Hyperbolic Functions

First created in 1989, retyped as MS WORD document on 20080609 by Mr. Francis Hung

Reference: Techniques of Mathematical Analysis by C.T. Tranter p. 160 - p.165, p.263 - p.266

Euler's Function: $e^{ix} = \cos x + i \sin x$; $e^{-ix} = \cos x - i \sin x$

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix});\ \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

$$e^{x+iy} = e^x(\cos y + i \sin y)$$

Hyperbolic Function: $\cosh z = \frac{1}{2}(e^z + e^{-z})$ $\qquad$ $\text{sech } z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}}$

$\sinh z = \frac{1}{2}(e^z - e^{-z})$ $\qquad$ $\text{cosech } z = \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}}$

$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}$ $\qquad$ $\coth z = \frac{1}{\tanh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}}$

$\cosh z + \sinh z = e^z$ $\qquad$ $\cosh z - \sinh z = e^{-z}$

$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!} + \cdots$; $\qquad$ $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n-1}}{(2n-1)!} + \cdots$

Relation with real trigonometric functions:

$\cosh z = \cos iz$; $\qquad$ $\sinh z = -i \sin iz$

$\tanh z = -i \tan iz$; $\qquad$ $\coth z = i \cot iz$

$\text{sech } z = \sec iz$; $\qquad$ $\text{cosech } z = i \text{ cosec } iz$

Identities: $\cosh^2 z - \sinh^2 z = 1$

$\text{sech}^2 z = 1 - \tanh^2 z$

$\text{cosech}^2 z = -1 + \coth^2 z$

Exercise: Find sinh(–z); cosh(–z); sinh 0; cosh 0

Addition Formulae: $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

Double Angle Formulae: $\sinh 2z = 2 \sinh z \cosh z$

$\cosh 2z = \cosh^2 z + \sinh^2 z = 2\cosh^2 z - 1 = 2\sinh^2 z + 1$

Special manipulation: If x and y are real numbers,

$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

$\cos(x + iy) = \cos x \cosh y - i \cos x \sinh y$

$\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$

$\cosh(x + iy) = \cosh x \cos y + i \cosh x \sin y$

Inverse hyperbolic function: z is a complex number, $\text{Tanh}^{-1} z = \frac{1}{2}\text{Log}\left(\frac{1+z}{1-z}\right)$

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$ $\qquad$ $\text{cosech}^{-1} x = \ln\left(\frac{x \pm \sqrt{x^2+1}}{x}\right)$, $x \neq 0$

($x > 0$, take +; $x < 0$, take –)

$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$, $x \geq 1$ $\qquad$ $\text{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$, $0 < x \leq 1$

$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $-1 < x < 1$ $\qquad$ $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$, $|x| > 1$

Differentiation:

$$\frac{d\sinh x}{dx} = \cosh x \qquad \frac{d\,\text{cosech}\, x}{dx} = -\text{cosech}\, x \coth x$$

$$\frac{d\cosh x}{dx} = \sinh x \qquad \frac{d\,\text{sech}\, x}{dx} = -\text{sech}\, x \tanh x$$

$$\frac{d\tanh x}{dx} = \text{sech}^2 x \qquad \frac{d\coth x}{dx} = -\text{cosech}\, x$$

$$\frac{d\sinh^{-1} x}{dx} = \pm\frac{1}{\sqrt{1+x^2}} \quad (x \ge 0, \text{ take } +;\ x < 0, \text{ take } -)$$

$$\frac{d\cosh^{-1} x}{dx} = \pm\frac{1}{\sqrt{x^2-1}} \quad (\text{for all } x \ge 0, \text{ take both } + \text{ and } -)$$

$$\frac{d\tanh^{-1} x}{dx} = \pm\frac{1}{1-x^2} \quad (x \ge 0, \text{ take } +;\ x < 0, \text{ take } -)$$

$$\frac{d\,\text{cosech}^{-1} x}{dx} = \mp\frac{1}{x\sqrt{1+x^2}} \quad (x > 0, \text{ take } -;\ x < 0, \text{ take } +)$$

$$\frac{d\,\text{sech}^{-1} x}{dx} = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d\coth^{-1} x}{dx} = \mp\frac{1}{x^2-1} \quad (x \ge 0, \text{ take } -;\ x < 0, \text{ take } +)$$

Integration: The inverse process of differentiation

$$\int \sinh x\,dx = \cosh x + c \qquad \int \text{cosech}\, x \coth x\,dx = -\text{cosech}\, x + c$$

$$\int \cosh x\,dx = \sinh x + c \qquad \int \text{sech}\, x \tanh x\,dx = -\text{sech}\, x + c$$

$$\int \text{sech}^2 x\,dx = \tanh x + c \qquad \int \text{cosech}^2 x\,dx = -\coth x + c$$

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln\left|x+\sqrt{x^2+1}\right| = \sinh^{-1} x + c$$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \ln\left|x+\sqrt{x^2-1}\right| = \cosh^{-1} x + c$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = \tanh^{-1} x + c$$

$$\int \frac{1}{x\sqrt{1+x^2}}\,dx = -\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right| = -\text{cosech}^{-1} x + c$$

$$\int \frac{1}{x\sqrt{1-x^2}}\,dx = -\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| = -\text{sech}^{-1} x + c$$

$$\int \frac{1}{x^2-1}\,dx = -\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| = -\coth^{-1} x + c$$

Remember $\int \frac{1}{x}\,dx = \ln|x| + c$

then $\int \sec x\,dx = \ln|\sec x + \tan x| + c = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + c$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + c_1$$

$$= \ln|\csc x - \cot x| + c_2 = \ln\left|\tan\frac{x}{2}\right| + c_2$$