned}
 \text{Note that } \lim_{x \rightarrow 0^+} (1+x)^{1/x} &= \lim_{y \rightarrow \infty} \left(1 + \frac{1}{y}\right)^y \\
 &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \lim_{x \rightarrow 0^-} (1+x)^{1/x} &= \lim_{y \rightarrow -\infty} \left(1 + \frac{1}{y}\right)^y \\
 &= \lim_{z \rightarrow \infty} \left(1 - \frac{1}{z}\right)^{-z}, \text{ where } y = -z \\
 &= \lim_{z \rightarrow \infty} \frac{1}{\left(\frac{z-1}{z}\right)^z} \\
 &= \lim_{z \rightarrow \infty} \left(\frac{z}{z-1}\right)^z \\
 &= \lim_{z \rightarrow \infty} \left(1 + \frac{1}{z-1}\right)^{z-1} \left(1 + \frac{1}{z-1}\right) \\
 &= \lim_{z \rightarrow \infty} \left(1 + \frac{1}{z-1}\right)^{z-1} \cdot 1 \\
 &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n, \text{ where } n = z-1 \\
 &= e
 \end{aligned}$$

$$\therefore \lim_{x \rightarrow 0^+} (1+x)^{1/x} = \lim_{x \rightarrow 0^-} (1+x)^{1/x} = e$$

$$\text{Now let } y = (1+x)^{1/x}$$

$$\ln y = \frac{\ln(1+x)}{x}$$

Differentiate w.r.t. x

$$\frac{y'}{y} = \frac{\frac{x}{1+x} - \ln(1+x)}{x^2}$$

$$y' = \frac{x - (1+x)\ln(1+x)}{x^2(1+x)} \cdot (1+x)^{1/x} \dots\dots (*)$$

$$\lim_{x \rightarrow 0} \frac{e - (1+x)^{1/x}}{x} \quad (\text{type } \frac{e-e}{0} = \frac{0}{0})$$

$$= \lim_{x \rightarrow 0} -\frac{d}{dx} (1+x)^{1/x} \quad (\text{L' Hôpital's rule})$$

$$\begin{aligned}
 &= -\lim_{x \rightarrow 0} \frac{x - (1+x)\ln(1+x)}{x^2(1+x)} \cdot (1+x)^{\frac{1}{x}} && \text{(by (*))} \\
 &= -e \lim_{x \rightarrow 0} \frac{x - (1+x)\ln(1+x)}{x^2(1+x)} && \text{(type } \frac{0}{0} \text{)} \\
 &= -e \lim_{x \rightarrow 0} \frac{1 - 1 - \ln(1+x)}{2x + 3x^2} && \text{(L' Hôpital's rule)} \\
 &= e \lim_{x \rightarrow 0} \frac{\ln(1+x)}{2x + 3x^2} && \text{(type } \frac{0}{0} \text{)} \\
 &= e \lim_{x \rightarrow 0} \frac{\frac{1}{1+x}}{2 + 6x} && \text{(L' Hôpital's rule)} \\
 &= \frac{e}{2}
 \end{aligned}$$

Exercise 1.3 If $\lim_{x \rightarrow 0} \frac{x(1 + a \cos x) - b \sin x}{x^3} = 1$, find a and b . (type $\frac{0}{0}$)

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{(1 + a \cos x) - ax \sin x - b \cos x}{3x^2} &= 1, \quad \text{(L' Hôpital's rule)} \\
 \lim_{x \rightarrow 0} \frac{1 + (a - b) \cos x - ax \sin x}{3x^2} &= 1
 \end{aligned}$$

In order that the limit exist, and equal to 1, it must be of the type $\frac{0}{0}$

$$\begin{aligned}
 \therefore 1 + a - b &= 0 \dots\dots\dots (1) \\
 \lim_{x \rightarrow 0} \frac{1 - \cos x - ax \sin x}{3x^2} &= 1, \quad \text{(type } \frac{0}{0} \text{)} \\
 \lim_{x \rightarrow 0} \frac{\sin x - a \sin x - ax \cos x}{6x} &= 1, \quad \text{(L' Hôpital's rule)} \\
 \lim_{x \rightarrow 0} \frac{(1 - a) \sin x - ax \cos x}{6x} &= 1, \quad \text{(type } \frac{0}{0} \text{)} \\
 \lim_{x \rightarrow 0} \frac{(1 - a) \cos x - a \cos x + ax \sin x}{6} &= 1, \quad \text{(L' Hôpital's rule)} \\
 \lim_{x \rightarrow 0} \frac{(1 - 2a) \cos x + ax \sin x}{6} &= 1, \\
 \frac{(1 - 2a)}{6} &= 1 \dots\dots\dots (2) \\
 a &= -\frac{5}{2} \\
 \text{sub into (1), } b &= 1 + a = -\frac{3}{2}
 \end{aligned}$$

Exercise 2.1

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \frac{\log(1 + e^x)}{x} & \quad \text{(type } \frac{\infty}{\infty} \text{)} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + e^x} \cdot e^x}{1} && \text{(L' Hôpital's rule)} \\
 &= \lim_{x \rightarrow \infty} \frac{e^x}{1 + e^x} && \text{(type } \frac{\infty}{\infty} \text{)} \\
 &= \lim_{x \rightarrow \infty} \frac{e^x}{e^x} = 1 && \text{(L' Hôpital's rule)}
 \end{aligned}$$

Exercise 2.2 $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\log \tan^2 x}{\log(x - \frac{\pi}{2})^2}$ (type $\frac{\infty}{\infty}$)

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{2 \tan x \sec^2 x}{\frac{\tan^2 x}{2(x - \frac{\pi}{2})^2}}$$

(L' Hôpital's rule)

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{(x - \frac{\pi}{2})}{\sin x \cos x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\sin x} \cdot \lim_{x \rightarrow \frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\cos x}$$

(type $\frac{0}{0}$)

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{-\sin x} = -1$$

(L' Hôpital's rule)

Exercise 3.1 $\lim_{x \rightarrow a} (a - x) \tan\left(\frac{\pi x}{2a}\right)$ (type $0 \times \infty$)

$$= \lim_{x \rightarrow a} \frac{\tan\left(\frac{\pi x}{2a}\right)}{\frac{1}{a-x}}$$

(type $\frac{\infty}{\infty}$)

$$= \lim_{x \rightarrow a} \frac{\frac{\pi}{2a} \sec^2\left(\frac{\pi x}{2a}\right)}{\frac{1}{(a-x)^2}}$$

$$= \lim_{x \rightarrow a} \frac{\pi}{2a} \cdot \frac{(a-x)^2}{\cos^2\left(\frac{\pi x}{2a}\right)}$$

(type $\frac{0}{0}$)

$$= \lim_{x \rightarrow a} \frac{\pi}{2a} \cdot \frac{-2(a-x)}{-2 \cos\left(\frac{\pi x}{2a}\right) \sin\left(\frac{\pi x}{2a}\right) \cdot \frac{\pi}{2a}}$$

(L' Hôpital's rule)

$$= \lim_{x \rightarrow a} \frac{2(a-x)}{2 \cos\left(\frac{\pi x}{2a}\right) \sin\left(\frac{\pi x}{2a}\right)}$$

$$= 2 \lim_{x \rightarrow a} \frac{a-x}{\sin\left(\frac{\pi x}{a}\right)}$$

(type $\frac{0}{0}$)

$$= 2 \lim_{x \rightarrow a} \frac{-1}{\frac{\pi}{a} \cos\left(\frac{\pi x}{a}\right)}$$

(L' Hôpital's rule)

$$= \frac{2a}{\pi}$$

Exercise 3.2 $\lim_{x \rightarrow 0} \sin(\sin x) \tan\left(x - \frac{\pi}{2}\right)$ (type $0 \times -\infty$)

$$= \lim_{x \rightarrow 0} \sin(\sin x) \cdot (-\cot x)$$

$$= -\lim_{x \rightarrow 0} \frac{\sin(\sin x)}{\tan x}$$

(type $\frac{0}{0}$)

$$= -\lim_{x \rightarrow 0} \frac{\cos(\sin x) \cos x}{\sec^2 x}$$

(L' Hôpital's rule)

$$= -\lim_{x \rightarrow 0} \cos(\sin x) \cos^3 x = -1$$

Exercise 4.1 $\lim_{x \rightarrow 0} \left(\frac{a}{x} - \cot \frac{x}{a}\right)$

$$= \lim_{x \rightarrow 0} \frac{a \tan \frac{x}{a} - x}{x \tan \frac{x}{a}}$$

(type $\frac{0}{0}$)

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \frac{a \sec^2 \frac{x}{a} \cdot \frac{1}{a} - 1}{\tan \frac{x}{a} + x \sec^2 \frac{x}{a} \cdot \frac{1}{a}} && \text{(L' Hôpital's rule)} \\
&= \lim_{x \rightarrow 0} \frac{\tan^2 \frac{x}{a}}{\tan \frac{x}{a} + \frac{x}{a} \sec^2 \frac{x}{a}} && (\because \sec^2 \frac{x}{a} - 1 = \tan^2 \frac{x}{a}) \\
&= \lim_{x \rightarrow 0} \frac{\sin^2 \frac{x}{a}}{\sin \frac{x}{a} \cos \frac{x}{a} + \frac{x}{a}} \\
&= \lim_{x \rightarrow 0} \frac{\frac{1}{2}(1 - \cos \frac{2x}{a})}{\frac{1}{2} \sin \frac{2x}{a} + \frac{x}{a}} \\
&= \lim_{x \rightarrow 0} \frac{1 - \cos \frac{2x}{a}}{\sin \frac{2x}{a} + \frac{2x}{a}} && \text{(type } \frac{0}{0} \text{)} \\
&= \lim_{x \rightarrow 0} \frac{\frac{2}{a} \sin \frac{2x}{a}}{\frac{2}{a} \cos \frac{2x}{a} + \frac{2}{a}} = 0 && \text{(L' Hôpital's rule)}
\end{aligned}$$

Exercise 4.2

$$\begin{aligned}
&\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{2x}{\pi} \sec x - \tan x \right) \\
&= \lim_{x \rightarrow \frac{\pi}{2}} \frac{2x - \pi \sin x}{\pi \cos x} && \text{(type } \frac{0}{0} \text{)} \\
&= \lim_{x \rightarrow \frac{\pi}{2}} \frac{2 - \pi \cos x}{-\pi \sin x} && \text{(L' Hôpital's rule)} \\
&= -\frac{2}{\pi}
\end{aligned}$$

Exercise 4.3

$$\begin{aligned}
&\lim_{x \rightarrow 0} \left(\frac{x-1}{2x^2} + \frac{e^{-x}}{2x \sinh x} \right), \text{ where } \sinh z = \frac{1}{2}(e^z - e^{-z}). \\
&= \lim_{x \rightarrow 0} \frac{(x-1) \sinh x + xe^{-x}}{2x^2 \sinh x} \\
&= \lim_{x \rightarrow 0} \frac{(x-1) \cdot \frac{1}{2}(e^x - e^{-x}) + xe^{-x}}{2x^2 \cdot \frac{1}{2}(e^x - e^{-x})} \\
&= \lim_{x \rightarrow 0} \frac{(x-1)(e^x - e^{-x}) + 2xe^{-x}}{2x^2(e^x - e^{-x})} \\
&= \lim_{x \rightarrow 0} \frac{(x-1)e^x + (x+1)e^{-x}}{2x^2(e^x - e^{-x})} \\
&= \lim_{x \rightarrow 0} \frac{(x-1)e^{2x} + x + 1}{2x^2(e^{2x} - 1)} && \text{(type } \frac{0}{0} \text{)} \\
&= \frac{1}{2} \lim_{x \rightarrow 0} \frac{e^{2x} + 2(x-1)e^{2x} + 1}{x(e^{2x} - 1) + x^2(2e^{2x})} && \text{(L' Hôpital's rule)} \\
&= \frac{1}{4} \lim_{x \rightarrow 0} \frac{(2x-1)e^{2x} + 1}{(x^2 + x)e^{2x} - x} && \text{(type } \frac{0}{0} \text{)} \\
&= \frac{1}{4} \lim_{x \rightarrow 0} \frac{2e^{2x} + (2x-1) \cdot 2 \cdot e^{2x}}{(2x+1)e^{2x} + 2(x^2 + x)e^{2x} - 1} && \text{(L' Hôpital's rule)} \\
&= \frac{1}{4} \lim_{x \rightarrow 0} \frac{4xe^{2x}}{(2x^2 + 4x + 1)e^{2x} - 1}
\end{aligned}$$

$$= \lim_{x \rightarrow 0} \frac{x}{2x^2 + 4x + 1 - e^{-2x}} \quad (\text{type } \frac{0}{0})$$

$$= \lim_{x \rightarrow 0} \frac{1}{4x + 4 + 2e^{-2x}} \quad (\text{L' Hôpital's rule})$$

$$= \frac{1}{6}$$

Exercise 5.1 $\lim_{x \rightarrow 0} (1 + \sin x)^{\cot x}$ (type 1^∞)

Let $y = (1 + \sin x)^{\cot x}$

$\log y = \cot x \log(1 + \sin x)$

$\log y = \frac{\log(1 + \sin x)}{\tan x}$

$\lim_{x \rightarrow 0} \log y = \lim_{x \rightarrow 0} \frac{\log(1 + \sin x)}{\tan x}$ (type $\frac{0}{0}$)

$= \lim_{x \rightarrow 0} \frac{\frac{\cos x}{1 + \sin x}}{\sec^2 x}$ (L' Hôpital's rule)

$= \lim_{x \rightarrow 0} \frac{\cos^3 x}{1 + \sin x} = 1$

$\log \lim_{x \rightarrow 0} y = 1 \Rightarrow \lim_{x \rightarrow 0} y = e$

Exercise 5.2 $\lim_{x \rightarrow \infty} x^{1/x}$ (type ∞^0)

Let $y = x^{1/x}$

$\log y = \frac{\log x}{x}$

$\lim_{x \rightarrow \infty} \log y = \lim_{x \rightarrow \infty} \frac{\log x}{x}$ (type $\frac{\infty}{\infty}$)

$= \lim_{x \rightarrow \infty} \frac{1}{x} = 0$ (L' Hôpital's rule)

$\log \lim_{x \rightarrow \infty} y = 0 \Rightarrow \lim_{x \rightarrow \infty} y = e^0 = 1$

Exercise 5.3 $\lim_{x \rightarrow a^+} (x - a)^{x-a}$ (type 0^0)

Let $y = (x - a)^{x-a}$

$\log y = (x - a) \log(x - a)$

$\lim_{x \rightarrow a^+} \log y = \lim_{x \rightarrow a^+} \frac{\log(x - a)}{\frac{1}{x-a}}$ (type $\frac{\infty}{\infty}$)

$\log \lim_{x \rightarrow a^+} y = \lim_{x \rightarrow a^+} \frac{1}{x-a} - \frac{1}{(x-a)^2}$ (L' Hôpital's rule)

$= - \lim_{x \rightarrow a^+} (x - a) = 0$

$\lim_{x \rightarrow a^+} y = e^0 = 1$